Notes on EPR "paradox", Bell's Inequality, and Experimental Tests

by

Douglas Ross

University of Southampton

1 Einstein-Rosen-Podolski Paper (1935):

In 1926 Albert Einstein wrote in a letter to Max Born:
"Die Quantenmechanik ist sehr achtunggebietend. Aber eine innere Stimme sagt mir, daß das noch nicht der wahre Jakob ist. Die Theorie liefert viel, aber dem Geheimnis des Alten bringt sie uns kaum näher. Jedenfalls bin ich überzeugt, daß der nicht würfelt." [1]

In 1935, he wrote a paper with Nathan Rosen and Boris Podolski [1] in which they considered two particles moving in one dimension with coordinates x_1, x_2 in an entangled state with wavefunction, $\Psi(x_1, x_2)$. Although the wavefunction cannot be factorised, it can be expanded as a series in the eigenfunctions, $u_n(x_1)$ of an operator $\hat{A}$ acting on particle 1,

$$\Psi(x_1, x_2) \;=\; \sum_n u_n(x_1)\, f_n(x_2), \tag{1}$$

where $f_n(x_2)$ are the coefficients of the expansion If, when the two particles are well-separated, a measurement is made on particle 1 of the quantity A and it is found to be in the n^{th} eigenstate, then the wavefunction collapses to the untangled state

$$\Psi'(x_1, x_2) = u_n(x_1)\, f_n(x_2). \tag{2}$$

Alternatively, we may expand the (initial) wavefunction in terms of the eigenfunctions, $v_m(x_1)$ of a different operator, $\hat{B}$ acting on particle 1:

$$\Psi(x_1, x_2) \;=\; \sum_m v_m(x_1)\, g_m(x_2), \tag{3}$$

where $g_m(x_2)$ are the coefficients of the expansion If instead of measuring the quantity A, we measure the quantity B of particle 1, and find it to be in its m^{th} eigenstate, then the wavefunction collapses to the state

$$\Psi''(x_1, x_2) = v_m(x_1)\, g_m(x_2). \tag{4}$$

Now it is perfectly possible that the coefficients $f_n(x_2)$ and $g_m(x_2)$ are themselves eigenfunctions of two non-commuting operators $\hat{C}$ and $\hat{D}$, corresponding to two properties of particle 2, which, according to the uncertainty principle cannot be measured simultaneously. Nevertheless, a measurement of either A or B on particle 1, which is well-separated from particle 2, determines whether particle 2 is in a state of well-defined C or well-defined D.

This, they claimed, was either evidence of "spooky (instantaneous) action at a distance", which contravenes Special Relativistic causality or a manifestation that Quantum Mechanics is incomplete and that the system has "hidden variables", which determine all the possible properties of the system (both particles).

[1] "Quantum Mechanics is worthy of close attention. However, an inner voice tells me that this is not yet the whole story. The theory delivers a lot, but it hardly brings us any closer to the secrets of the Old Man. In any case, I am convinced that He does not play dice."

As an example they considered a wavefunction for two particles moving in one dimension (ignoring normalisation constant)

$$\Psi(x_1, x_2) \; = \; \int_{-\infty}^{\infty} dp\, e^{ip(x_1 - x_2 - X)/\hbar}, \tag{5}$$

where X is some constant.

If we measure the momentum of particle 1 and obtain the value p_0, then the wavefunction becomes (up to an overall constant phase)

$$\Psi'(x_1, x_2) \; = \; \phi_{p_0}(x_1)\; \phi_{-p_0}(x_2),$$

where $\phi_p(x)$ are momentum eigenfunctions with eigenvalue p. This means that after the measurement of momentum on particle 1, we (instantaneously) know with absolute certainty what a measurement of the momentum of particle 2 will yield, even if the particles are separated so that no signal can be sent from particle 1 to particle 2 after measurement of its momentum but before the measurement of the momentum of particle 2.

On the other hand, we can also write the original wavefunction (up to an overal constant) as

$$\Psi(x_1, x_2) = \int_{-\infty}^{\infty} dx\, \delta\left(x - x_1)\right) \delta\left(x_2 - X + x\right). \tag{6}$$

If now, a measurement is performed on the position of particle 1 and found to be x_0, then the (untangled) wavefunction for particle 2 becomes

$$\psi(x_2) \; = \; \delta\left(x_2 - X + x_0\right). \tag{7}$$

This is a position eigenfunction with eigenvalue $(X - x_0)$ meaning that a measurement of the position of particle 1 instantaneously collapses the wavefunction of particle 2 into an eigenstate of position with eigenvalue $(X - x_0)$, so that its position is then known with absolute certainty.

1.1 Einstein's Interpretation:

Einstein argued that since the position and momentum of a particle were properties of its "reality", then Quantum Mechanics was incomplete and there had to be some "hidden parameters" which determined the position and momentum of a particle (and any other properties which contributed to its "local reality"), even though the two properties could not be measured simultaneously.

1.2 Bohr's Rebuttal:

Niels Bohr refuted this interpretation, claiming that one could not talk about "local reality" and that the properties of a system such as the momentum or position of a particle were simply not defined until they were measured, and at that point the measuring apparatus becomes part of the quantum system. This very abstract idea is known as the "Copenhagen interpretation."

At the fifth Solvsay Conference in 1927, Bohr told Einstein " Stop telling God what to do."

1.3 Bohm–Aharonov Development:

In 1957 David Bohm and Yakir Aharonov [2] applied this idea to dichotomic systems – i.e. to systems which have properties that can only take two values, i.e. the system can only be in one of two states.

The most obvious of these is the case of a spin-zero particle which decays into two spin-$\frac{1}{2}$ particles. By conservation of angular momentum, if one particle has component of spins $+\frac{1}{2}\hbar$ in some direction $\hat{\mathbf{n}}$, then the other must have component $-\frac{1}{2}\hbar$ in the same direction. Furthermore the total spin is zero, so the electrons are in an entangled state

$$\Psi_S \;=\; \frac{1}{\sqrt{2}}\left(|+\rangle|-\rangle - |-\rangle|+\rangle\right). \tag{8}$$

This can be expanded into eigenstates of the components of spin in *any* different direction

$$\Psi_S \;=\; \frac{1}{\sqrt{2}}\left(|+\rangle'|-\rangle' - |-\rangle'|+\rangle'\right). \tag{9}$$

If we measure the component of spin in that direction ($\hat{\mathbf{n}}'$) and find it to be $+\frac{1}{2}\hbar$ then we know with absolute certainty that the component of spin in the same direct is $-\frac{1}{2}\hbar$. Since we have a choice of which direction $\hat{\mathbf{n}}'$ to measure the spin-component then, according to Einstein's concept of the reality of the spin-$\frac{1}{2}$ particles, either there is some hidden variable which determines the component of spin in any given direction, or there is instantaneous communication between the particle whose spin is measured and the other particle. This is demonstrated in Fig.1.

Bohm and Aharonov realized that this would be difficult to test experimentally for spin-$\frac{1}{2}$ particles such as electrons, but observed that one could just as easily create a pair of entangled photons from the decay of a spin-zero particle. The polarization of these photons would be correlated in the same way. Unlike electrons, photons are bosons, so that if the spatial part of the wavefunction of the two photons is symmetric (as is the case if they are in a zero orbital angular momentum state) then the polarization part of the wavefunction is symmetric[2] in their direction of

$$\Psi_{pol.} \;=\; \frac{1}{\sqrt{2}}\left(|L\rangle|L\rangle + |R\rangle|R\rangle\right), \tag{10}$$

where $|L\rangle$ and $|R\rangle$ indicated left- and right- hand polarized photons.

[2]Note that if the two photons are back-to back and the total spin is zero, then they have the *same* helicity.

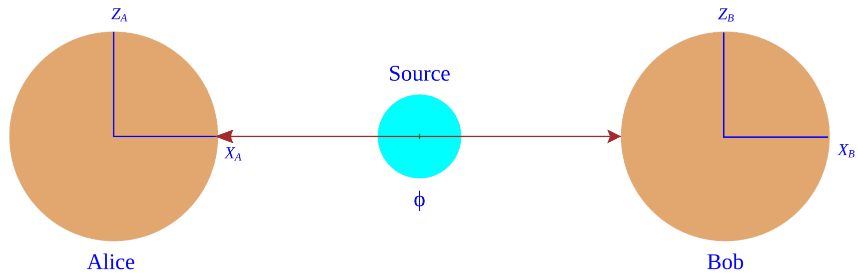

Figure 1: An electron-positron pair is produced from the decay of a spin-zero particle. One heads towards Alice and the other towards Bob. By conservation of angular momentum, if Alice measures the component of spin in the z-direction and obtains the value $+\frac{1}{2}$, then she knows with absolute certainty that a measurement of the z-component of spin by Bob on his particle with yields the value $-\frac{1}{2}$ and vice-versa. On the other hand, Alice could have chosen to measure the component of spin in the x-direction, in which case she would have absolute knowledge about the result of a measurement by Bob of the component of spin in the x-direction. It would then "appear" that either a measurement by Alice instantaneously affects Bob's particle – "spooky action at a distance" – which seems to violate causality in Special Relativity, , or that there are hidden variables which pre-determine what the result of a measurement in any direction.

If the two photons are moving along the z-direction in opposite directions (back to back) then this entangled state can be expressed in terms of an entangled state in the basis of linearly polarized states, $|x\rangle$, $|y\rangle$, for which the electric field vector in the x or y-direction. For photons moving along the z-direction, these linearly polarized states are related to the circular polarized states by

$$\begin{aligned} |L,+p_z\rangle &= \frac{1}{\sqrt{2}}\left(|x\rangle + i|y\rangle\right), \quad |L,-p_z\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle - i|y\rangle\right) \\ |R,+p_z\rangle &= \frac{1}{\sqrt{2}}\left(|x\rangle - i|y\rangle\right), \quad |R,-p_z\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle + i|y\rangle\right) \end{aligned} \tag{11}$$

so that the polarization part of the two-photon wavefunction may also be written

$$\Psi_{pol.} = \frac{1}{\sqrt{2}}\left(|x\rangle|x\rangle + |y\rangle|y\rangle\right), \tag{12}$$

If photon 1 is transmitted through a polarizer with its axis in the x-direction, the wavefunction collapses to

$$\Psi' = |x\rangle|x\rangle,$$

and we can predict with certainty the resultant of a measurement of the state of polarization of the second photon, event if the measurement events have a space-like separation - so that no signal can be sent from one photon to the other.

Bohm and Aharonov suggested a "fix" for the apparent "paradox" i.e. the contradiction between local reality and quantum mechanics. They suggested that a pair of particles produced in an entangled state only remained entangled whilst there was substantial overlap of their wavefunctions, but when they were widely separated their wavefunction is reduced to the product of wavefunctions for each of the two particles.

However, they were able to show that their model was not experimentally verified. They considered the coincidence events for two entangled photons, each of which underwent Compton scattering off electrons. They compared the number number of events in which the scattering planes of the two photon scatterings were co-planar and when they were perpendicular, i.e. when the relative azimuthal angle of the two scatterings was zero and 90° The dependence on the azimuthal angle of the scattering is a function of the direction of polarization of the incoming photon. They considered two possibility for the untangles states:

1. the two photons are linearly polarized in perpendicular directions, averaged over the polarization direction of one of the photons.
2. the two photon are circularly polarized in opposite senses.

Using the Klein-Nishina formula for (polarized) Compton scattering they were able to show that in case 1 the ratio of events in which the scattering planes were perpendicular to the events in which the planes were co-planar could not be larger than 1.5, and that for the case 2 the ratio was one. On the other hand for entangled photons the ratio was 2. The experiment was conducted by C.S. Wu [3] with the result 2.04 ± 0.08, compatible with the ratio expected if the two photons were still entangled before scattering.

1.4 Bell's Inequality

In 1964 John Bell [4] derived limits on the correlation between the measurements of a dichotomic property a of one particle and a different dichotomic property b on the other (correlated) particle, under the hypothesis that both these quantities are determined by a local hidden variable.

We consider two "correlated particles" with a number of dichotomic properties, O_a, labelled by a. The value of the a^{th} such property for particle is denoted by $A(a)$ and takes the value ± 1. The particles are correlated, but *not* necessarily quantum entangled, in the sense that a measurement of the *same* property yields the value $B(a)$ such that

$$A(a)B(a) \; = \; -1. \tag{13}$$

An example of this could be (twice) the component of spin in a direction, $\mathbf{n}_a$ labelled by a.

Importantly, a measurement of O_a on particle 1 and O_b on particle 2 yields the result

$$P(a,b) \; \equiv \; A(a)\, B(b)$$

which can take either value ± 1, with the restriction that it is -1 when $a = b$.

The hidden (local) variable hypothesis assumes that the system of two particles depends on one or more variables $\boldsymbol{\lambda}\, (\equiv \lambda_1, \lambda_2, \cdots)$, which determines the values of $A(a)$ and $B(a)$

though functions $A(\boldsymbol{\lambda}, a)$, $B(\boldsymbol{\lambda}, a)$, and which occurs with some probability function $\rho(\boldsymbol{\lambda})$. The assumption of "locality" implies that the result of a measurement on particle 1 does not influence the result of a measurement on particle 2 or vice versa. With that assumption, the average value, $P(a, b)$ of the product $A(a)B(a)$ is given by

$$\langle P(a,b)\rangle = \int d\boldsymbol{\lambda}\rho(\boldsymbol{\lambda})A(\boldsymbol{\lambda}, a)B(\boldsymbol{\lambda}, b). \tag{14}$$

If we make a large number, $N_{tot.}$, of measurements then defining N_{ij} to be the number of events in which the measurement on particle 1 yields i and the measurement on particle 2 yields j $(i, j = \pm 1)$ then we have

$$\langle P(a,b)\rangle = \frac{N_{++} + N_{--} - N_{+-} - N_{-+}}{N_{tot.}}. \tag{15}$$

Now consider the quantity

$$\begin{aligned} \langle P(a,b)\rangle - \langle P(a,c)\rangle &\equiv \int d\boldsymbol{\lambda}\rho(\boldsymbol{\lambda}) \left[A(\boldsymbol{\lambda}, a)B(\boldsymbol{\lambda}, b) - A(\boldsymbol{\lambda}, a)B(\boldsymbol{\lambda}, c)\right] \\ &= \int d\boldsymbol{\lambda}\rho(\boldsymbol{\lambda})A(\boldsymbol{\lambda}, a)B(\boldsymbol{\lambda}, b) \left[1 + A(\boldsymbol{\lambda}, b)B(\boldsymbol{\lambda}, c)\right], \end{aligned} \tag{16}$$

where in the last step, we have used (13).

The term in square parenthesis $[1 + A(\boldsymbol{\lambda}, b)B(\boldsymbol{\lambda}, c)]$ is either zero or $+2$. We therefore conclude that

$$|\langle P(a,b)\rangle - \langle P(a,c)| \leq \int d\boldsymbol{\lambda}\rho(\boldsymbol{\lambda}) \left[1 + A(\boldsymbol{\lambda}, b)B(\boldsymbol{\lambda}, c)\right],, \tag{17}$$

where the equality pertains if for all values of $\boldsymbol{\lambda}$ for which $[1 + A(\boldsymbol{\lambda}, b)B(\boldsymbol{\lambda}, c)]$ is not zero, $A(\boldsymbol{\lambda}, a)B(\boldsymbol{\lambda}, b)$ is *either* always positive *or* always negative. In any other case, i.e where $A(\boldsymbol{\lambda}, a)B(\boldsymbol{\lambda}, b)$ is sometimes positive and sometimes negative, absolute value of the RHS of (16) is always less than the RHS of (17).

Using the conservation of probability

$$\int d\boldsymbol{\lambda}\rho(\boldsymbol{\lambda}) = 1,$$

this leaves us with Bell's inequality (in the form expressed in [4])

$$|\langle P(a,b)\rangle - \langle P(a,c)| \leq 1 + \langle P(b,c)\rangle. \tag{18}$$

1.4.1 Violation of Bell's Inequality

There are examples of quantum systems, for which quantum mechanics implies that there are cases in which this inequality can be violated.

In order to investigate this, we need to calculate $\langle P(a,b)\rangle$ using Quantum Mechanics.

1. **Two spin-$\frac{1}{2}$ particles:**
We consider two spin-$\frac{1}{2}$ particles, produced in an orbital angular-momentum $l = 0$ (uniform angular distribution) from the decay of a spin-zero particle. By conservation of angular momentum the spin-part of the wavefunction for the two particles is given by
$$\Psi \;=\; \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right). \tag{19}$$
The quantity $\langle P(a,b)\rangle$ is the expectation value of the operator
$$S_{\mathbf{n}_a} \otimes S_{\mathbf{n}_b},$$
where $\frac{1}{2}S_{\mathbf{n}_a}$ is the component of the spin-operator in the direction $\mathbf{n}_a$ and $\frac{1}{2}S_{\mathbf{n}_b}$ is the component of the spin-operator in the direction $\mathbf{n}_b$, We choose $\mathbf{n}_a$ to be in the $x-y$ plane, making an angle α with the x-axis. In this case we have
$$S_{\mathbf{n}_a} \;=\; U(\alpha) S_x U^{-1}(\alpha), \tag{20}$$
where
$$S_x \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{21}$$
and
$$U(\alpha) \;=\; \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}, \tag{22}$$
is the rotation through angle α about the z-axis acting on a spin$=\frac{1}{2}$ particle. so that
$$S_{\mathbf{n}_a} \;=\; \begin{pmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix} \tag{23}$$
and similarly
$$S_{\mathbf{n}_b} \;=\; \begin{pmatrix} 0 & e^{i\beta} \\ e^{-i\beta} & 0 \end{pmatrix}, \tag{24}$$
where $\mathbf{n}_b$ is also ion the $x-y$ plane making an angle β with the x-axis. This gives us
$$S_{\mathbf{n}_a} \otimes S_{\mathbf{n}_b}|\uparrow\downarrow\rangle \;=\; e^{i(\alpha-\beta)}|\downarrow\uparrow\rangle, \tag{25}$$
and
$$S_{\mathbf{n}_a} \otimes S_{\mathbf{n}_b}|\downarrow\uparrow\rangle \;=\; e^{-i(\alpha-\beta)}|\uparrow\downarrow\rangle, \tag{26}$$
so that
$$\langle P(a,b)\rangle \;\equiv\; \langle\Psi|S_{\mathbf{n}_a} \otimes S_{\mathbf{n}_b}|\Psi\rangle \;=\; -\cos(\alpha-\beta). \tag{27}$$
In order to test Bell's inequality, we also need to consider measuring the component of spin at a further angle γ such that
$$\langle P(a,c)\rangle \;=\; -\cos(\alpha-\gamma)$$

Bell's inequality is violated if we can find three angles α, β, γ such that

$$|\cos(\alpha - \beta) - \cos(\alpha - \gamma)| - \cos(\beta - \gamma) \; > \; 1.$$

There are ranges of angles for which the inequality is violated. A particular example is

$$\alpha \; = \; \frac{\pi}{2}, \;\; \beta \; = \; \frac{3\pi}{4}, \;\; \gamma = 0,$$

for which

$$|\cos(\alpha - \beta) - \cos(\alpha - \gamma)| - \cos(\beta - \gamma) \; = \; \sqrt{2} \; (> \; 1).$$

2. **Two photons:**
We consider two entangled photons assumed to be moving in opposite direction along the z-axis, arising from the decay of a spin-zero particle. The total component of spin in the z-direction must be zero and so either both photons are left-circularly polarized or both are right-circularly polarized. For the total spin of the two photons to be zero, as required by conservation of angular momentum the photons, assumed to have zero orbital angular momentum must be in an entangled state given by given by

$$|\Psi\rangle \; = \; \frac{1}{\sqrt{2}}\left(|L, L\rangle + |R, R\rangle\right), \tag{28}$$

where R, L refer to clockwise and anticlockwise rotation of the electric field (recall that as the photons are moving in opposite directions, opposite z-component of spin corresponds to the same helicity and hence the same sense of circular polarization).

If the two photons are back-to-back, moving along the positive and negative z-axis, respectively, then this entangled state can be expressed in terms of linear polarization

$$|\Psi\rangle \; = \; \frac{1}{\sqrt{2}}\left(|x, x\rangle + |y, y\rangle\right), \tag{29}$$

where x, y refer to the direction of the electric field for plane-polarized light. [3] We can proceed as in the case of correlated electrons by defining the operator E_x, which returns +1 if the photon is polarized in the x-direction and -1 if the photon is polarized in the y-direction. In the representation in which we write

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \;\; |y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{30}$$

(we suppress the z-components as the photons are transverse and the component in the z-direction is always zero), this operator may be written

$$E_x \; = \; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{31}$$

[3] In this case the photons are correlated in such a way that the directions of polarization's are equal rather than opposite.

Define the operator

$$E_\alpha \;=\; U_z^{-1}(\alpha) E_x U_z(\alpha), \tag{32}$$

where

$$U_z(\alpha) \;=\; \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \tag{33}$$

is a rotation through α about the z-axis.

$$E_\alpha \;=\; \begin{pmatrix} \cos(2\alpha) & \sin 2\alpha \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix} \tag{34}$$

$$\langle P(a,b)\rangle \;=\; \langle\Psi|E_\alpha \otimes E_\beta|\Psi\rangle \;=\; \cos\left(2\left(\alpha-\beta\right)\right). \tag{35}$$

It is useful to determine the components, N_{++}, N_{--}, N_{+-} N_{-+}. for this particular entangled state, $|\Psi\rangle$. We calculate the corresponding amplitudes, $\mathcal{A}_{ij}$

$$\begin{aligned}
\mathcal{A}_{++} &= \langle + + |U_\alpha \otimes U_\beta|\Psi\rangle \;=\; \frac{1}{\sqrt{2}}\cos\left(\alpha-\beta\right). \\
\mathcal{A}_{--} &= \langle - - |U_\alpha \otimes U_\beta|\Psi\rangle \;=\; \frac{1}{\sqrt{2}}\cos\left(\alpha-\beta\right). \\
\mathcal{A}_{+-} &= \langle + - |U_\alpha \otimes U_\beta|\Psi\rangle \;=\; \frac{1}{\sqrt{2}}\sin\left(\alpha-\beta\right). \\
\mathcal{A}_{-+} &= \langle - + |U_\alpha \otimes U_\beta|\Psi\rangle \;=\; \frac{1}{\sqrt{2}}\sin\left(\alpha-\beta\right).
\end{aligned} \tag{36}$$

so that if the total number of coincidence events is $N_{tot.}$

$$\begin{aligned}
\frac{N_{++}}{N_{tot.}} &= |\mathcal{A}_{++}|^2 \;=\; \frac{1}{2}\cos^2\left(\alpha-\beta\right), \\
\frac{N_{--}}{N_{tot.}} &= |\mathcal{A}_{--}|^2 \;=\; \frac{1}{2}\cos^2\left(\alpha-\beta\right), \\
\frac{N_{+-}}{N_{tot.}} &= |\mathcal{A}_{+-}|^2 \;=\; \frac{1}{2}\sin^2\left(\alpha-\beta\right), \\
\frac{N_{-+}}{N_{tot.}} &= |\mathcal{A}_{-+}|^2 \;=\; \frac{1}{2}\sin^2\left(\alpha-\beta\right).
\end{aligned} \tag{37}$$

Bell's inequalilty is violated for a system of two entangled photons passing through polarizers set at angles α, β, γ such that

$$\left|\cos\left(2\left(\alpha-\beta\right)\right) + \cos\left(2\left(\alpha-\gamma\right)\right)\right| - \cos\left(2\left(\beta-\gamma\right)\right) \;>\; 1.$$

An example of this is

$$\alpha \;=\; \frac{\pi}{4},\;\; \beta \;=\; \frac{3\pi}{8},\;\; \gamma = 0,$$

for which

$$\left|\cos\left(2\left(\alpha-\beta\right)\right) + \cos\left(2\left(\alpha-\gamma\right)\right)\right| - \cos\left(2\left(\beta-\gamma\right)\right) \;=\; \sqrt{2}\;\;(>\;\;.1)$$

2 Clauser-Horn-Shimony-Holt (CHSH) Inequality

In 1969, John Hauser, Michael Horne, Abner Shimony, and Robert Holt [5] proposed a variation of Bell's original theorem, which is more amenable to experimental verification.

In this scenario, each experimenter (Alice and Bob), have the choice of performing a measurement of one of two dichotomic properties. Thus Alice measures either $A(a)$ or $A(b)$ on her particle, whereas Bob measures either $B(c)$ or $B(d)$ on his (correlated) particle. The hidden variable postulate assumes that *both* $A(a)$ and $A(b)$ have a given value, even if only one of these is measured. The same is true of $B(c)$ and $B(d)$. We therefore have four quantities

$$A(a)B(c), \ A(b)B(c), \ A(b)B(c), \text{ and } A(b)B(d)$$

all of which take the value ± 1.

Now consider the combination

$$\begin{aligned} S(a,b,c,d) &\equiv A(a)B(c) + A(b)B(d) + A(b)B(c) - A(a)B(d) \\ &= (A(a) + A(b))\, B(c) + (A(b) - A(a))\, B(d). \end{aligned} \tag{38}$$

Either

$$(A(a) + A(b)) \ = \ 0, \quad (A(b) - A(a)) \ = \ \pm 2$$

or

$$(A(a) + A(b)) \ = \ \pm 2, \quad (A(a) - A(b)) \ = \ 0.$$

If we now repeat this N times,

$$\sum_{i=1}^{N} S_i(a,b,c,d) \ \equiv \sum_{i=1}^{N} \left[(A_i(a) + A_i(b))\, B(c) + (A_i(a) - A_i(b))\, B_i(d) \right], \tag{39}$$

has a minimum value of $-2N$ and a maximum value of $+2N$.

Dividing by N we get an inequality for the absolute value of this particular combination of pairs of measurements

$$|\langle A(a)B(c)\rangle + \langle A(b)B(d)\rangle + \langle A(b)B(c)\rangle - \langle A(a)B(d)\rangle| \ \leq \ 2$$

or using the quantities $P(a,b)$ etc. defined in (15)

$$\overline{S}(a,b,c,d) \ \leq \ 2, \tag{40}$$

where

$$\overline{S}(a,b,c,d) \ \equiv \ |\langle \ P(a,c)\rangle + \langle P(b,d)\rangle + \langle P(b,c)\rangle - \langle P(a,d)\rangle| \tag{41}$$

In the example of two spin$\frac{1}{2}$ particles, we again identify the properties a, b, c, d as the components of spin in the $x - y$ plane making angles α, β, γ, δ with the x-axis and the inequality becomes using the quantities $P(a,b)$ etc. given in (35)

$$|\cos(\alpha - \gamma) + \cos(\beta - \delta) + \cos(\beta - \gamma) - \cos(\alpha - \delta)| \ \leq \ 2 \tag{42}$$

An example of the violation of this inequality occurs for the values

$$\alpha = 0,\ \beta = \frac{\pi}{2},\ \gamma = \frac{\pi}{4},\ \delta = \frac{3\pi}{4},$$

for which

$$\cos(\alpha - \gamma) + \cos(\beta - \delta) + \cos(\beta - \gamma) - \cos(\alpha - \delta) \ = \ 2\sqrt{2}.$$

In the case of entangled photons (moving in the z-direction, with the polarization measured in directions α, β, γ, δ to the x-axis), the inequality is

$$|\cos(2(\alpha - \gamma)) + \cos(2(\beta - \delta)) + \cos(2(\beta - \gamma)) - \cos(2(\alpha - \delta))| \ \leq \ 2$$

In this case we select

$$\alpha = 0,\ \beta = \frac{\pi}{4},\ \gamma = \frac{\pi}{8},\ \delta = \frac{3\pi}{8}, \tag{43}$$

for which

$$|\cos(2(\alpha - \gamma)) + \cos(2(\beta - \delta)) + \cos(2(\beta - \gamma)) - \cos(2(\alpha - \delta))| \ = \ 2\sqrt{2}.$$

This set of angles leads to a Quantum Mechanics prediction with maximum violation of Bell's inequalities.

More generally, if we set

$$|\alpha - \gamma| = |\beta - \gamma| = |\beta - \delta| \ = \ \phi, \quad |\alpha - \delta| \ = \ 3\phi, \tag{44}$$

we can plot $\overline{S}$ against ϕ as shown in Fig.2. We see that for small values of ϕ and values of ϕ close to 90°), the value of $|\overline{S}|$ exceeds the bounds set by the Bell inequalities.

In the early experiments the polarization was determined using a filter, which blocks the photon in the state $|-\rangle$. This means that the number of coincidence photons observed is N_{++}. The other combinations can be inferred by performing a measurement of $N_{+\infty}$, $N_{\infty+}$ in which one or the other of the polarization filters is removed, to obtain

$$N_{+\infty} \ \equiv \ N_{++} + N_{+-}, \tag{45}$$

and

$$N_{\infty+} \ \equiv \ N_{++} + N_{-+}. \tag{46}$$

Furthermore, we have

$$N_{--} \ = \ N_{tot.} - N_{++} - N_{+-} - N_{-+} \ = \ N_{tot.} + N_{++} - N_{+\infty} - N_{\infty+}. \tag{47}$$

This gives us an expression for $\langle P(a,b)\rangle$ in terms of the measured quantities, $N_{++}(a,b)$, $N_{+\infty}(a)$ $N_{\infty+}(c)$,

$$\langle P(a,b)\rangle \ = \ 4R(a,b) - 2R_1(a) - 2R_2(b) + 1, \tag{48}$$

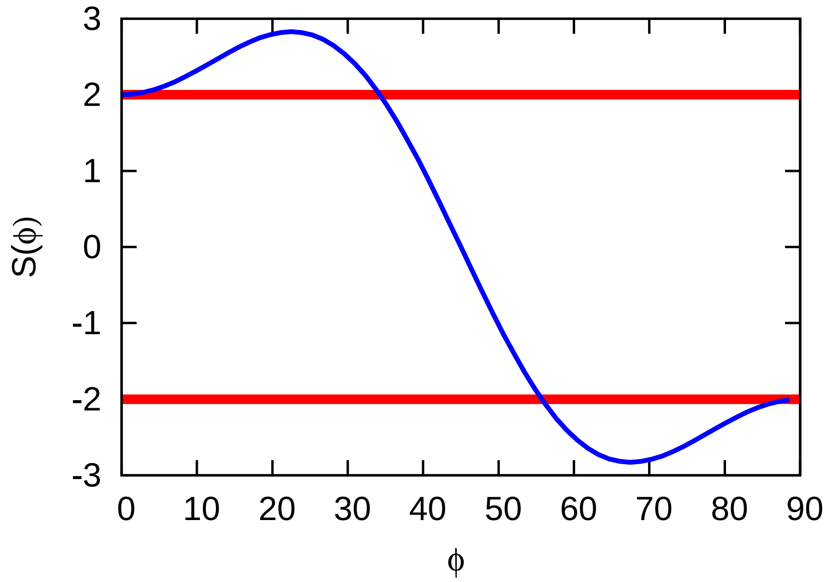

Figure 2: Plot of $S(\phi)$ against ϕ, where $|\alpha-\gamma| = |\beta-\gamma| = |\beta-\delta| \;=\; \phi, \;\; |\alpha-\delta| \;=\; 3\phi$, we see that for values of ϕ close to zero or 90° the bounds (shown as red horizontal lines) imposed by Bell's (CSHS) inequality are violated.

where

$$\begin{aligned} R(a,b) &\equiv \frac{N_{++}(a,b)}{N_{tot.}}, \\ R_1(a) &\equiv \frac{N_{+\infty}(a)}{N_{tot.}} \\ R_2(b) &\equiv \frac{N_{\infty+}(b)}{N_{tot.}}. \end{aligned} \tag{49}$$

In other words, $R(a,b)$ is the fraction of coincidence events arising from both photons being transmitted by polarizers at angles α and β, $R_1(a)$ is the fraction of coincidence events arising from a polarizer at angle α in front of beam 1 only and likewise $R_2(b)$ is the fraction of coincidence events arising from a polarizer at angle β in front of beam 2 only.

The CHSH inequality can therefore be written

$$\begin{aligned} |\langle P(a,c)\rangle + \langle P(b,d)\rangle + \langle P(b,c)\rangle - \langle P(a,d)\rangle| &= \\ |4\left[R(a,c)+R(b,d)+R(b,c)-R(a,d)-R_1(c)-R_2(b)\right]+2| &\leq 2, \end{aligned} \tag{50}$$

which can be rewritten as

$$-1 \;\leq\; \tilde{\mathcal{S}}(a,b,c,d) \;\leq\; 0. \tag{51}$$

where [4]

$$\tilde{\mathcal{S}}(a,b,c,d) \;\equiv\; [R(a,c) + R(b,d) + R(b,c) - R(a,d) - R_2(b) - R_1(c)] \tag{52}$$

In the case of entangled photons, assume that $N_{++}(a,b)/N_{tot.}$ depends only on the relative angle, $|\alpha - \beta|$ of the two polarization analyzers. If these polarization analyzers are set at angles which are related to each other by (44), then (51) simplifies to

$$-1 \;\leq\; 3R(\phi) - R(3\phi) - R_1(\beta) - R_2(\gamma) \;\leq 0 \tag{53}$$

Note that $N_{+\infty}(\beta)$ is the number of coincidence counts with a polarizing filter only in front of particle 1. This will depend on the angle of the polarizing filter relative to the direction of polarization of the filtered photon, but the inequality still holds if we take the average this angle. The same applies to $N_{+\infty}(\gamma)$ and so we can replace $R_1(\alpha)$, $R_2(\beta)$ by their $R_1(\cdot)$, $R_2(\cdot)$, which are their averages over the angles α, β respectively.

We now take a new angle difference $\phi' = 3\phi$ and (53) is replaced by

$$-1 \;\leq\; 3R(3\phi) - R(9\phi) - R_1(\cdot) - R_2(\cdot) \;\leq 0 \tag{54}$$

Reversing the sign of this inequality gives

$$0 \;\leq\; R(9\phi) - 3R(3\phi) + R_1(\cdot) + R_2(\cdot) \;\leq 1 \tag{55}$$

We can now add the inequalities (53) and (55). The coincidence counts with one polarization analyzer removed cancel out and do not need to be measured. We obtain

$$-1 \;\leq\; R(9\phi) + 3R(\phi) - 4R(3\phi) \;\leq\; 1 \tag{56}$$

If we take the value $\phi = \pi/8$ and use the fact that the coincident count rate is unaffected by a rotation of the polarization analyzer through π, so that

$$R(\phi) \;=\; R(\phi + \pi),$$

this inequality simplifies (dividing by 4) to the Freedman inequality [6]

$$\delta \;\equiv\; |R(\pi/8) - R(3\pi/8)| - \frac{1}{4} \;\leq\; 0 \tag{57}$$

For two entangled photons, Quantum Mechanics predicts

$$R(\phi) \;=\; \frac{1}{4}\,(1 + \cos(2\phi)) \tag{58}$$

so that the prediction of δ defined in (57) is

$$\delta \;=\; \frac{\sqrt{2} - 1}{4}$$

in violation of the Freedman inequality (57).

[4]Unfortunately, the symbol S is used in the literature to denote two related but different quantities.. We use $\tilde{\mathcal{S}}(a,b,c,d)$ to denote this variable which is related to $\overline{S}$ by $\tilde{\mathcal{S}} = |\overline{S} - 2|/4$.

3 Experimental Tests

3.1 Correlated Photons vs. Correlated Electrons

For an experimental test of Bell's inequality, one needs to be able to prepare pairs of particles that possess correlated dichotomic properties. Two examples which immediately come to mind are

1. Two electrons whose spin-components in a given direction are correlated by conservation of angular momentum. The Bell test then consists of determining the probabilities of obtaining the same result from the measurement of the component of spin in different directions for two electrons

2. Two photons whose circular polarizations are related by conservation of angular momentum. The Bell test then consists of determining the probabilities of both photons being transmitted through polarization filters set in different directions for the two photons.

In each case, in order to test the CHSH inequality, the observation needs to be conducted for two different directions for each of the two particles.

There are several reasons why experiments with photons are more practical than with electrons.

1. polarization is easier to measure than spin component, which requires some sort of Stern-Gerlach apparatus,

2. photons only interact weakly with their environment. They can travel large distances so that "simultaneous" (i.e. space-like separated) coincidence detection can be carried out.

3. changes in the path of photons due to fluctuations in refractive index can be prevented by surrounding the photon path with a tube (which could be evacuated if necessary).

In practice, allowance has to be made for experimental imperfections:

- The transmissivity, $\epsilon^{\parallel}$, of a polarization filter in front of a photon whose polarization direction is exactly the same as the polarization analyzer axis, is less than unity.

- The transmissivity, $\epsilon^{\perp}$, of a polarization filter in front of a photon whose polarization direction is perpendicular the polarization analyzer axis, is greater than zero.

- The photons are detected within some non-zero opening angle, θ, so that the photons are not all exactly back-to-back. Their circular polarizations are correlated due to angular momentum conservation, but this does not translate exactly into entangled states in terms of linear polarization.

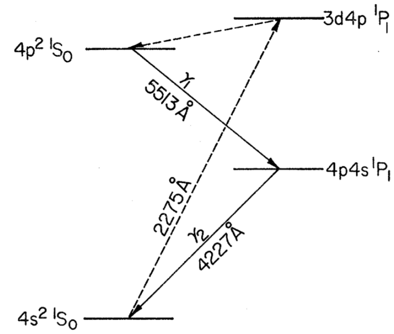

Figure 3: Two-stage decay of excited state of calcium.

This affects the Quantum Mechanics prediction for two entangled photons. The expression (35) for the average value of the product $A(a)B(b)$, where a and b are angles α and β of the polarization analyzers in front of the two beams is modified to

$$\langle P(a,b)\rangle \;=\; F(\theta)\frac{\left(\epsilon_1^{\parallel}-\epsilon_1^{\perp}\right)}{\left(\epsilon_1^{\parallel}+\epsilon_1^{\perp}\right)}\frac{\left(\epsilon_2^{\parallel}-\epsilon_2^{\perp}\right)}{\left(\epsilon_2^{\parallel}+\epsilon_2^{\perp}\right)}\cos\left(2\left(\alpha-\beta\right)\right), \tag{59}$$

where $F(\theta) < 1$ is the reduction factor due to the non-zero opening angle, θ, of the detected photons (and the suffices on ϵ refer to photons 1 and 2).

Similarly, the probability that the photons are both transmitted though polarization analyzers whose axes are oriented at an angle ϕ to each other, (58) is modified to

$$R(\phi) \;=\; \frac{1}{4}\left(\left(\epsilon_1^{\parallel}+\epsilon_1^{\perp}\right)\left(\epsilon_2^{\parallel}+\epsilon_2^{\perp}\right)+\left(\epsilon_1^{\parallel}-\epsilon_1^{\perp}\right)\left(\epsilon_2^{\parallel}-\epsilon_2^{\perp}\right)F(\theta)\cos(2\phi)\right).,\tag{60}$$

and the Quantum Mechanics prediction for the quantity δ, which should be negative according to Bell's inequality is

$$\delta \;=\; \frac{1}{4}\left[\left(\epsilon_1^{\parallel}-\epsilon_1^{\perp}\right)\left(\epsilon_2^{\parallel}-\epsilon_2^{\perp}\right)F(\theta)\sqrt{2}-1\right].\tag{61}$$

Polarization Analyzers are quite efficient so that typically $\epsilon^{\parallel} \sim 0.98$, $\epsilon^{\perp} \sim 0.02$ and $F(\theta) \sim 0.98$. Nevertheless these correction all go in the same direction, so that these corrections lead to a reduction in the Quantum Mechanics prediction of $\langle P(a,b)\rangle$ by around 10%.

3.2 Preparing Pairs of Entangled Photons

Most of the experiments which tested the Bell inequality prepared the entangled photon pairs from a two-stage cascade of decays of an excited atomic state.

An example is the two-stage decay of the $4p^2\,{}^1\mathrm{S}_0$ excited state of calcium to its ground-state $4s^2\,{}^1\mathrm{S}_0$. Since the initial and final states both have $J = 0$, this decay is strictly forbidden by conservation of angular momentum.. The decay therefore proceeds though an intermediate state $4p1s\,{}^3\mathrm{P}_1$ state, emitting a photon with wavelength 551.3 nm, and subsequently back to the ground state, emitting a photon with wavelength 422.7 nm, as shown in Fig.3. By momentum conservation the photons are back-to-back (in the rest frame of the excited atom, whose recoil is negligible as the mass of the atom is much larger than the energy of the photons). The total angular momentum of the photons is zero and so they must be produced with the same circular polarization, with total angular momentum zero. We therefore have an entangled state

$$|\Psi\rangle \;=\; \frac{1}{\sqrt{2}}\left(|L,L\rangle + |R,R\rangle\right).$$

In terms of linear polarization (taking the direction of the photons to be along the z-axis, this may be written

$$|\Psi\rangle \;=\; \frac{1}{\sqrt{2}}\left(|x,x\rangle + |y,y\rangle\right).$$

The excited state, $4p^2\,{}^1\mathrm{S}_0$, can be reached by illumination with UV radiation of wavelength 227.5 nm, which promotes an electron to the state $3d4p\,{}^1\mathrm{P}_1$, which has slightly higher energy and which subsequently decays (some of the time) to the desired $4p^2\,{}^1\mathrm{S}_0$ state.

The delay between the first and second transitions is of the order of a few nanoseconds, so that these photon pairs can be detected by "coincidence counting" in which both photons are observed within a time interval of a few nanoseconds.

3.3 Photon Detection

The photons were detected using photomultiplier tubes connected to an electronic coincidence counter, capable of detecting a "coincidence" count of within ~ 5 ns. The photomultipliers had to be sufficiently separated so that the detection of the two photons occurred at a space-like separation.

Photomultiplier tubes can detect single photons, but the detection efficiency is way below unity. Furthermore, only a small fraction of photon pairs are detected because of the small opening angle of the detector.

In an experiment using a single path polarization analyzer (i.e. one that transmits the photon only if its direction of polarization is aligned with the axis of the polarization analyzer), it is impossible to say if a single photon detection is due to the other photon having been blocked by its polarization analyzer or because of the detection efficiency of the photomultiplier tube. It is therefore necessary to measure the coincidence rate with the polarization analyzers removed, and even then one is making the assumption of "unbiased sampling", i.e the assumption that the detection efficiency does *not* depend on the polarization of the detected photon, or indeed on any other possible hidden variable. This generates a loophole in the interpretation the result of an experiment, in which Bell's inequality appears to

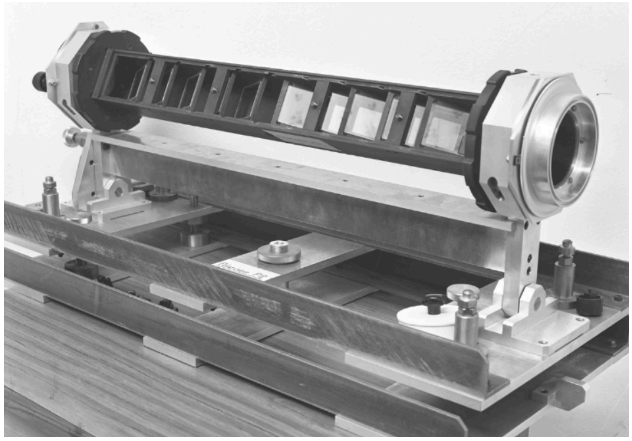

Figure 4: Pile of plates polarization analyzer.

be violated and the definitive conclusion that there cannot be local hidden variables. The use of two-channel polarization analyzers in later experiments greatly reduces (but does not totally eliminate) such loopholes. In any case, it is necessary to measure the coincidence counting rate with the polarization analyzers removed in order to determine the transmission probabilities. In the case of single-channel polarization analyzers, set at angles α, β what is measured is $N_{++}(\alpha, \beta)$. To determine the probability of transmission through both analyzers, this needs to be divided by $N_{tot.}$, which is measured by counting the coincidence rate with the polarization analyzers removed.

A further potential difficulty with photomultiplier tubes is that they can sometimes produce a false positive due to an electron being liberated from the inside of the tube by thermal fluctuations. However, the probability of a false positive coincidence count - i.e. where both photomultipliers give a false signal within a period of a few nanoseconds is negligible.

The more recent experiments used avalanche photo-diode detectors, which have a detection efficiency of over 50%, and modern superconducting single-wire photon detectors have an efficiency close to 100%.

3.4 Polarization Analyzers

The simplest polarization analyzer is a sheet of polarizing film (from which polarizing filters are made) which blocks light polarized in the direction perpendicular to the axis of polarization of the film. Such film blocks over 99% of the perpendicular polarized light, it usually only transmits only about 40% of the parallel polarized light.

A more efficient polarization analyzer is the pile-of-plates polarizer, a photograph of which is shown in Fig.4. A series of glass plates are oriented so that the angle of incidence is Brewster's angle, ϕ_b where

$$\tan \phi_B \;=\; \frac{n_2}{n_1},$$

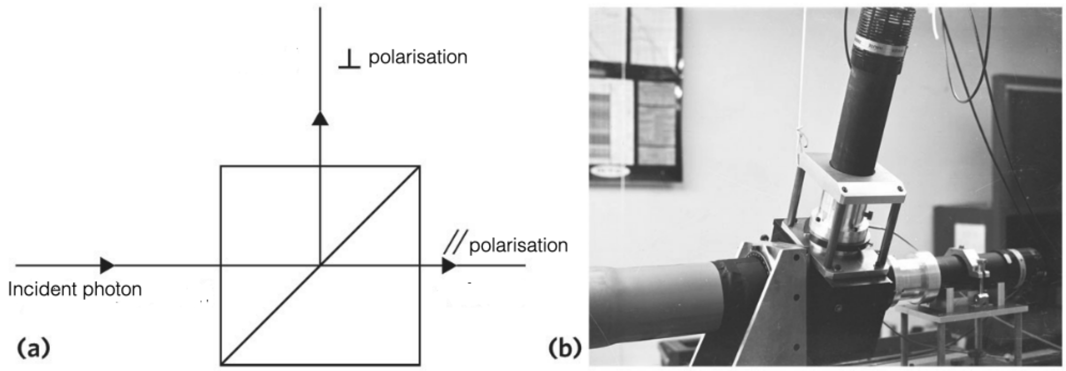

Figure 5: Two channel beam-splitter polarization analyzer.

n_1 and n_2 being the refractive index of air and glass respectively. At this angle of incidence only light polarized perpendicular to the scattering plane ("s-polarization") is reflected. The reflection coefficient at each such surface is about 16% so that after passing 10 such plates - i.e. 20 reflections, the transmission of the s-polarized light is $(1-0.16)^{20}$ which is less than 3%.

The pile-of-plates polarizer is an example of a single channel polarization analyzer, which blocks the light polarized perpendicular to the polarization axis (the plane of incidence of the light beam and the refracting surfaces). For experiments using this type of polarizer, it is necessary to sample the coincidence rates with one or both of the polarizers removed. Such experiments are sensitive to any violation of the assumption of unbiased sampling, and can only measure the quantity $\tilde{\mathcal{S}}$ defined in (52) or δ defined in (57). On the other hand, later experiments used a polarizing beam-splitter of the type shown in Fig.5 which transmits light polarized in the plane of incidence (p-polarized) but reflects (at 90°) light polarized perpendicular to the pane of incidence. These beam-splitters consist of two right-angle prisms with a dielectric substrate between them. The substrate consists of layers of high ad low refractive index material which are arranged such that the multiple reflections for the transmitted p-polarization light interfere constructively, whereas for the s-polarization light they interfere destructively. This is possible because the p-polarized light undergoes a phase reversal upon reflection, whereas the s-polarized light does not.

The later experiments, which used two-channel polarizers and were able to measure the probabilities of transmission for polarization states $|++\rangle$, $|+-\rangle$, $|-+\rangle$, $|--\rangle$, for any orientation of the polarizers for the two beams and therefore could measure directly the quantity $\overline{S}(a,b,c,d)$ defined in (41).

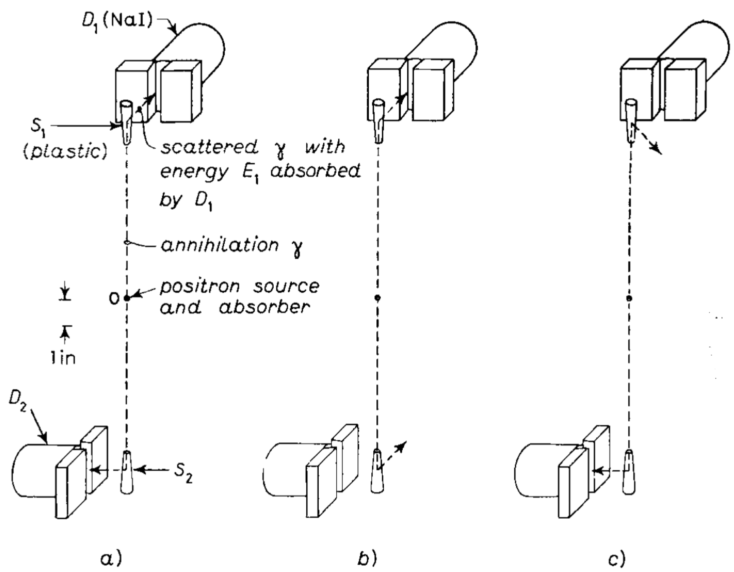

Figure 6: Experimental setup of Kasday, Ullman and Wu, (with azimuthal angles set at right-angles in thiso case) , showing the three possible types of events: (a) both photons detected, (b) photon 1 only detected, (c) photon 2 only detected.

3.5 Early Experiments

Kasday, Ullman, Wu (1970)

The first experimental test of Bell's inequality was performed by Leonard Kasday, Jack Ullman and Chien-Shiung Wu in (1970) [7].

They obtained entangles photons from $e^+ e^-$ annihilation in the state

$$\Psi = \frac{1}{\sqrt{2}} \left(|L\rangle|R\rangle - |R\rangle|L\rangle \right). \tag{62}$$

Each of the photons underwent Compton scattering off an electron. The Compton scattering cross-section depends on the (azimuthal) angle between the scattering plane and the axis of polarization of the incoming photon so that the azimuthal distribution of the scattered photons in a measure of the correlation of polarization of the two entangled photons.

They set two detectors for the two Compton-scattered photons at azimuthal angles ϕ_1, ϕ_2, as shown schematically in Fig.6. For each azimuthal setting, they plotted the ratio

$$R \equiv \frac{\text{No. of events detecting both photons}}{(\text{No. of events detecting only photon 1}) \times (\text{No. of events detecting only photon 2})}$$

For two photons in orthogonal polarization states, it can be shown that the quantity R is of the form

$$R = 1 + B\cos(\phi_1 - \phi_2) \tag{63}$$

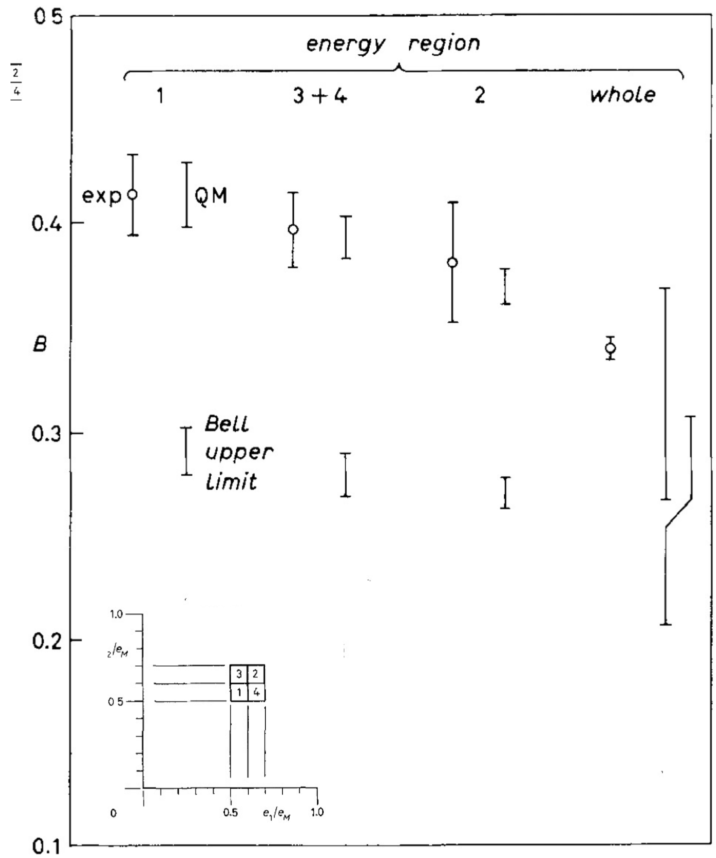

Figure 7: Values of B for different energy of the two scattered photons: The theoretical prediction from Quantum mechanics, the experimental values and the limits from Bell's inequalities.

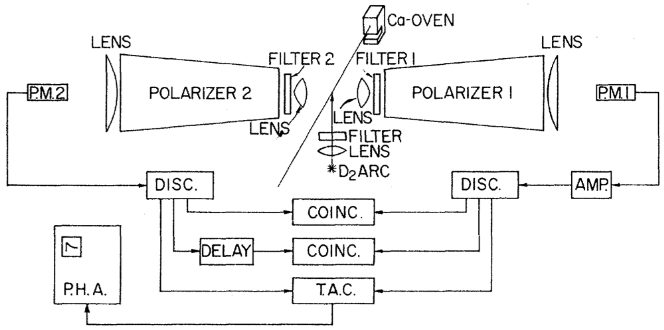

Figure 8: Experimental setup of experiment by Clauser and Freedman (1972).

The value of the coefficient B depends on the energies of the two scattered photons. The experimental results, together with the prediction from Quantum Mechanics, and the limits set by Bell's inequalities, for several ranges of scattered photon energies.

The results are in good agreement with Quantum Mechanics and appear to exceed limits set by Bell's inequality. However, the "measurement" of the polarization state of the photons assumes the correctness of the Klein-Nishina formula for Compton-scattering cross-sections - i.e. it assumes QED. This experiment *cannot* be interpreted as a falsification of the postulate of hidden variables. Experiments designed to do tbis must use direct polarizers to measure the polarization state of the photons

Clauser and Freedman (1972)

The first experiment using a cascade of atomic transition was conducted by John Clauser and Stuart Freedman in 1972. [9].

They used the cascade decay between the levels $4p^2\,{}^1\mathrm{S}_0$ and $4s^2\,{}^1\mathrm{S}_0$ of calcium.

A schematic diagram of their experimental setup shown in Fig.8. Calcium atoms emerging from an oven were irradiated by a deuterium arc lamp. Resonance absorption at 227.5 nm promoted the atoms to $3d4p\,{}^1\mathrm{P}_1$. state which decayed to the $4p^2\,{}^1\mathrm{S}_0$ state (see Fig.3).

Single channel pile-of-plates polarizers were used, and the coincidence counting rate $R(\phi)$ was measured with various different angles between the axes of the polarizers. These were normalized by measuring the coincidence rate, R_0, with both polarizers removed. The results are shown in Fig.9.

From these results, at angles of 22.5° and 67.5°, they were able to construct the quantity δ, which, as shown in (57) should be negative in any system of hidden variables.

They found

$$\delta \;=\; 0.050 \pm 0.008,$$

– a violation of Bell's inequality by $6\,\sigma$.

The photons were collected within a cone of angle $\theta = 30°$ for which $F(\theta) = 0.99$

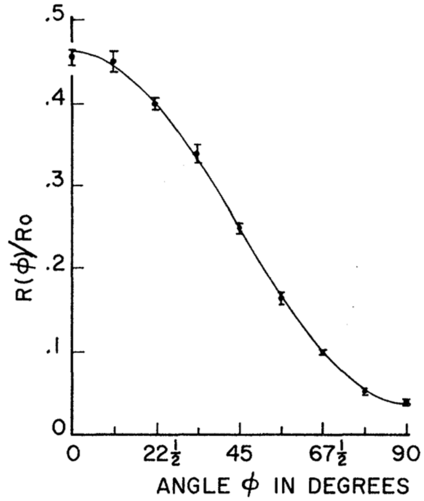

Figure 9: $R(\phi)/R_0$ for various different relative angles, ϕ, of the polarizers.

and the transmissivities of the polarizers were $\epsilon_1^{\parallel} = 0.97 \pm 0.01$, $\epsilon_1^{\perp} = 0038 \pm 0.004$, $\epsilon_2^{\parallel} = 0.96 \pm 0.01$, $\epsilon_2^{\perp} = 0.037 \pm 0.004$. Inserting these values into (61), the Quantum Mechanics prediction is

$$\delta_{QM} \;=\; 0.052 \pm 0.001,$$

in agreement (within errors) with the measured value.

Holt and Pipkin (1973)

At the same time, Richard Holt, a PhD student at Harvard and his supervisor Francis-Pipkin carried out a similar, but not identical experiment. [10]

Their entangled photon pairs were obtained from $9\,{}^1\mathrm{P}_1 \rightarrow 7\,{}^3\mathrm{S}_1 \rightarrow 6\,{}^3\mathrm{P}_0$ in mercury, emitting a photon of wavelength 567.6 nm and a photon of wavelength 404.6 nm.

The atoms were not projected in a jet, but kept in an evacuated chamber and excited by ultraviolet radiation. Furthermore. rather than the pile-of-plates polarizers they used more conventional birefringent calcite prisms made of material which has a different refractive index for light polarized in the plane of incidence (p-polarized) than for light polarized perpendicular to the plane of incidence (s-polarized).

The results of their experiment were consistent with Bell's inequality, measuring a value of δ

$$\delta \;=\; -0.034,$$

which is negative, in keeping the the limit set by Freedman's inequality (57).

Clauser (1976)

In 1976, John Clauser repeated the experiment of Holt and Pipkin using the same cascade in mercury [11]. However, he used pile-of-plates polarization analyzers, which have a far greater efficiency than birefringence polarizers.

He *did* obtain a violation of Freedman's inequality, obtaining a result for δ

$$\delta = 0.0385 \pm 0.0093.$$

This is also consistent with the Quantum Mechanics prediction, which for the efficiencies of the polarizers used was 0.0348.

Fry and Thomson (1976)

In the same year, Edward Fry and Randall Thomson repeated the experiment using the cascade in mercury [12]

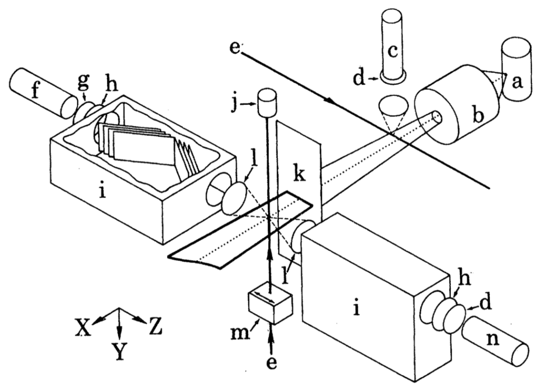

Figure 10: Schematic of the apparatus. (a) Hg oven; (b) solenoid electron gun; (c) HCA 8575; (d) 4358-A filter; (e) 5461-A laser beam; (f) Amperex 56 DUVP/OB; (g) 2587-A filter; (b) focusing lens; (i) pile-of-plates polarizer; (j) laser beam trap; (k) atomic beam defining slit; (1) light collecting lens; (m) crystal polarizer; (n) RCA 8850

A diagram of their experimental setup is shown in Fig.10.

They also used pile-of-plates polarizers. They also obtained a value of δ in violation of freedman's inequality:

$$\delta = 0.046 \pm 0.014.$$

The Quantum Mechanics prediction, using their polarizer efficiencies and photon acceptance opening angle was 0.044.

3.6 Aspect's Experiments:

Between 1980 and 1982, Alain Aspect and his collaborators carried out three experiments which greatly increased the sensitivity of the measurement of the Bell inequalities [13]. This was due to a number of brilliant improvements. The first such improvement was to increase

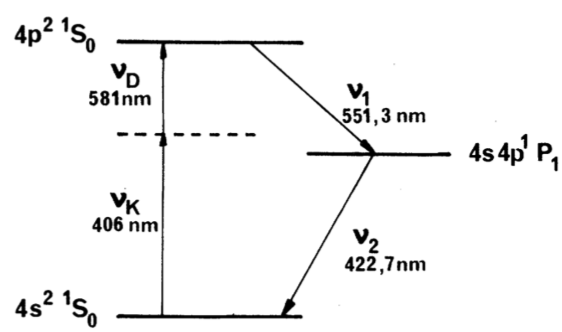

Figure 11: Excitement of $4p^2\,{}^1\mathrm{S}_0$ by irradiation from a krypton laser of fixed wavelength 406.7 nm and a Rhodamine dye laser tuned to 581 nm

the separation of the polarizers, so that each one was six metres from the source of the entangled photons. This meant that for any genuine coincidence count, the measurements of the state of polarization of the two photons definitely occurred at a space-like separation so that the measurement of the polarization of one photon could have no causal effect on the polarization state of the other.

First Experiment (1980) [14]

Aspect found a robust new way to excite the $4p^2\,{}^1\mathrm{S}_0$ from the ground state $4s^2\,{}^1\mathrm{S}_0$. The transition is strictly forbidden by absorption of a single photon, but it can be effected by simultaneous absorption of two photons whose plane polarizations axes are aligned (see Fig.11). One photon comes from a krypton laser which emits photons of wavelength 406.7 nm and the other is a tunable Rhodamine laser. Resonance absorption occurs when this laser is tuned to 581 nm. The calcium atoms emerge from an oven and move along the axis of a cylindrical vacuum chamber, where they are subject to radiation fro the two laser beams.

A schematic diagram of the apparatus is shown in Fig.(12)

The quantity $R(\phi)$ defined in (49) was measured for various different values of the angle ϕ between the polarizers in front of the two beams. The results are shown in Fig.13. From this they deduced the value of δ to be

$$\delta \;=\; 0.0572 \pm 0.0043,$$

in violation of Freedman's inequality by 13 standard deviations.

The polarizer transmissivities were $\epsilon_1^{\parallel} = 0.971 \pm 0.005$, $\epsilon_1^{\perp} = 0.029 \pm 0.005$ and $\epsilon_2^{\parallel} = 0.968 \pm 0.005$, $\epsilon_1^{\perp} = 0.028 \pm 0.005$. The factor $F(\theta) = 0.984$.

With these values the prediction of Quantum Mechanics is

$$\delta_{QM} \;=\; 0.058 \pm 0.002$$

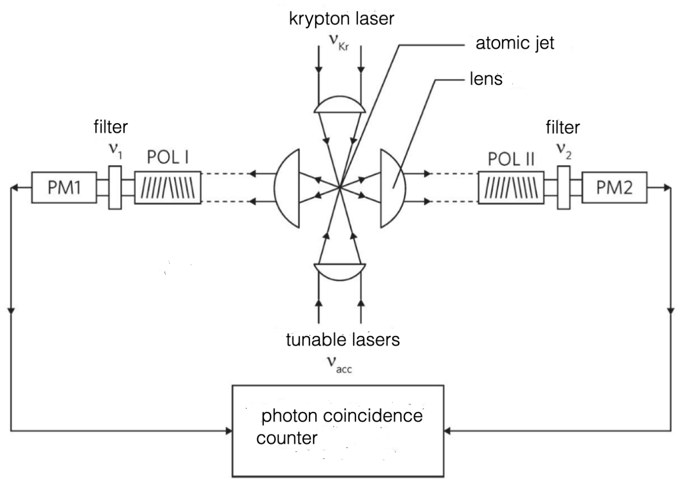

Figure 12: Schematic of the apparatus: Aspect's first experiment

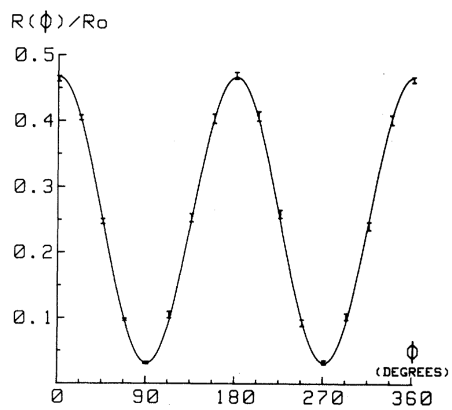

Figure 13: Results for correlation function $R(\phi)$, where ϕ is the angle between the polarizers of the two beams

They also measured directly the fraction of coincidence counts with a polarizer in front of one of the beams only, so that they could directly measure the value $\tilde{\mathcal{S}}$, with polarizer angles given by (43), obtaining the value

$$\tilde{\mathcal{S}} = 0.126 \pm 0.014,$$

violating Bell's inequality by 9 standard deviations. The Quantum Mechanics prediction for this quantity, for the polarizer efficiencies and opening angle correction, used in this experiment was

$$\tilde{\mathcal{S}}_{QM} = 0.118 \pm 0.005,$$

in agreement (within errors) with the experimental result.

Second Experiment (1981) [15]

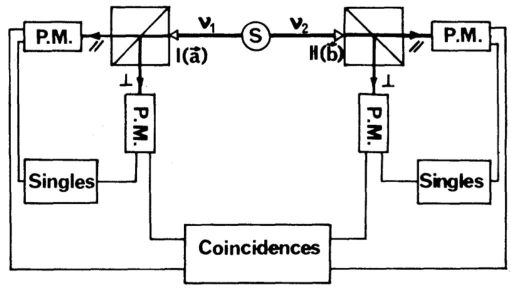

Figure 14: Diagram of apparatus with two-channel polarization analyzers and four coincidence counting circuits.

In the second experiment the pile-of-plates polarizers were replaced by two-channel polarization beam-splitters (see Fig.5), which sent the photon into different channels depending on whether its axis of polarization was parallel or perpendicular to the axis of the beam-splitter. Each channel was fitted with a photomultiplier tube whose which sent a signal to a coincidence counting circuit.

A schematic diagram of the apparatus is shown in Fig.14. There were four separated coincidence counting circuits which recorded the coincidence events for parallel-parallel, parallel-perpendicular, perpendicular-parallel, and perpendicular-perpendicular polarizations of the two photons. This enables direct measurement of the quantities $N_{++}(\alpha,\gamma)$, $N_{+-}(\alpha,\gamma)$, $N_{-+}(\alpha,\gamma)$, $N_{--}(\alpha,\gamma)$ for the polarizers set to angles α, γ. There was no necessity to measure the coincidence rates with one of the polarizers removed. The only remaining assumption that was made was that the efficiency of the photomultipliers (which was around 20-30%) did it not depend on the polarization of the detected photons. This experiment did, however, remove the ambiguity present in all previous experiments which used single channel polarizers, arising from being unable to determine whether an event in which only a single photon is recorded arises from a coincidence event in which one of the photons is not detected by the photomultiplier.

This experiment enabled the direct measurement of the quantity $\overline{S}(\alpha,\beta,\gamma,\delta)$ defined in (15), and which according to (CHSH) Bell's inequality cannot take a value larger than 2. With the polarizers angles set at (43) for which the Quantum Mechanics prediction of the violation of Bell's inequality is maximum, they found

$$\overline{S}\left(0,\frac{\pi}{4},\frac{\pi}{8},\frac{3\pi}{8}\right) = 2.697 \pm 0.015$$

in violation of Bell's inequality by 40 σ!

Their polarizer efficiencies were $\epsilon_1^{\parallel} = 0.95$, $\epsilon_1^{\perp} = 0.007$ and $\epsilon_2^{\parallel} = 0.93$, $\epsilon_1^{\perp} = 0.007$ and $F(\theta) = 0.984$ so that the Quantum mechanics prediction is

$$\overline{S}_{QM}\left(0,\frac{\pi}{4},\frac{\pi}{8},\frac{3\pi}{8}\right) = 2.7,$$

in remarkable agreement with the experimental result.

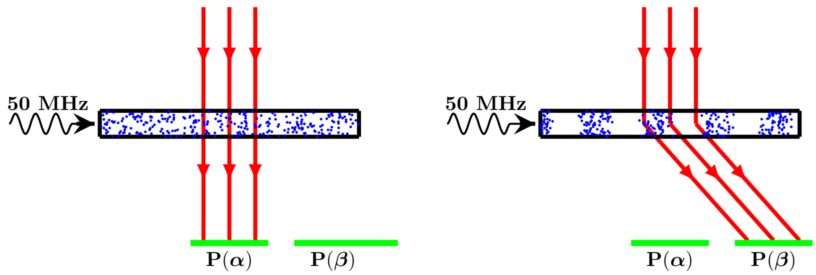

Figure 15: Diffraction from acoustic standing wave, frequency 50 MHz.

Third experiment (1982) [16]

The remaining "loophole" which provides the possibility of interpreting all of the previous experiments in terms of hidden variables, arises because the angles of the polarizers are fixed before the photon pair is produced and it is possible that the presence and settings of the polarizer in front of one of the beams influences that hidden properties of the other photon. This is possible since such an interaction would not violate causality.

To eliminate this possibility, one needs to be able to devise an experiment in which the settings of the polarizers are events which are separated from the pair production event by a space-like separation. Even with the polarizers set at six meters from the calcium beam from which the photon pairs are produced, it is impossible to rotate the polarizers within the required timescale of 20 ns.

Instead, Aspect and his collaborators devised an ingenious opto-acoustic switch, which was able to deflect the path of either photon (or both) on a time-scale of around 5 ns. Acoustic standing waves with a frequency of 50 MHz are set up in a tube containing water. The standing waves are waves of the density of the water. When the wave is flat, the water is evenly distributed and there is no diffraction – the photon goes straight through as shown on the left of diagram Fig.15. However, one quarter of a period (5 ns) later, when the standing wave is at maximum the density of water oscillates with position in the tube, and this acts as a diffraction grating, with the first diffraction maximum at the Bragg angle, as shown in the right of the diagram Fig.15. The photon is the directed to polarizer $P(\alpha)$ or $P(\beta)$ depending on the phase of the standing waves. Although this is not a random choice of polarization angle, it is switched within the time taken for the photons to reach the polarizer.

A schematic diagram of the apparatus is shown in Fig.16. The Bragg angle is less than 0.3° (even at the high frequency of 50 MHz, the diffraction grating produced is equivalent to one of only 60 grooves per mm.). The deflected photon is then reflected from two mirrors in order to produce sufficient separation for it to pass through a different polarizer.

For the polarizers set at the angles (43), maximizing the violation of Bell's inequality,

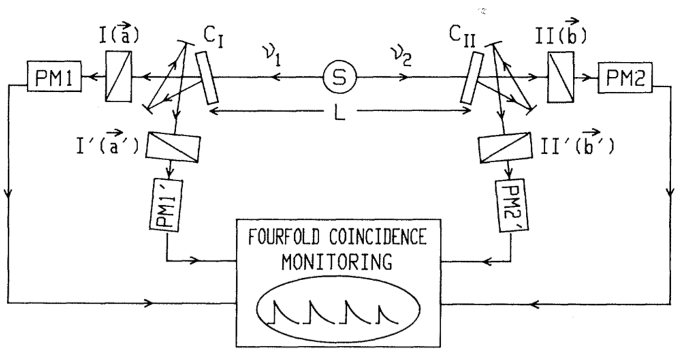

Figure 16: Schematic diagram of apparatus for the experiment in which the photon is directed towards one polarizer or another at a switching interval which is smaller than the time taken for the photons to reach the polarizers.

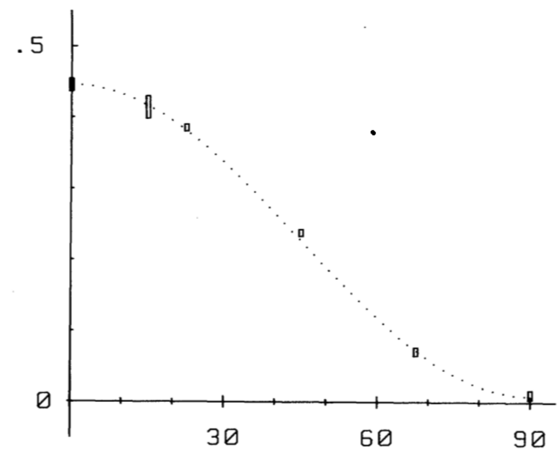

Figure 17: Plot of $R(\phi)$ against ϕ for different relative angles of the polarizers. The dotted line is the Quantum Mechanics prediction for this quantity as a function of ϕ.

they found

$$\tilde{S} \;=\; 0.101 \pm 0.02,$$

a violation of Bell's inequality by $5\,\sigma$.

They repeated the experiment for various different relative angles ϕ between the axes of the polarizers, in each case allowing for switching between polarizers set at different angles. In each case they determined the quantity $R(\phi)$. The results are shown in Fig.17, in which the dotted line is the prediction of Quantum Mechanics.

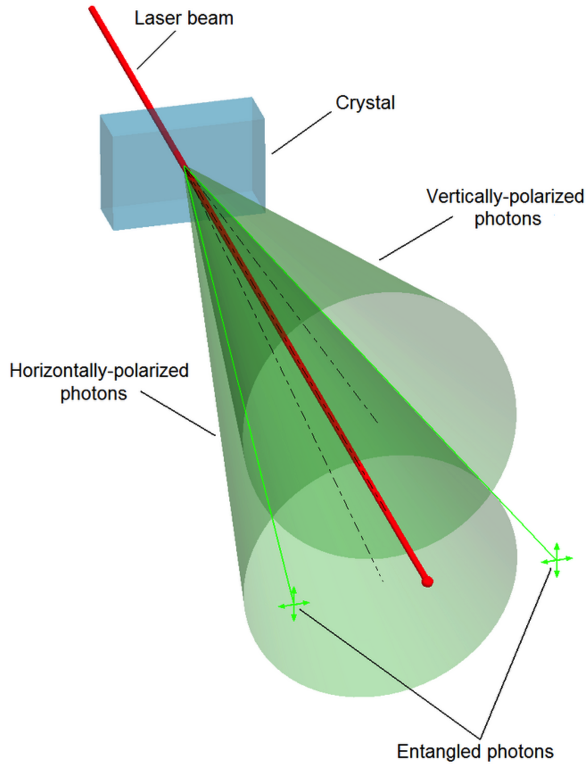

Figure 18: One of the photons emerging from the non-linear crystal is polarized horizontally and the other is vertically polarized. They travel along paths which are on the surfaces of cones with different aces of symmetry. Photons travelling along the intersection of these two cones are entangled.

3.7 Zeilinger's Experiment (1998)

In 1998, Gregor Weihs, Thomas Jennewein, Christoph Simon, Harald Weinfurter, and Anton Zeilinger carried out the definitive experimental test of Bell's inequality, which was free of any "loopholes" in which the results of the experiment could still be explained by a model of local hidden variables. [17]

To begin with, the photons from an entangled pair each travelled 200 metres (across the science campus of the University of Innsbruck) metres before they were observed at the two observation stations. This meant that it was possible to randomly choose the orientation of both the polarizers without one choice having a causal effect on the other polarizer, provided this could be done at a time scale of less than 1.3 μs.

The pairs of entangled photons were not generated by an atomic decay cascade but by spontaneous parametric down conversion. This is a process in which a crystal made from a birefringent material (in this case a beta-barium-borate crystal) converts a single photon (in this case from an argon-ion laser) into two photons with lower frequency whose linear polarization axes are anti-correlated - one being polarized parallel to the crystal's optical axis,

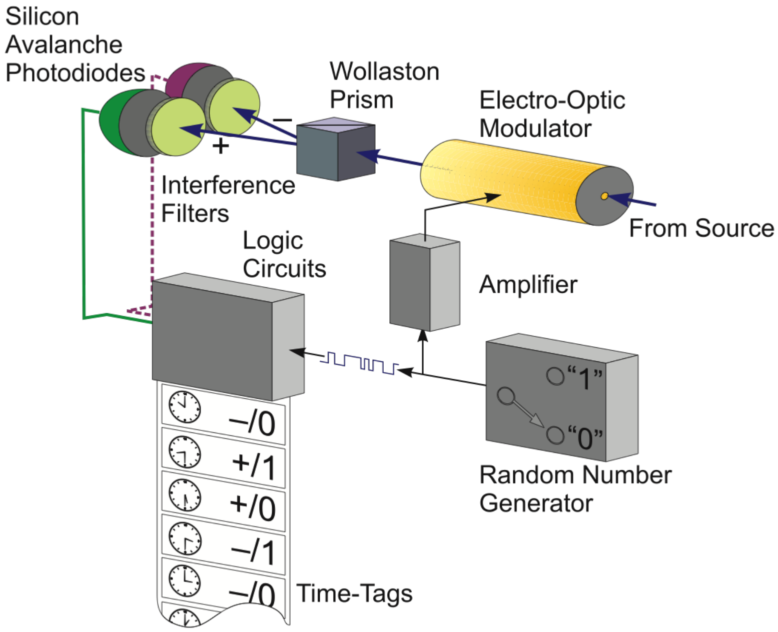

Figure 19: Schematic diagram of one of the two observation stations.

$|H\rangle$, and the other perpendicular to the optic axis, $|V\rangle$. The two photons are constrained (by energy and momentum conservation) to travel along a path which is on the surface of one of two cones. Photons travelling along the two paths defined by the intersection of the two cones are entangled (see Fig.18 so that the wavefunction for the two photons is

$$|\Psi\rangle \;=\; \frac{1}{\sqrt{2}}|H,V\rangle + \eta|V,H\rangle,$$

where η is a phase, which can be selected by passing the photons through appropriate compensating birefringent crystal plates.

This has the enormous advantage. The directions of the two photons are known, so that all of the entangled photons can be directed towards their respective observation stations. In the case of entangled photons arising from an atomic decay cascade the photons are homogeneously distributed in all directions (assuming zero orbital angular momentum) so that if the photons are collected with an opening a angle, θ, only a small fraction $\sim \theta^2/2$ of the photons are observed.

A schematic diagram of the apparatus at one of the observing stations is shown in Fig.19. The photon passes through an electro-optic modulator which rotates the direction of linear polarization of the photon. Upon receiving a "1" from a random binary number generator the timer polarization is rotated by further $\pi/4$. Each observing station has such a random number generator, so the choice of direction in which the polarization is measured is truly random and, importantly, causally independent. The photon emerges from the electro-optic modulator into a Wollaston prism which acts as a beam splitter, sending the photon into one of two detectors depending on its state of linear polarization. The detectors are silicon avalanche detectors, as opposed to the photomultiplier tubes used in previous experiments. These have the advantage of having an efficiency of around 90%, compared with 20-30% for a photomultiplier tube, and a much higher signal to noise ratio, implying a much smaller number of false signals.

For every coincidence event, each photon is tagged electronically with “+” or “-”, depending on which detector fired (indicating the polarization state of the photon), as well as “0” or “1” from the random number generator, indicating the direction in which polarization was measured, and the time of the event was recorded using a atomic clock. For each of the four possible settings of ”0” or ”1” for the two photons (from which one can deduce the angles $\alpha, \cdots \delta$) of (43), there is one a four possible coincidences (+,+), (+,-) (-,+) and (-,-) (a coincidence defined as two detectors firing within a time interval of 5 ns). The numbers of such coincidences are N_{++}, N_{+-}, N_{-+} and N_{--}, respectively. The total delay time of the electronic system was about 75 ns, which is an order of magnitude less than the time taken for an electromagnetic signal to travel from one observation station to the other.

They were able to collect 14700 coincidences in 10 s. They found

$$\overline{S} \ = \ 2.73 \pm 0.02,$$

violating Bell’s inequality by 35 standard deviations. This agrees well with the Quantum Mechanics prediction, once the asymmetries of the detectors and errors in the voltages of the electro-optic modulators, are taken into account.

Whenever an experiment is conducted, which “falsifies” a given theory, there are always those who continue to seek loopholes in order to revive the theory. Whereas this is often motivated by an irrational attachment to the theory on the grounds of its elegance, completeness, or its ability to explain otherwise unexplained phenomena, it generates a healthy debate amongst scientists and helps to ensure rigour, scientific integrity, and critical analysis. This is also the case for the hypothesis of local hidden variables. Nevertheless the overwhelming consensus is that this experiment together with its predecessors demonstrates that the Bell inequality is violated so that the hypothesis of local hidden variables is disproven. On the other hand, the remarkable quantitative agreement with the predictions of Quantum Mechanics provides strong support for its validity.

4 Afterword

Alain Aspect, John Clauser, and Anton Zeiliinger shared the 2022 Nobel Prize for physics.

John Bell died from a brain haemorrhage in 1990, aged 62. Unbeknownst to him, he had been nominated for the Nobel Prize in Physics and was considered to be a frontrunner. Nobel prizes are never awarded posthumously.

References

[1] A.Einstein, B. Podolski and N. Rosen, Phys. Rev. **47**, 777 (1935).

[2] D. Bohm and Y. Aharonov, Phys.Rev **108** 1070 (1957)

[3] C. S. Wu, Phys. Rev. **77** , 136 (1950)

[4] J.S. Bell, Physics. Vol. 1 No. 3 195 (1964).

[5] J.F. Clauser, M.A. Horne, A. Shimony, and R.A. Holt, Phys. Rev. Lett. **23** 880 (1969)

[6] S. J. Freedman, Lawrence Berkeley Laboratory Report 391 (1972).

[7] L.R. Kasday, J. D. Ullman and C. S. Wu, Bull. Am. Phys. Soc. **15**, 586 (1970).

[8] O. Klein and Y. Nishina, . Z. Phys. **52** 853 and 869, (1929)

[9] S.J. Freedman and J.f. Clauser, Phys. Rev. Lett. **28** 938 (1972)

[10] Holt, R. A., 'Atomic Cascade Experiments' (Ph.D. Thesis), Harvard University, 1973, unpublished.

[11] J. Clauser, Phys. Rev. Lett. **36** 1223 (1976)

[12] E.S. Fry and R.C. Thomson, Phys. Ref. Lett. **37** 465 (1976)

[13] A. Aspect "Si Einstein Avait Su", Odile Jacob (2025)

[14] A. Aspect, P. Grangier and G. Roger, Phys. Rev. Lett. **47** 460 (1981)

[15] A. Aspect, P. Grangier and G. Roger, Phys. Rev. Lett. **49** 91 (1981)

[16] A. Aspect, J. Dalibard and G. Roger, Phys. Rev. Lett. **49** 1804 (1982)

[17] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, A, Zeilinger, Phys. Rev. Lett. **81** 5039 (1998)