QUANTUM FIELD THEORY 1

lectures to first year SHEP postgraduate students
given by
D.A. Ross

http://www.hep.phys.soton.ac.uk/hepwww/staff/D.Ross/ft1/

Syllabus

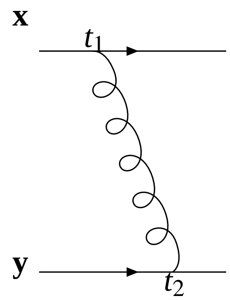
- Introduction
- Free Scalar Field
- Energy-momentum tensor, momentum and angular momentum
- Green Functions
- Complex Scalar Field
- Interactions - the LSZ reduction formula
- Perturbation Theory
- Feynman Diagrams
- Lorientz-invariant phase-space
- Inelastic processes
- The Electromagnetic Field
- Electromagnetic interactions
- Examples of electromagnetic interactions of scalar particles
- The Dirac Equation
- The Dirac field - Quantum Electrodynamics
- Examples of electromagnetic interactions of fermions.

1 Introduction

Relativistic Quantum Field Theory is an attempt to synthesise Quantum Mechanics and Special Relativity.

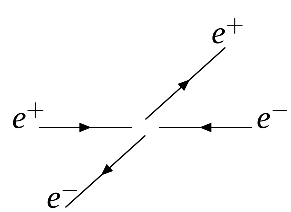
There are two main qualitative differences between classical and relativistic mechanics.

1. Whereas there exists a relativistic generalisation of kinetic energy - we know how energy and momentum transform under Lorentz transformations - there is no equivalent generalisation of potential since the concept of potential implies an action at a distance, which violates the relativity principle of causality. For a particle at a point **x** to influence a particle at a point **y**, something has to travel from **x** to **y** with a speed no greater than the speed of light. The picture we have of two charged particles interacting is that they are exchanging particles of light (photons)

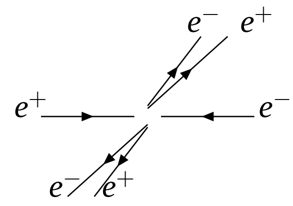
   It can be shown that this gives rise to the familiar Coulomb potential in the non-relativistic limit. Therefore we drop the concept of potential entirely and consider systems in which particles emit and absorb other particles at particular points in space-time.

2. In non-relativistic Quantum Mechanics the number of particles is conserved, but in relativity energy and mass are conserved together. If we have energy larger than some mass m [†] then it is possible to create a particle with mass m. For example if we scatter an electron and a positron at low energy we have an electron and a positron in the final state.

   but if the energy is larger than $4m_e$, we can have a final state consisting of two electrons and two positrons.

[†] In these lectures the speed of light c is consistently set to 1.



Likewise an electron-positron pair can annihilate into two photons, with the mass-energy of the electron and positron taken up by high-energy photons.

Actually we have already introduced the concept of particle non-conservation in the picture of charged-particle interactions. In the figure an electron emits a photon at time t_1 so the number of particles increases from 2 to 3 at t_1 and then goes back to 2 at t_2 when the photon is absorbed.

The Schroedinger equation gives the amplitude for a particle to be at a particular point in space and the probability (square amplitude) integrates to one. This is inadequate for a relativistic description and so we seek a formalism in which particles (lumps of energy) can be created and destroyed.

1.1 The Harmonic Oscillator

Mass=1, angular frequency=ω. † The Lagrangian is

$$L \equiv T - V = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2\phi^2$$

Canonical momentum:

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}$$

Hamiltonian

$$H \equiv \pi\dot{\phi} - L = \frac{1}{2}\left(\pi^2 + \omega^2\phi^2\right)$$

In terms of the raising and lowering operators

$$a^\dagger = \frac{1}{\sqrt{2\omega}}(\pi + i\omega\phi)$$

$$a = \frac{1}{\sqrt{2\omega}}(\pi - i\omega\phi)$$

we have

$$H = \omega a^\dagger a + \frac{1}{2}\omega$$

†In these lectures we also use units in which $\hbar$ is set equal to 1

The raising and lowering operators obey the commutation relations

$$\left[a, a^{\dagger}\right] = 1$$

If $|n\rangle$ is the eigenstate with energy E_n, then $a^{\dagger}|n\rangle$ is the eigenstate with energy $E_n + \omega$ and $a|n\rangle$ is the eigenstate with energy $E_n - \omega$. (up to a factor of n which arises because there are n quanta of energy in the state $|n\rangle$ that the operator a can lower.)

Thus we can interpret the raising and lowering operators as operators that create and destroy a "lump of energy" ω.

1.2 Coupled Harmonic Oscillators

Let us consider a more complicated system - a system of N coupled harmonic oscillators a distance Δx apart, with a mass Δx, a harmonic potential $\frac{1}{2}m^2\Delta x\phi_j^2$ (ϕ_j is the displacement of the j^{th} harmonic oscillator), and a coupling between nearest neighbours of $1/(2\Delta x)\left(\phi_{j+1} - \phi_j\right)^2$

The Lagrangian is

$$L = \sum_{j=0}^{N-1}\left(\frac{\Delta x}{2}\dot{\phi}_j^2 - \frac{1}{2\Delta x}(\phi_{j+1} - \phi_j)^2 - \frac{m^2}{2}\Delta x\phi_j^2\right)$$

We impose the periodicity condition

$$\phi_N = \phi_0,$$

for convenience (we will later take $N \to \infty$ so this is not important)

The classical equations of motion are obtained from the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}_j} = \frac{\partial L}{\partial\phi_j},$$

giving

$$\frac{\partial^2\phi}{\partial t^2} = \frac{1}{(\Delta x)^2}\left(\phi_{j+1} + \phi_{j-1} - 2\phi_j\right) - m^2\phi_j^2$$

The canonical momenta π_j are given by

$$\pi_j = \dot{\phi}_j$$

and the Hamiltonian is

$$H = \sum_{j=0}^{N-1}\left(\frac{\Delta x}{2}\pi_j^2 + \frac{1}{2\Delta x}(\phi_{j+1} - \phi_j)^2 + \frac{m^2}{2}\Delta x\phi_j^2\right)$$

The system is quantised by imposing the commutation relations

$$\left[\pi_k(t), \phi_j(t)\right] \;=\; -i\delta_{jk}$$

N.B. We are working in the Heisenberg representation, so that these operators must be considered to be time dependent - so that these commutation operators apply at equal times (called "equal time commutation relations").

The oscillators can be decoupled by taking a discrete Fourier transform

$$\phi_j \;=\; \sum_{k=0}^{N-1} \frac{\sqrt{2}}{\sqrt{N\Delta x}} \sin\left(\frac{2\pi k j}{N}\right) \tilde{\phi}_k$$

Inserting this into the Hamiltonian and performing some algebra this gives

$$H \;=\; \sum_{k=0}^{N-1} \left(\frac{1}{2}\tilde{\pi}_k^2 + \frac{1}{2}\left(\frac{4}{\Delta x^2}\sin^2\left(\frac{\pi k}{N}\right) + m^2\right)\tilde{\phi}_k^2\right)$$

This is a sum of harmonic oscillators with frequency ω_k where

$$\omega_k^2 \;=\; \frac{4}{(\Delta x)^2}\sin^2\left(\frac{\pi k}{N} + m^2\right)$$

We quantise this as before by defining creation and annihilation operators

$$a_k^\dagger \;=\; \left(\tilde{\pi}_k + i\omega_k\tilde{\phi}_k\right) e^{-i\omega_k t}$$

$$a_k \;=\; \left(\tilde{\pi}_k - i\omega_k\tilde{\phi}_k\right) e^{+i\omega_k t}$$

We have explicitly written out the factors $e^{\pm i\omega t}$ so that $a_k^\dagger$ and a_k are time independent. Furthermore we have adjusted the normalisation so that they obey the commutation relation

$$\left[a_k, a_l^\dagger\right] \;=\; 2\omega_k\Delta_{kl}$$

In terms of these operators the Hamiltonian becomes

$$H \;=\; \sum_{k=0}^{N-1} \frac{1}{2\omega_k}\left(a_k^\dagger a_k + \frac{1}{2}\right)$$

We now have a Hamiltonian in which the raising and lowering operators, create and destroy particles with different energy ω_k.

Here there is a problem. We would like the ground state $|0\rangle$, for which $a_k|0\rangle = 0$, to represent the vacuum (in which there were no particles) and have zero energy. For this to be the case we need to subtract the ground state energy of all the oscillators and write

$$H = \sum_{k=0}^{N-1} \frac{1}{2\omega_k} \left(a_k^\dagger a_k \right).$$

In terms of the original variables ϕ_j and π_j, we can achieve this by introducing the concept of "normal ordering", which means that the annihilation operators always occur on the right of the creation operators, so that the Hamiltonian acting on the vacuum gives zero. Thus we have

$$H = \sum_{j=0}^{N-1} : \left(\frac{\Delta x}{2} \pi_j^2 + \frac{1}{2\Delta x} (\phi_j - \phi_{j+1})^2 + \frac{1}{2} m^2 \phi_j^2 \right) :,$$

where the notation $:\cdots:$ indicates this normal ordering.

Now if we take $N \gg k$, ω_k may be approximated to

$$\omega_k = \left(\frac{4\pi^2 k^2}{N^2 \Delta x^2} + m^2 \right)^{1/2}.$$

The "lumps of energy" created and destroyed by $a_k^\dagger$ and a_k are the energies of relativistic particles with mass m and momenta

$$p = \frac{2\pi k}{N \Delta x}.$$

We interpret these as particles and we have a formalism for describing the creation and annihilation of relativistic moving particles. The quantum state

$$|n_1, n_2 \cdots\rangle$$

is the state with n_1 particles with momentum $2\pi/(N\Delta x)$, n_2 particles with momentum $4\pi/(N\Delta x)$, etc.

$$a_k |n_1, \cdots n_k, \cdots\rangle = n_k |n_1, \cdots n_k - 1, \cdots\rangle,$$

and

$$a^\dagger |n_1, \cdots n_k - 1, \cdots\rangle = |n_1, \cdots n_k, \cdots\rangle,$$

so that the number operator is given by

$$N = \sum_{k=0}^{N-1} a_k^\dagger a_k,$$

which counts the number of particles with energy ω_k and sums over all possible energies.

1.3 Continuum Limit

In order to allow for *all* possible values of momentum we need to take the limit $N \to \infty$ and $\Delta x \to 0$. In that case the discrete set of variables $\phi_j(t)$ becomes a "field", $\phi(x,t)$, and the quantity

$$\frac{\phi_{j+1} - \phi_j}{\Delta x} \;\to\; \frac{\partial \phi(x,t)}{\partial x}.$$

The Lagrangian becomes

$$L \;=\; \int dx \mathcal{L},$$

where $\mathcal{L}$ is the "Lagrangian density" given by

$$\mathcal{L} \;=\; \frac{1}{2}\dot{\phi}(x,t)^2 - \frac{1}{2}\left(\frac{\partial \phi(x,t)}{\partial x}\right)^2 - \frac{1}{2}m^2 \phi(x,t)^2$$

The classical equations of motion are

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - m^2 \phi \;= 0.$$

The canonical momentum is also now a function of both x and t,

$$\pi(x,t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \;=\; \dot{\phi}(x,t)$$

The Hamiltonian is

$$H \;=\; \int dx \left(\pi \dot{\phi} - \mathcal{L}\right) \;=\; \int dx \left(\frac{1}{2}\pi(x,t)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2}m^2 \phi^2\right)$$

The creation and annihilation operators are now functions of momentum $a^\dagger(p)$, $a(p)$ and we can expand $\phi(x,t)$ as

$$\phi(x,t) \;=\; \int \frac{dp}{(2\pi)2E_p}\left(a(p)e^{-i(E_p t - px)} + a^\dagger(p)e^{+i(E_p t - px)}\right)$$

with $E_p = \sqrt{p^2 + m^2}$. The field operator $\phi(x,t)$ annihilates a particle with positive energy (time dependent $e^{-iE_p t}$) or creates a particle with negative energy.

The field operator and the conjugate momentum obey the equal time commutation relations

$$[\pi(x,t), \phi(y,t)] \;=\; -i\delta(x-y),$$

which is equivalent to the commutation relations between creation and annihilation operators

$$\left[a(p), a^\dagger(p')\right] \;=\; (2\pi)2E_p \delta(p - p')$$