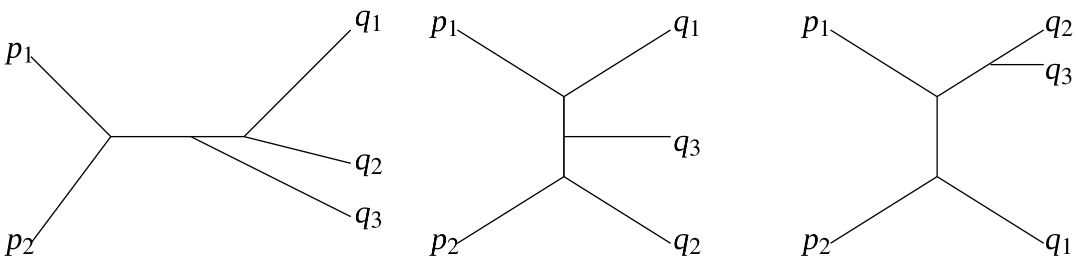
10 Inelastic Processes

So far we have been considering a theory with only one real field ϕ, so all particles are the same. The two-to-two-body process is an elastic scattering process in which the outgoing particles are the same as the incoming ones, but with different momenta.

At sufficiently high energies we could have inelastic processes in which two incoming particles scatter into three or more final particles. Feynman graphs for such a process would be



+ permutations of q_1, q_2, q_3

Alternatively, we can consider a theory with two fields: ϕ corresponding to a particle of mass m and χ corresponding to a particle of mass M and an interaction Lagrangian

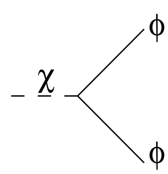
$$\mathcal{L}_I \; = = -\frac{g}{2}\chi\phi^2$$

which couples the fields and allows the two types of partcile to interact with each other.

The total Lagrangian density for such a system is

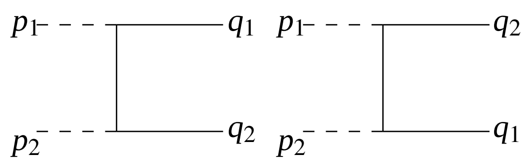
$$\mathcal{L} \; = \; \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}M^2\chi^2 - \frac{g}{2}\chi\phi^2$$

The interaction vertex



has a Feynman rule $-ig$ (the factor of $\frac{1}{2}$ cancels against the two ways of coupling the two ϕ's in the same way that the $\frac{1}{3!}$ cancelled in the previous ϕ^3 case.)

Consider the scattering of two χ-particles with momenta p_1, p_2 into two ϕ particles with momenta q_1, q_2. There are two Feynman diagrams for this process:



Note that the s-channel diagram is missing since this would require an interaction of the form ϕ^3 or χ^3, which we choose *not* to include in this model.

Following the Feynman rules the contributions from these two graphs are

$$(-ig)^2 \frac{i}{(p_1-q_1)^2-m^2} \;=\; (-ig)^2 \frac{i}{(t-m^2)}$$

and

$$(-ig)^2 \frac{i}{(p_1-q_2)^2-m^2} \;=\; (-ig)^2 \frac{i}{(u-m^2)}$$

Recall that the quantity u may be expressed in terms of s and t and the masses as

$$u = 2M^2 + 2m^2 - s - t$$

.

The cross-section is given by

$$\begin{aligned}\sigma \;&=\; \int \frac{d^3\mathbf{q_1}}{(2\pi)^3 2E_{q_1}} \frac{d^4 q_2}{(2\pi)^2} \delta(q_2^2-m^2)\theta(q_2^0)(2\pi)^4\delta^4(p_1+p_2-q_1-q_2) \\ &\quad \times \frac{1}{F} g^4 \left(\frac{1}{(t-m^2)} + \frac{1}{(u-m^2)}\right)^2\end{aligned}$$

The flux factor F is given by

$$F = 4\sqrt{(p_1\cdot p_2)^2 - M^4} \;=\; 2\sqrt{s(s-4M^2)},$$

where we have used

$$s = (p_1+p_2)^2 = 2M^2 + 2p_1\cdot p_2$$

Now integrating over q_2 and absorbing the energy-momentum conserving delta-function this leaves

$$\sigma \;=\; \int \frac{d^3\mathbf{q_1}}{(2\pi)^2 2E_{q_1}} \frac{1}{2\sqrt{s(s-4M^2)}} g^4 \left(\frac{1}{(t-m^2)} + \frac{1}{(u-m^2)}\right)^2 \delta\left((p_1+p_2-q_1)^2 - m^2\right)$$

Now move to the centre-of-mass frame for which we set the magnitude of the three-momentum of q_1 to q and the magnitude of the three momentum of p_1 is p and use

$$q\,dq = E_{q_1} dE_{q_1}$$

and

$$t = m^2 + M^2 - 2E_{p_1}E_{q_1} + 2pq\cos\theta$$

so that

$$dt = 2pqd\cos\theta$$

This gives

$$\frac{d^3\mathbf{q_1}}{2E_{q_1}} = d\phi d\cos\theta\frac{q^2dq}{2E_{q_1}} = d\phi d\cos\theta qdE_{q_1} = d\phi\frac{dt}{4p}dE_{q_1}$$

The remaining delta function is now

$$\delta((p_1+p_2-q_1)^2 - m^2) = \delta(s - 2\sqrt{s}E_{q_1})$$

so that we can use the integration over E_{q_1} to absorb this delta-function giving a factor of $1/(2\sqrt{s})$ and the integration over the azimuthal ϕ gives a factor of 2π so we end up with

$$\sigma = \frac{g^4}{32\pi s}\int dt\frac{1}{p\sqrt{s-4M^2}}\left(\frac{1}{(t-m^2)} + \frac{1}{(2M^2+m^2-s-t)}\right)^2$$

and finally the magnitude of the incoming three-momentum in the centre-of mass frame is given by

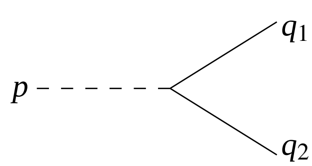
$$p = \frac{1}{2}\sqrt{s-4M^2}$$

so we end up with a differential cross-section with respect to t

$$\frac{d\sigma}{dt} = \frac{g^4}{16\pi s(s-4M^2)}\left(\frac{1}{(t-m^2)} + \frac{1}{(2M^2+m^2-s-t)}\right)^2$$

10.1 Decay Rates

If $M > 2m$ then a χ particles can decay into two ϕ particles. The Feynman graph for such a decay is



There is *no* internal propagator, so the Feynman rules for this graph give a matrix-element

$$M = -ig(2\pi)^4\delta(p - q_1 - q_2)$$

The calculation of the decay rate proceeds along the same lines as the calculation for scattering cross-section. Again, in order to avoid obtaining the square of a delta function, the incoming state is smeared with a distribution function peaked at p, whose Fourier transform is he wavefunction $\psi(x)$ of the incoming χ particle. The transition probability is then

$$W = \int d^4x \frac{|\psi(x)|^2}{2E_p} |\mathcal{M}|^2 (2\pi)^4 \delta^4(p - q_1 - q_2)$$

The $2E_P$ in the denominator arises from the relativistic normalisation of the incoming state.

The transition rate per unit volume is

$$\frac{dW}{d^3\mathbf{x}dt} = d\Gamma \times |\psi(x)|^2$$

$|\psi(x)|^2$ is the probability of finding the decaying particle in unit volume and $d\Gamma$ is the differential decay rate of the χ-particle into two ϕ-particles with momenta q_1 and q_2.

The total decay rate is obtained by integrating this quantity over the DLIPS for the decay products.

$$\begin{aligned}
\Gamma &= \frac{1}{2E_p} \int \frac{d^3\mathbf{q_1}}{(2\pi)^3 2E_{q_1}} \frac{d^4q_2}{(2\pi)^3} \delta(q_2^2 - m^2)(2\pi)^4 \delta^4(p - q_1 - q_2) |\mathcal{M}|^2 \\
&= \frac{g^2}{2E_p} \frac{1}{(2\pi)^2} \int \frac{d^3\mathbf{q_1}}{2E_{q_1}} \delta((p - q_1)^2 - m^2)
\end{aligned}$$

Decay rates (inverse lifetimes) are *not* Lorentz invariant, but transform like $1/E$. Lifetimes are usually quoted in rest-frame of the decaying particle and in such a frame we replace E_p by M and the argument of the remaining delta-function is $M^2 - ME_{q_1}$.

Again, we have (setting the magnitude of the three-momentum of the q_1 to q in the rest-frame of the parent particle)

$$\frac{d^3\mathbf{q_1}}{2E_{q_1}} = q^2 \frac{dq}{2E_{q_1}} d\Omega = \frac{q dE_{q_1}}{2} d\Omega$$

The integration over the solid angle Ω gives a factor of 4π, so that we have

$$\Gamma = \frac{g^2}{8\pi M} \int dE_{q_1} q \delta(M^2 - 2ME_{q_1})$$

We perform the integration over E_{q_1} absorbing the delta function getting a factor of $(1/(2M)$ and the solution $E_{q_1} = \frac{1}{2}M$ gives us

$$q = \frac{\sqrt{M^2 - 4m^2}}{2},$$

so that finally we end up with

$$\Gamma = \frac{g^2}{16\pi M^2} \sqrt{M^2 - 4m^2}.$$