11 The Electromagnetic Field

The photon has spin one and is represented by a 4-vector field A_μ. In terms of this field the components of the electric and magnetic field are given by first constructing the antisymmetric tensor

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu$$

where the components of the electric and magnetic field E_i, B_i are

$$F_{0i} = -F_{i0} \;=\; E_i \;\; (i = 1, \cdots 3)$$

and

$$\varepsilon_{ijk} F_{jk} = -B_i.$$

The Lagrangian density for this field is

$$\mathcal{L}_A \;=\; -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Note that there is no mass term since the photon is massless.

The Euler Lagrange equations of motion are

$$\partial^\mu F_{\mu\nu} \;=\; 0$$

This reproduces two of Maxwell's equations in free space (i.e. with no electric charge density or current)

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{B} &= \frac{d}{dt}\mathbf{E} \end{aligned}$$

The other Euler-Lagrange equation is

$$\varepsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 0,$$

where we have introduced the totally antisymmetric (Levi-Civita) tensor in 4-dimensions

$$\begin{aligned} \varepsilon_{\mu\nu\rho\sigma} &= 1 \;\; \text{for } \{\mu,\nu,\rho,\sigma\} \text{ even permutation of } \{1,2,3,0\} \\ &= -1 \;\; \text{for } \{\mu,\nu,\rho,\sigma\} \text{ even permutation of } \{0,1,2,3\} \\ &= 0 \;\; \text{otherwise} \end{aligned} \tag{11.1}$$

Take care that because of the form of the Minkowski metric

$$\varepsilon^{\mu\nu\rho\sigma} \;\equiv\; g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} g^{\sigma\sigma'} \varepsilon_{\mu'\nu'\rho'\sigma'} \;=\; -\varepsilon_{\mu\nu\rho\sigma}$$

This reproduces the other two Maxwell equations

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = -\frac{\partial \mathbf{E}}{\partial t}$$

The canonical momentum is again defined by

$$\pi_\mu \;=\; \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu}$$

Here we have

$$\pi_0 = 0,$$

meaning that care needs to be taken in quantising the field since the zero component does not possess a conjugate momentum. For the spae-like components,

$$\pi_i \;= E_i$$

Using this canonical momentum we can construct the Hamiltonian density

$$\mathcal{H} \;=\; \frac{1}{2}\pi_i^2 + \frac{1}{4}\left(\partial_i A_j - \partial_j A_i\right)^2 \;=\; \frac{1}{2}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right)$$

For the space-like components, the equal time commutation relations are

$$\left[\pi_i(\mathbf{x},t), A_j(\mathbf{y},t)\right] \;=\; -i\delta^3(\mathbf{x}-\mathbf{y})\delta_{ij}$$

Since the photon has spin, we define a photon state not only by its momentum $\mathbf{p}$ but also specifiy its helicity, λ, (the component of its spin in its direction of motion). The field A_μ can be expanded in terms of creation and annihilation operators for photons of momentum $\mathbf{p}$ helicity λ, which obey the commutation relations

$$\left[a(\mathbf{p},\lambda)a^\dagger(\mathbf{p}',\lambda')\right] \;=\; (2\pi)^3 2E_p \delta^3(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'}$$

The photon, being massless, can only have two possible helicities ± 1, in contrast to a massive spin-one particle (described by a vector field with a mass term) which has three, corresponding to the three possible components of its spin in its direction of motion.

The expansion of the photon field is

$$A_\mu(x) \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \sum_{\lambda=\pm 1} \left(a(\mathbf{p},\lambda)\varepsilon_\mu(p,\lambda)e^{-ip\cdot x} + a^\dagger(\mathbf{p},\lambda)\varepsilon_\mu^*(p,\lambda)e^{+ip\cdot x},\right)$$

For a photon moving along the z-direction the polarisation vectors may be written

$$\varepsilon_\mu(\mathbf{p},\mp 1) \;=\; \frac{1}{\sqrt{2}}(0,1,\pm i,0)),$$

having components perpendicular to the momentum of the photon. The $\pm i$ in the y-direction correspond to a photon with right/left circular polarisation.

However there is some ambiguity in these polarisation vectors, arising form the fact that we have a"gauge invariance", i..e under the transformation

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x),$$

which can be seen to leave the tensor $F_{\mu\nu}$ unchanged and so it has no effect on the physical electric and magnetic fields. This freedom allows us to make corresponding changes to the polarisation vectors the polarisation vectors by adding a term proportional to the photon momentum p_μ

The above choice of representation for the polarisation vectors is called the "Coulomb gauge" and in this gauge

$$\sum_{\lambda=\pm 1} \varepsilon_i(p,\lambda)\varepsilon_j^*(p,\lambda) = \delta_{ij} - \frac{p_i p_j}{|\mathbf{p}|^2}$$

This is not terribly convenient and a gauge can be found in which this sum can be written in covariant notation

$$\sum_{\lambda=\pm 1} \varepsilon_\mu(p,\lambda)\varepsilon_\nu^*(p,\lambda) = -g_{\mu\nu} + \xi \frac{p_\mu p_\mu}{p^2},$$

where ξ is the "gauge parameter" and can take any value without changing the result of the calculation of any physical process (this is a consequence of the gauge invariance).

From now on we will choose the Feynman gauge for which $\xi = 0$ is that

$$\sum_{\lambda=\pm 1} \varepsilon_\mu(p,\lambda)\varepsilon_\nu^*(p,\lambda) = -g_{\mu\nu}.$$

The Feynman propagator

$$i\Delta_F(x,y)_{\mu\nu} = \langle 0|TA_\mu(x)A_\nu(y)|0\rangle >,$$

can be deduced from this expansion and is given by

$$\Delta_F(x,y)_{\mu\nu} = \lim_{\varepsilon\to 0} \int \frac{d^4p}{(2\pi)^4} \frac{\sum_{\lambda=\pm 1} \varepsilon_\mu(p,\lambda)\varepsilon_\nu^*(p,\lambda)}{p^2 + i\varepsilon}$$

In Feynman gauge this simply becomes

$$\Delta_F(x,y)_{\mu\nu} = \lim_{\varepsilon\to 0} \int \frac{d^4p}{(2\pi)^4} \frac{-g_{\mu\nu}}{p^2 + i\varepsilon}$$

As in the case of the scalar field, we can invert the expansion for the field A_μ and its time derivative to obtain expressions for the creation and annihilation operators for the "in" and "out" states in terms of these fields. We find

$$a_{in}(\mathbf{p},\lambda) = i\varepsilon_\mu(p,\lambda) \int d^3\mathbf{x} e^{ip\cdot x} \overleftrightarrow{\partial_0} A^\mu_{in}(x)$$

$$a^\dagger_{in}(\mathbf{p},\lambda) \;=\; -i\varepsilon^*_\mu(p,\lambda)\int d^3\mathbf{x} e^{-ip\cdot x}\overleftrightarrow{\partial_0} A^\mu_{in}(x),$$

with similar expressions for the "out" states.

From this, we can go through the same steps as we did for the scalar field to obtain the LSZ reduction for the S-matrix element for an incoming state of n photons with momenta $p_1\cdots p_n$ and helicities $\lambda_1\cdots\lambda_n$ and an outgoing state of m photons with momenta $q_1\cdots q_n$ and helicities $\lambda'_1\cdots\lambda'_n$. The expression is

$$\langle q_1,\lambda'_1,\cdots q_m,\lambda'_m,out|p_1,\lambda_1,\cdots p_n,\lambda_n,in\rangle \;=\; \left(\frac{i}{\sqrt{Z}}\right)^{n+m}\int d^4x_1\cdots d^4x_n d^4y_1\cdots d^4y_m$$
$$e^{-i\sum_{j=1}^n p_j\cdot x_j}e^{+i\sum_{k=1}^m q_k\cdot y_k}\varepsilon_{\nu'_1}(q_1,\lambda'_1)\cdots\varepsilon_{\nu'_m}(q_m,\lambda'_m)\varepsilon^*_{\mu'_1}(p_1,\lambda_1)\cdots\varepsilon^*_{\mu'_n}(p_n,\lambda_n)$$
$$\left(-g^{\mu'_1\mu_1}\Box_{x_1}\right)\cdots\left(-g^{\mu'_n\mu_n}\Box_{x_n}\right)\left(-g^{\nu'_1\nu_1}\Box_{y_1}\right)\cdots\left(-g^{\nu'_m\nu_m}\Box_{y_m}\right)\langle 0|TA_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)A_{\nu_1}(y_1)\cdots A_{\nu_m}(y_m)|0\rangle$$

As in the case of the scalar field the operators $g^{\mu\mu'}\Box$ acting on the external propagators between an external photon field and an intenral filed (coming form the interaction Lagrangian) gives a delta-function which is absorbed by the integration over d^4x.

11.1 Angular Momentum Operator

The angular momentum operator fopr a spin-1 field contains an extra term, not present in the case of a scalar field, which ccounts for the spin of the particles. Thus we have

$$M^{\mu\nu} \;=\; \int d^3\mathbf{x}\left(:x^\mu T^{\nu 0} - x^\nu T^{\mu 0}:\right) + \Sigma^{\mu\nu},$$

where

$$\Sigma^{\mu\nu} \;=\; \int d^3\mathbf{x} : \left(A^\mu(x)\dot{A}^\nu(x) - A^\nu(x)\dot{A}^\mu(x):\right)$$

In terms of creation and annihilation operators this is

$$\Sigma^{\mu\nu} \;=\; i\int\frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\sum_{\lambda=\pm 1}\left(\varepsilon^{\mu *}(\mathbf{p},\lambda)\varepsilon^\nu(\mathbf{p},\lambda) - \varepsilon^{\nu *}(\mathbf{p},\lambda)\varepsilon^\mu(\mathbf{p},\lambda)\right)a^\dagger(\mathbf{p},\lambda)a(\mathbf{p},\lambda)$$

For a particle moving along the z-direction with helicity λ

$$S^z|p_z,\lambda\rangle \;=\; \Sigma^{xy}|p_z,\lambda\rangle \;=\; i\left(\varepsilon^{x*}(\lambda)\varepsilon^y(\lambda) - \varepsilon^{y*}(\lambda)\varepsilon^x(\lambda)\right)$$

setting $\varepsilon^x(\lambda) = 1/\sqrt{2}$, and $\varepsilon^y(\lambda) = -i\lambda/\sqrt{2}$ we see that this measures the helicity λ, as required.