14 The Dirac Field

The Lagrangian density for a free spin-$\frac{1}{2}$ particle is

$$\mathcal{L} \;=\; \overline{\Psi}^{\alpha}\,(i\partial\!\!\!/ - m)_{\alpha}^{\beta}\,\Psi_{\beta}$$

The Euler-Lagrange equation is the Dirac equation.

The canonical momentum π is given by

$$\pi \;=\; \frac{\partial\mathcal{L}}{\partial\dot{\Psi}} \;=\; i\overline{\Psi}\gamma^0 \;=\; i\Psi^{\dagger}$$

This gives a Hamiltonian density

$$\mathcal{H} \;=\; \overline{\Psi}i\gamma^j\partial_j\Psi + m\overline{\Psi}\Psi.$$

Because the Dirac field represents a particle which obeys Fermi statistics, the canonical equal-time commutation relations are replaced by equal-time anti-commutation relations

$$\{\pi_{\alpha}(x),\Psi_{\beta}(y))\}_{x_0=y_0} \;=\; i\delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta},$$

which is equivalent to

$$\left\{\Psi^{\dagger}_{\alpha}(x),\Psi_{\beta}(y)\right)\}_{x_0=y_0} \;=\; \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}.$$

The free-field, Ψ may be expanded in terms of an annihilation operator $b(p,\lambda)$ for a particle with momentum p and helicity λ and a creation operator $d^{\dagger}(p,\lambda)$ for an anti-particle with momentum p and helicity λ as

$$\Psi_{\alpha}(x) \;=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\sum_{\lambda=\pm1}\left(b(p,\lambda)e^{-ip\cdot x}u_{\alpha}(p,\lambda) + d^{\dagger}(p,\lambda)e^{+ip\cdot x}v_{\alpha}(p,\lambda)\right)$$

and the conjugate field $\overline{\Psi}$ is expanded as

$$\overline{\Psi}^{\alpha}(x) \;=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\sum_{\lambda=\pm1}\left(d(p,\lambda)e^{-ip\cdot x}\overline{v}^{\alpha}(p,\lambda) + b^{\dagger}(p,\lambda)e^{+ip\cdot x}\overline{u}^{\alpha}(p,\lambda)\right)$$

The creation and annihilation operators for the particle and antiparticle obey anti-commutation relations

$$\left\{b(p,\lambda),b^{\dagger}(p',\lambda')\right\} \;=\; 2E_p(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'}$$

and

$$\left\{d(p,\lambda),d^{\dagger}(p',\lambda')\right\} \;=\; 2E_p(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'},$$

all other anti-commutators being zero.

Thus, for example, we have

$$b^\dagger((p,\lambda)b^\dagger((p,\lambda)|0\rangle \;=0$$

meaning that we cannot create a state with two fermions with the same momentum and helicity, in keeping with the Pauli exclusion principle.

In order to verify that the anti-commutation relations for the creation and annihilation operators lead to the canonical equal-time anti-commutation relations between Ψ and $\Psi^\dagger$ we need to use the projection operators

$$\sum_{\lambda=\pm 1} u_\alpha(\mathbf{p},\lambda)\overline{u}^\beta(\mathbf{p},\lambda) \;=\; (\not{p}+m)^\beta_\alpha$$

$$\sum_{\lambda=\pm 1} v_\alpha(\mathbf{p},\lambda)\overline{v}^\beta(\mathbf{p},\lambda) \;=\; (\not{p}-m)^\beta_\alpha$$

In terms of creation and annihilation operators the (normal-ordered) Hamiltonian may be written

$$H \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} E_p \sum_{\lambda=\pm 1} \left(b^\dagger(p,\lambda)b(p,\lambda) + d^\dagger(p,\lambda)d(p,\lambda)\right),$$

which counts the total number of particles and antiparticles with momentum $\mathbf{p}$ and multiplied by their energy E_p.

The expansions for Ψ and $\overline{\Psi}$ can be inverted to give expressions for the creation and annihilation operators for the particle and antiparticle

$$b(p,\lambda) \;=\; \int d^3\mathbf{x} e^{ip\cdot x}\overline{u}(\mathbf{p},\lambda)\gamma^0\Psi(x)$$

$$d(p,\lambda) \;=\; \int d^3\mathbf{x} e^{ip\cdot x}\overline{\Psi}(x)\gamma^0 v(\mathbf{p},\lambda)$$

and similarly

$$b^\dagger(p,\lambda) \;=\; \int d^3\mathbf{x} e^{-ip\cdot x}\overline{\Psi}(x)\gamma^0 u(\mathbf{p},\lambda)$$

$$d^\dagger(p,\lambda) \;=\; \int d^3\mathbf{x} e^{-ip\cdot x}\overline{v}((\mathbf{p},\lambda))\gamma^0\Psi(x)$$

Following steps similar to those used to derive the LSZ reduction formula for scalar fields we can derive a similar formula for the S-matrix element between states of fermion fields. In the case of an incoming state consisting of a particle with momentum $\mathbf{p_1}$ and helicity λ_1 and an antiparticle with momentum $\mathbf{p_2}$ and helicity λ_2 and an outgoing state consisting of a particle with momentum $\mathbf{q_1}$ and helicity λ_1' and an antiparticle with momentum $\mathbf{q_2}$ and helicity λ_2', we find

$$\langle \mathbf{q_1},\lambda_1',\mathbf{q_2},\lambda_2',out|\mathbf{p_1},\lambda_1,\mathbf{p_2},\lambda_2,in\rangle \;=\; \left(\frac{i}{\sqrt{Z}}\right)^2\left(\frac{-i}{\sqrt{Z}}\right)^2\int d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{i(q_1\cdot x_1+q_2\cdot x_2-p_1\cdot y_1-p_2\cdot y_2)}$$

$$\overline{u}^{\alpha}(\mathbf{q_1},\lambda_1')\overline{v}^{\delta}(\mathbf{p_2},\lambda_2)\,(i\partial\!\!\!/_{x_1}-m)^{\alpha'}_{\alpha}\,(i\partial\!\!\!/_{y_2}-m)^{\delta'}_{\delta}\,\langle 0|T\Psi_{\alpha'}(x_1)\Psi_{\delta'}(y_2)\overline{\Psi}_{\beta'}(x_2)\overline{\Psi}_{\gamma'}(y_1)|0\rangle$$
$$(-i\overleftarrow{\partial\!\!\!/}_{x_2}-m)^{\beta'}_{\beta}(-i\overleftarrow{\partial\!\!\!/}_{y_1}-m)^{\gamma'}_{\gamma}v_{\beta}(\mathbf{q_2},\lambda_2')u_{\gamma}(\mathbf{p_1},\lambda_1).$$

This generalises to any number of fermions. Note that this LSZ reduction formula tells us that we treat an initial state antiparticle the same way as an outgoing particle and vice versa.

Because of the anti-commuting properties of the fields, the time ordering operator has the interpretation

$$\begin{aligned} T\left(\Psi_{\alpha}(x)\Psi_{\beta}(y)\right) &= \Psi_{\alpha}(x)\Psi_{\beta}(y),\ \ (x_0>y_0) \\ &= -\Psi_{\beta}(y)\Psi_{\alpha}(x),\ \ (y_0>x_0) \end{aligned}$$

The propagator for a Dirac field is defined as

$$i\Delta_F(x,y)^{\beta}_{\alpha} \equiv \langle 0|T\Psi_{\alpha}(x)\overline{\Psi}^{\beta}(y)|0\rangle,$$

Note the Ψ propagates into $\overline{\Psi}$ †.

Inserting the expansion of the Dirac field into this and using the anti-commutation relations between the creation and annihilation operators, this becomes

$$\begin{aligned} i\Delta_F(x,y)^{\beta}_{\alpha} &= \int\frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\sum_{\lambda=\pm1}\Big(u_{\alpha}(\mathbf{p},\lambda)\overline{u}^{\beta}(\mathbf{p},\lambda)e^{-ip\cdot(x-y)}\theta(x_0-y_0) \\ &\qquad -v_{\alpha}(\mathbf{p},\lambda)\overline{v}^{\beta}(\mathbf{p},\lambda)e^{+ip\cdot(x-y)}\theta(y_0-x_0)\Big) \end{aligned}$$

Using

$$\begin{aligned} \sum_{\lambda=\pm1}u_{\alpha}(\mathbf{p},\lambda)\overline{u}^{\beta}(\mathbf{p},\lambda) &= (p\!\!\!/+m)^{\beta}_{\alpha} \\ \sum_{\lambda=\pm1}v_{\alpha}(\mathbf{p},\lambda)\overline{v}^{\beta}(\mathbf{p},\lambda) &= (p\!\!\!/-m)^{\beta}_{\alpha}, \end{aligned}$$

we may write this as

$$\begin{aligned} i\Delta_F(x,y)^{\beta}_{\alpha} &= (i\partial\!\!\!/x+m)^{\beta}_{\alpha}\int\frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left(e^{-ip\cdot(x-y)}\theta(x_0-y_0)+e^{+ip\cdot(x-y)}\theta(y_0-x_0)\right) \\ &= (i\partial\!\!\!/x+m)^{\beta}_{\alpha}\lim_{\varepsilon\to0} i\int\frac{d^4\mathbf{p}}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{1}{(p^2-m^2+i\varepsilon)} \\ &= i\int\frac{d^4\mathbf{p}}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{(p\!\!\!/+m)}{(p^2-m^2+i\varepsilon)} \end{aligned}$$

The Wick theorem applied to fermions also carries a minus sign for every odd permutation of the fermion fields.

† The vacuum expectation value of $T\Psi((x)\Psi(y)$ or $T\overline{\Psi}((x)\overline{\Psi}(y)$ vanishes.

$$\langle 0|T\Psi_{\alpha_1}(x_1)\cdots\overline{\Psi}_{\alpha_n}(x_n)|0\rangle = \sum_{pairings}(-1)^{N_p}\prod_{pairs}\langle 0|T\Psi_{\alpha_i}(x_i)\overline{\Psi}_{\alpha_j}(x_j)|0\rangle,$$

where N_p is the number of permutations of fermion fields for that pairing. For example

$$\begin{aligned}\langle 0|T\Psi_\alpha(x_1)\overline{\Psi}^\beta(x_2)\Psi_{\gamma_1}(x_3)\overline{\Psi}^\delta(x_4)|0\rangle &= \langle 0|T\Psi_\alpha(x_1)\overline{\Psi}^\beta(x_2)0\rangle\langle 0|T\Psi_\gamma(x_3)\overline{\Psi}^\delta(x_4)0\rangle \\ &- \langle 0|T\Psi_\alpha(x_1)\overline{\Psi}^\delta(x_4)0\rangle\langle 0|T\Psi_\gamma(x_3)\overline{\Psi}^\beta(x_2)0\rangle\end{aligned}$$

In terms of Feynman graphs, this sign means that there is a relative minus sign between the contributions to the S-matrix elements from two graphs which are related by the interchange of two identical particles (or two identical anti-particles).

14.1 Angular Momentum

The energy-momentum tensor for a Dirac field is

$$T^{\mu\nu} = \partial^\nu\overline{\Psi}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\Psi})}+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\partial^\nu\Psi - g^{\mu\nu}\mathcal{L}$$

Note the ordering of the fields.

It is useful to exploit an integration by parts in the action ($S\equiv\int d^4x\mathcal{L}$) to rewrite the Lagrangian density as

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu\partial_\mu\Psi - \frac{i}{2}\left(\partial_\mu\overline{\Psi}\right)\gamma^\mu\Psi - m\overline{\Psi}\Psi$$

This leads to

$$T^{\mu\nu} = \frac{i}{2}\overline{\Psi}\gamma^\mu\overset{\leftrightarrow}{\partial}{}^\nu\Psi$$

The angular momentum operator is

$$M^{\mu\nu} = \int d^3\mathbf{x}\left(\left(x^\mu T^{\nu 0}-x^\nu T^{\mu 0}\right)+\frac{1}{2}\Psi^{\dagger\alpha}\left(\sigma^{\mu\nu}\right)_\alpha^\beta\Psi_\beta\right)$$

The first part is the orbital angular momentum and is the same as in the scalar field case, whereas the last term generates rotations (generalised Lorentz transformations) on the spinor.

14.2 Quantum Electrodynamics (QED)

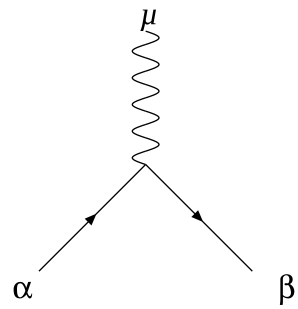
The Lagrangian density for a Dirac field and a photon field interacting with each other (the QED Lagrangian density) is

$$\mathcal{L}_{QED} = \overline{\Psi}\left(i\gamma^\mu\left(\partial_\mu+ieA_\mu\right)-m\right)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu},$$

so the interaction Lagrangian density is

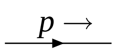
$$\mathcal{L}_I \;=\; -e\overline{\Psi}\gamma^\mu A_\mu \Psi$$

This leads to a Feynman rule for the interaction of the charged spin-$\frac{1}{2}$ particle with the photon

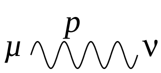


$-ie\,(\gamma^\mu)^\beta_\alpha$

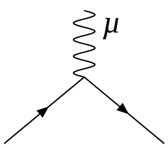
14.3 Feynman Rules for spin-$\frac{1}{2}$ QED



$i\,(\not{p}+m)\,/\,\big((p^2-m^2+i\varepsilon\big)$ for each internal spin-$\frac{1}{2}$ line.



$-ig^{\mu\nu}\,/\,\big(p^2+i\varepsilon\big)$ for each internal photon (Feynman gauge).



$-ie\gamma^\mu$ for each vertex.

The momentum p for the fermion propagator runs along the charge line - so it is in the direction of motion for a particle but opposite to the direction of motion for an anti-particle.

Spinors:

- $u(\mathbf{p},\lambda)$ (on the right) for an incoming particle with momentum $\mathbf{p}$ and helicity λ.
- $\overline{u}(\mathbf{p},\lambda)$ (on the left) for an outgoing particle with momentum $\mathbf{p}$ and helicity λ.
- $\overline{v}(\mathbf{p},\lambda)$ (on the left) for an incoming antiparticle with momentum $\mathbf{p}$ and helicity λ.
- $v(\mathbf{p},\lambda)$ (on the right) for an outgoing antiparticle with momentum $\mathbf{p}$ and helicity λ.