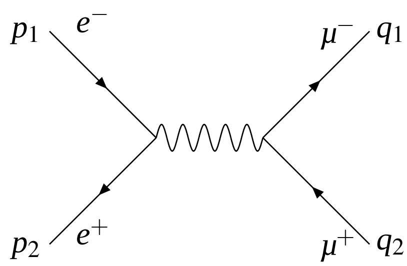
15 Applications of Quantum Electrodynamics (QED)

Application 1:
Electron + positron annihilation into muon + anti-muon.

$$e^+ + e^- \rightarrow \mu^+ \mu^-$$

To order e^2 in the scattering amplitude there is only one Feynman graph



Note the convention that the arrows on the fermion lines are always drawn in the same direction and follow the (negative) electron charge, so that the momenta assigned to the positron and μ^+ are understood to be in the opposite direction from the direction of the arrows.

Following the Feynman rules the matrix-element for this process (dropping the energy-momentum conserving delta-function) is:

$$\mathcal{M} = \overline{u}^{\alpha}_{(\mu)}(\mathbf{q_1},\lambda_1)\,(ie\gamma^{\mu})^{\beta}_{\alpha}\,v_{(\mu)}(\mathbf{q_2},\lambda_2)_{\beta}\frac{(-ig_{\mu\nu})}{(p_1+p_2)^2+i\varepsilon}\overline{v}^{\gamma}_{(e)}(\mathbf{p_2},\lambda_2^{\prime})\,(ie\gamma^{\nu})^{\delta}_{\gamma}\,u_{(e)}(\mathbf{p_1},\lambda_1^{\prime})_{\delta},$$

where λ_1, λ_2 are the helicities of the outgoing muons and $\lambda_1^{\prime}, \lambda_2^{\prime}$ are the helicities of the incoming electrons.

Contracting indices this may be written using matrix notation as

$$\mathcal{M} = ie^2\overline{u}_{(\mu)}(\mathbf{q_1},\lambda_1)\gamma^{\mu}v_{(\mu)}(\mathbf{q_2},\lambda_2)\frac{1}{(p_1+p_2)^2}\overline{v}_{(e)}(\mathbf{p_2}\lambda_2^{\prime})\gamma_{\mu}u_{(e)}(\mathbf{p_1}\lambda_1^{\prime}).$$

Here the order of the terms matters - the rule is to start at the end of each fermion line and work backwards writing down each vertex or internal fermion propagator as one moves back toward the beginning of the fermion line.

We need to multiply this amplitude by its complex conjugate in order to find the square-matrix-element. Using the fact that γ^0 is Hermitian whereas γ^i are anti-Hermitian and the definition $\overline{u} \equiv u^{\dagger}\gamma^0$ (and similarly for v), we can show (using the anti-commutation relations of the γ-matrices) that

$$\left(\overline{u}\Gamma u^{\prime}\right)^* = \overline{u}^{\prime}\overline{\Gamma}u,$$

where u and u' stand for any two spinors (u or v), Γ stands for any string of γ-matrices and $\overline{\Gamma}$ is the string of γ-matrices in reverse order (in this case we only have one γ-matrix between the spinors.).

Therefore for the square-matrix-element, we have (setting $(p_1+p_2)^2 = s$)

$$\begin{aligned}|\mathcal{M}|^2 \equiv \mathcal{M}\mathcal{M}^\dagger &= \frac{e^4}{s^2}\overline{v}_{(\mu)}(\mathbf{q_2},\lambda_2)\gamma^\nu u_{(\mu)}(\mathbf{q_1},\lambda_1)\overline{u}_{(\mu)}(\mathbf{q_1},\lambda_1)\gamma^\mu v_{(\mu)}(\mathbf{q_2},\lambda_2) \\ &\quad \overline{u}_{(e)}(\mathbf{p_1},\lambda_1')\gamma_\nu v_{(e)}(\mathbf{p_2},\lambda_2')\overline{v}_{(e)}(\mathbf{p_2},\lambda_2')\gamma_\mu v_{(e)}(\mathbf{p_1},\lambda_1')\end{aligned}$$

In most experiments the direction of polarisation of the final-state fermions is not measured but since these are in principle measurable we sum the square-matrix-element over final-state helicities. Furthermore, we usually have unpolarised incoming fermion beams so we average over incoming helicities - i.e. we sum over the helicities of all four fermions and divide by 4.

Using the relations

$$\sum_{\lambda=\pm 1} u_\alpha(\mathbf{p},\lambda)\overline{u}^\beta(\mathbf{p},\lambda) = (\not{p}+m)^\beta_\alpha$$

and

$$\sum_{\lambda=\pm 1} v_\alpha(\mathbf{p},\lambda)\overline{v}^\beta(\mathbf{p},\lambda) = (\not{p}-m)^\beta_\alpha$$

we see that

$$\sum_{\lambda,\lambda'} \overline{u}(\mathbf{p},\lambda)\Gamma v(\mathbf{p}',\lambda')\overline{v}(\mathbf{p}',\lambda')\overline{\Gamma}u(\mathbf{p},\lambda) = \mathrm{Tr}\left\{(\not{p}+m)\Gamma\left(\not{p}'-m\right)\overline{\Gamma}\right\}$$

Likewise

$$\sum_{\lambda,\lambda'} \overline{v}(\mathbf{p},\lambda)\Gamma u(\mathbf{p}',\lambda')\overline{u}(\mathbf{p}',\lambda')\overline{\Gamma}v(\mathbf{p},\lambda) = \mathrm{Tr}\left\{(\not{p}-m)\Gamma\left(\not{p}'+m\right)\overline{\Gamma}\right\}$$

and

$$\sum_{\lambda,\lambda'} \overline{u}(\mathbf{p},\lambda)\Gamma u(\mathbf{p}',\lambda')\overline{u}(\mathbf{p}',\lambda')\overline{\Gamma}u(\mathbf{p},\lambda) = \mathrm{Tr}\left\{(\not{p}+m)\Gamma\left(\not{p}'+m\right)\overline{\Gamma}\right\}$$

$$\sum_{\lambda,\lambda'} \overline{v}(\mathbf{p},\lambda)\Gamma v(\mathbf{p}',\lambda')\overline{v}(\mathbf{p}',\lambda')\overline{\Gamma}v(\mathbf{p},\lambda) = \mathrm{Tr}\left\{(\not{p}-m)\Gamma\left(\not{p}'-m\right)\overline{\Gamma}\right\}$$

So that the sum over fermion helicities is reduced to the calculation of a trace of a string of γ-matrices

In the case we are considering we have

$$\frac{1}{4}\sum_{helicities}|\mathcal{M}|^2 = \frac{e^4}{s^2}\mathrm{Tr}\left\{\left(\not{q}_2-m_\mu\right)\gamma^\nu\left(\not{q}_1+m_\mu\right)\right)\gamma^\mu\right\}\mathrm{Tr}\left\{(\not{p}_1+m_e)\gamma_\nu\left(\not{p}_2-m_e\right)\gamma_\mu\right\}$$

15.1 Trace Theorems

Traces of long strings of γ-matrices can be performed using algebraic manipulation packages. Traces of strings of up to four γ-matrices can be done by hand, using the following properties of strings of γ-matrices, which we state without proof (they can be proved using the anti-commutation relations of γ-matrices and the cyclic property of traces).

1. $$\not{p}\not{p} = p\cdot p I$$
   where I is the 4×4 unit matrix.

2. $$\gamma^\mu\gamma_\mu \;=\; 4I,$$

3. $$\gamma^\mu\gamma^\nu\gamma_\mu \;=\; -2\gamma^\nu$$
   or
   $$\gamma^\mu\not{p}\gamma_\mu \;=\; -2\not{p}$$

4. $$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu \;=\; 4g^{\nu\rho}I$$
   or
   $$\gamma^\mu\not{p}\not{q}\gamma_\mu \;=\; 4p\cdot qI$$

5. $$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu \;=\; -2\gamma^\sigma\gamma^\nu\gamma^\rho$$
   or
   $$\gamma^\mu\not{p}\not{q}\not{r}\gamma_\mu \;=\; -2\not{r}\not{q}\not{p}$$

6. The trace of a string of an odd number of γ-matrices is zero. In particular
   $$\mathrm{Tr}\not{p} \;=\; 0$$
   $$\mathrm{Tr}\{\not{p}\not{q}\not{r}\} \;=\; 0$$

7. $$\mathrm{Tr}\{\gamma^\mu\gamma^\nu\} \;=\; 4g^{\mu\nu}$$
   or
   $$\mathrm{Tr}\{\not{p}\not{q}\} \;=\; 4p\cdot q$$

8. $$\mathrm{Tr}\{\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\} \;=\; \sum_{j=2}^{2n}(-1)^j g^{\mu_1\mu_j}\mathrm{Tr}\{\gamma^{\mu_2}\cdots\gamma^{\mu_{j-1}}\gamma^{\mu_{j+1}}\cdots\gamma^{\mu_{2n}}\}$$
   In particular
   $$\mathrm{Tr}\{\not{p}\not{q}\not{r}\not{s}\} \;=\; 4\left(p\cdot q\, r\cdot s - p\cdot r\, q\cdot s + p\cdot s\, q\cdot r\right)$$

9.

$$\mathrm{Tr}(\gamma^5) = \mathrm{Tr}\left(\gamma^5\gamma^\mu\right) = \mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right) = \mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\right) = 0$$

10.

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4i\varepsilon^{\mu\nu\rho\sigma}$$

or

$$\mathrm{Tr}\left(\gamma^5 \not{p}\not{q}\not{r}\not{s}\right) = 4i\varepsilon^{\mu\nu\rho\sigma} p_\mu q_\nu r_\rho s_\sigma$$

Returning to the reaction $e^+ e^- \to \mu^+ \mu^-$, we have the product of two traces

1.

$$\mathrm{Tr}\left\{\left(\not{q}_2 - m_\mu\right)\gamma^\nu\left(\not{q}_1 + m_\mu\right)\gamma^\mu\right\} = \mathrm{Tr}\{\not{q}_2\gamma^\nu\not{q}_1\gamma^\nu\} - m_\mu \mathrm{Tr}\{\gamma^\nu\not{q}_1\gamma^\mu\} + m_\mu \mathrm{Tr}\{\not{q}_2\gamma^\nu\gamma^\mu\} - m_\mu^2 \mathrm{Tr}\{\gamma^\mu\gamma^\nu\}$$

The middle two terms vanish as they are traces of an odd number of γ-matrices. The remaining terms give (using the above formulae)

$$4\left(q_2^\nu q_1^\mu + q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu} - m_\mu^2 g^{\mu\nu}\right)$$

2.

$$\begin{aligned}
\mathrm{Tr}\left\{\left(\not{p}_1 + m_e\right)\gamma_\nu\left(\not{p}_2 - m_e\right)\gamma_\mu\right\} &= \mathrm{Tr}\{\not{p}_1\gamma_\nu\not{p}_2\gamma^\nu\} + m_e \mathrm{Tr}\left\{\gamma_\nu\not{p}_2\gamma_\mu\right\} - m_e \mathrm{Tr}\left\{\not{p}_1\gamma_\nu\gamma_\mu\right\} - m_e^2 \mathrm{Tr}\left\{\gamma_\mu\gamma_\nu\right\} \\
&= 4\left(p_{2\nu}p_{1\mu} + p_{2\mu}p_{1\nu} - p_1\cdot p_2 g_{\mu\nu} - m_e g_{\mu\nu}\right)
\end{aligned}$$

Contracting these together and multiplying by the factor $e^4/(4s)$ we find

$$\frac{1}{4}\sum_{helicities}|\mathcal{M}|^2 = \frac{8e^4}{s^2}\left(p_1\cdot q_1\, p_2\cdot q_2 + p_1\cdot q_2\, p_2\cdot q_1 + m_e^2 q_1\cdot q_2 + m_\mu^2 p_1\cdot p_2 - 2m_e^2 m_\mu^2\right)$$

We can express this in terms of Mandelstam variables s and t using

$$s = 2\left(m_e^2 + p_1\cdot p_2\right) = 2\left(m_\mu^2 + q_1\cdot q_2\right)$$

$$t = m_e^2 + m_\mu^2 - 2p_1\cdot q_1 = m_e^2 + m_\mu^2 - 2p_2\cdot q_2$$

$$u = m_e^2 + m_\mu^2 - 2p_1\cdot q_2 = m_e^2 + m_\mu^2 - 2p_2\cdot q_1 = 2m_e^2 + 2m_\mu^2 - s - t,$$

to get

$$\frac{1}{4}\sum_{helicities}|\mathcal{M}|^2 = \frac{2e^4}{s^2}\left(2t^2 + 2st + s^2 - 4(m_e^2 + m_\mu^2)t + 2(m_e^2 + m_\mu^2)^2\right)$$

The total cross-section is therefore

$$\begin{aligned}
\sigma = \; & \frac{1}{F}\frac{2e^4}{s^2}\int \frac{d^3\mathbf{q_1}}{(2\pi)^3 2E_{q_1}}\frac{d^4 q_2}{(2\pi)^3}\delta(q_2^2 - m_\mu^2)\theta(q_2^0)(2\pi)^4\delta^4\left(p_1 + p_2 - q_1 - q_2\right) \\
& \times\left(2t^2 + 2st + s^2 - 4(m_e^2 + m_\mu^2)t + 2(m_e^2 + m_\mu^2)^2\right)
\end{aligned}$$

The flux factor $F = 2\sqrt{s(s-4m_e^2)}$ and we can write

$$\frac{d^3\mathbf{q_1}}{2E_{q_1}} = \frac{1}{4|\mathbf{p_1}|} dE_{q_1} dt d\phi$$

and after performing the integral over q_1 absorbing the energy-momentum conserving delta-function we are left with

$$\sigma = \frac{e^4}{8\pi^2 s^2 \sqrt{s(s-4m_e^2)}|\mathbf{p_1}|} \int d\phi dt dE_{q_1} \delta\left(s - 2\sqrt{s}E_{q_1}\right) \left(2t^2 + 2st + s^2 - 4(m_e^2 + m_\mu^2)t + 2(m_e^2 + m_\mu^2)^2\right)$$

The magnitude of the incoming three-momentum $|\mathbf{p_1}|$ is $\frac{1}{2}\sqrt{s-4m_e^2}$. Integrating over ϕ and integrating over E_{q_1} to absorb the remaining delta-function we are left with the differential cross-section:

$$\frac{d\sigma}{dt} = \frac{2\pi\alpha^2}{s^3(s-4m_e^2)} \left(2t^2 + 2st + s^2 - 4(m_e^2 + m_\mu^2)t + 2(m_e^2 + m_\mu^2)^2\right),$$

$(\alpha = e^2/(4\pi))$.

Of particular interest is the limit of this cross-section where s and $|t|$ are both much larger than the masses, which may then be neglected. In that case the kinematic limit on t (obtained form the values of t for which the scattering angle in the centre-of-mass frame is 0 or π), is

$$-s < t < 0,$$

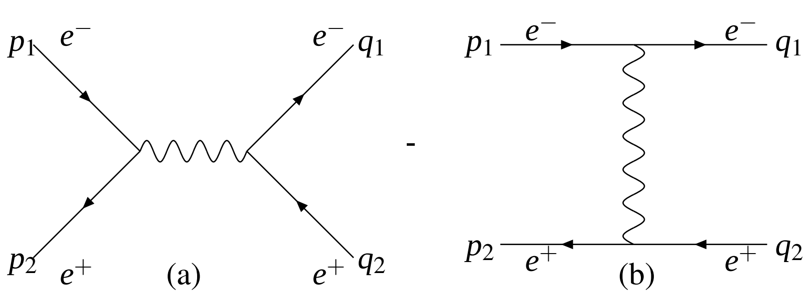
so that the total cross-section in this limit is

$$\sigma_{TOT}(e^+e^- \to \mu^+\mu^-) = \frac{2\pi\alpha^2}{s^4} \int_{-s}^{0} dt \left(2t^2 + s^2 + 2st\right) = \frac{4\pi\alpha^2}{3s}$$

Application 2:
Electron-positron (Bhabha) Scattering

$$e^+ e^- \to e^+ e^-$$

In this case there are two Feynman graphs



Again the momenta for the positrons is in the opposite direction to the arrows on the fermion lines. There is a relative minus sign between the two graphs because in the case of the second graphs there is an odd number of permutations of fermion fields in the second term of the Wick contraction.

We suppress the Dirac indices on the spinors and Dirac matrices, but ensure that we write these down in the correct order, starting form the end of each fermion line (final state) and working backwards to the beginning (initial state)

The contribution from the first graph is

$$\mathcal{M}_{(a)} = \bar{v}(\mathbf{p_2},\lambda_2)(ie\gamma^\mu)u(\mathbf{p_1},\lambda_1)\frac{-ig_{\mu\nu}}{(p_1+p_2)^2}\bar{u}(\mathbf{q_1},\lambda_1')(ie\gamma^\nu)v(\mathbf{q_2},\lambda_2'),$$

where $\lambda_{1(2)}$ are the helicities of the incoming electron and positron and $\lambda'_{1(2)}$ are the helicities of the outgoing electron and positron. †

The contribution from the second graph is

$$\mathcal{M}_{(b)} = -\bar{u}(\mathbf{q_1},\lambda_1')(ie\gamma^\nu)u(\mathbf{p_1},\lambda_1)\frac{-ig_{\mu\nu}}{(p_1-q_1)^2}\bar{v}(\mathbf{p_2},\lambda_2)(ie\gamma^\nu)v(\mathbf{q_2},\lambda_2').$$

When we construct the square matrix-element we get three terms (replacing $(p_1+p_2)^2$ by s and $(p_1-q_1)^2$ by t)

$$|\mathcal{M}_{(a)}|^2 = \frac{e^4}{s^2}\bar{v}(\mathbf{p_2},\lambda_2)\gamma^\mu u(\mathbf{p_1},\lambda_1)\bar{u}(\mathbf{p_1},\lambda_1)\gamma^\nu v(\mathbf{p_2},\lambda_2)\,\bar{u}(\mathbf{q_1},\lambda_1')\gamma_\nu v(\mathbf{q_2},\lambda_2')\bar{v}(\mathbf{q_2},\lambda_2')\gamma_\mu u(\mathbf{q_1},\lambda_1')$$

$$|\mathcal{M}_{(b)}|^2 = \frac{e^4}{t^2}\bar{v}(\mathbf{p_2},\lambda_2)\gamma^\nu v(\mathbf{q_2},\lambda_2')\bar{v}(\mathbf{q_2},\lambda_2')\gamma^\mu v(\mathbf{p_2},\lambda_2)\,\bar{u}(\mathbf{q_1},\lambda_1')\gamma_\nu u(\mathbf{p_1},\lambda_1)\bar{u}(\mathbf{p_1},\lambda_1)\gamma_\mu u(\mathbf{q_1},\lambda_1')$$

and the interference term between the two graphs

$$2\Re e\{\mathcal{M}_{(a)}^\dagger\mathcal{M}_{(b)}\} = -2\frac{e^4}{st}\bar{v}(\mathbf{p_2},\lambda_2)\gamma^\nu u(\mathbf{p_1},\lambda_1)\bar{u}(\mathbf{p_1},\lambda_1)\gamma^\mu u(\mathbf{q_1},\lambda_1')\bar{u}(\mathbf{q_1},\lambda_1')\gamma_\nu v(\mathbf{q_2},\lambda_2')\bar{v}(\mathbf{q_2},\lambda_2')\gamma_\mu v(\mathbf{p_2},\lambda_2)$$

Once again we do not measure the polarisations of the electrons and positrons so we sum over all of these and divide by 4 for the average over the incoming polarisations. making use of the polarisation matrices for the sums over helicities this gives

$$\frac{1}{4}\sum_{helicities}|\mathcal{M}_{(a)}|^2 = \frac{e^4}{4s^2}\mathrm{Tr}\{(\not{p}_2-m)\gamma^\mu(\not{p}_1+m)\gamma^\nu\}\,\mathrm{Tr}\{(\not{q}_1+m)\gamma_\nu(\not{q}_2-m)\gamma_\mu\}$$

$$\frac{1}{4}\sum_{helicities}|\mathcal{M}_{(b)}|^2 = \frac{e^4}{4t^2}\mathrm{Tr}\{(\not{p}_2-m)\gamma^\nu(\not{q}_2-m)\gamma^\mu\}\,\mathrm{Tr}\{(\not{q}_1+m)\gamma_\nu(\not{p}_1+m)\gamma_\mu\}$$

† Here we drop the $i\varepsilon$ in the propagator as this makes no difference when we take the $\varepsilon\to 0$ limit.

$$\frac{1}{2}\sum_{helicities}\Re e\{\mathcal{M}_{(a)}^{\dagger}\mathcal{M}_{(b)}\} \;=\; -\frac{e^4}{2st}\mathrm{Tr}\left\{(\not{p}_2-m)\gamma^{\nu}(\not{p}_1+m)\gamma^{\mu}(\not{q}_1+m)\gamma_{\nu}(\not{q}_2-m)\gamma_{\mu}\right\}$$

Henceforth we restrict ourselves to high energies where all the momenta are large compared with the electron mass, m and these simplify to

$$\frac{1}{4}\sum_{helicities}|\mathcal{M}_{(a)}|^2 \;=\; \frac{e^4}{4s^2}\mathrm{Tr}\{\not{p}_2\gamma^{\mu}\not{p}_1\gamma^{\nu}\}\,\mathrm{Tr}\{\not{q}_1\gamma_{\nu}\not{q}_2\gamma_{\mu}\}$$

$$\frac{1}{4}\sum_{helicities}|\mathcal{M}_{(b)}|^2 \;=\; \frac{e^4}{4t^2}\mathrm{Tr}\{\not{p}_2\gamma^{\nu}\not{q}_2)\gamma^{\mu}\}\,\mathrm{Tr}\{\not{q}_1\gamma_{\nu}\not{p}_1\gamma_{\mu}\}$$

$$\frac{1}{2}\sum_{helicities}\Re e\{\mathcal{M}_{(a)}^{\dagger}\mathcal{M}_{(b)}\} \;=\; -\frac{e^4}{2st}\mathrm{Tr}\{\not{p}_2\gamma^{\nu}\not{p}_1\gamma^{\mu}\not{q}_1\gamma_{\nu}\not{q}_2\gamma)\mu\}$$

Using the formula for the trace of a product of four γ-matrices we have

$$\frac{1}{4}\sum_{helicities}|\mathcal{M}_{(a)}|^2 \;=\; \frac{4e^4}{s^2}\left(p_2^{\mu}p_1^{\nu}+p_2^{\mu}p_1^{\nu}-p_1\cdot p_2 g^{\mu\nu}\right)\left(q_{2\mu}q_{1\nu}+q_{2\mu}q_{1\nu}-q_1\cdot q_2 g_{\mu\nu}\right)$$

$$\frac{1}{4}\sum_{helicities}|\mathcal{M}_{(b)}|^2 \;=\; \frac{4e^4}{t^2}\left(p_2^{\mu}q_2^{\nu}+p_2^{\mu}q_2^{\nu}-p_2\cdot q_2 g^{\mu\nu}\right)\left(p_{1\mu}q_{1\nu}+p_{1\mu}q_{1\nu}-p_1\cdot q_1 g_{\mu\nu}\right)$$

Contracting the Lorentz indices gives

$$\frac{1}{4}\sum_{helicities}|\mathcal{M}_{(a)}|^2 \;=\; \frac{8e^4}{s^2}\left(p_1\cdot q_1\, p_2\cdot q_2 \,+\, p_1\cdot q_2\, p_2\cdot q_1\right)$$

and

$$\frac{1}{4}\sum_{helicities}|\mathcal{M}_{(b)}|^2 \;=\; \frac{8e^4}{t^2}\left(p_1\cdot p_2\, q_1\cdot q_2 \,+\, p_1\cdot q_2\, p_2\cdot q_1\right)$$

In the high-energy limit the Mandelstam variables simplify to

$$s \;=\; 2p_1\cdot p_2 \;=\; 2q_1\cdot q_2$$

$$t \;=\; -2p_1\cdot q_1 \;=\; -2q_2\cdot p_2$$

$$u \;=\; -2p_1\cdot q_2 \;=\; -2q_1\cdot p_2$$

giving us

$$\frac{1}{4}\sum_{helicities}|\mathcal{M}_{(a)}|^2 \;=\; \frac{2e^4}{s^2}\left(t^2+u^2\right)$$

$$\frac{1}{4}\sum_{helicities}|\mathcal{M}_{(b)}|^2 \;=\; \frac{2e^4}{t^2}\left(s^2+u^2\right)$$

The interference term is a trace of the product of eight γ-matrices. However, this can be reduced using the γ-matrix relations

$$\gamma^{\nu}\not{p}_1\gamma^{\mu}\not{q}_1\gamma_{\nu} \;=\; -2\not{q}_1\gamma^{\mu}\not{p}_1$$

followed by

$$\gamma^\mu \not{p}_1^\mu \not{q}_2 \gamma_\mu \;=\; 4 p_1 \cdot q_2,$$

the interference term simplifies to

$$\frac{1}{2} \sum_{helicities} \Re e\{\mathcal{M}^\dagger_{(a)} \mathcal{M}_{(b)}\} \;=\; +\frac{4e^4}{st} p_1 \cdot q_2 \mathrm{Tr}\{\not{p}_2 \not{q}_1\} \;=\; +\frac{16e^4}{st} p_1 \cdot q_2\, p_2 \cdot q_1 \;=\; \frac{4e^2 u^2}{st}$$

The total square matrix-element is then

$$\frac{1}{4} \sum_{helicities} |\mathcal{M}|^2 \;=\; 2e^4 \left(\frac{t^2+u^2}{s^2} + \frac{s^2+u^2}{t^2} + \frac{2u^2}{st} \right)$$

We insert this into the phase-space integral over the two final-state particles in the usual way and divide by the flux-factor and in the limit where the electron mass can be neglected we end up with

$$\frac{d\sigma}{dt} \;=\; \frac{2\pi\alpha^2}{s^2} \left(\frac{t^2+u^2}{s^2} + \frac{s^2+u^2}{t^2} + \frac{2u^2}{st} \right).$$

In the high-energy limit the relation between t, u and the scattering angle in the centre-of-mass frame is simplified to

$$t \;=\; -s \frac{(1-\cos\theta)}{2}$$

$$u \;=\; -s \frac{(1+\cos\theta)}{2}$$

so that in terms of the centre-of-mass scattering angle we have the “Bhabha” cross-section for electron-positron scattering,

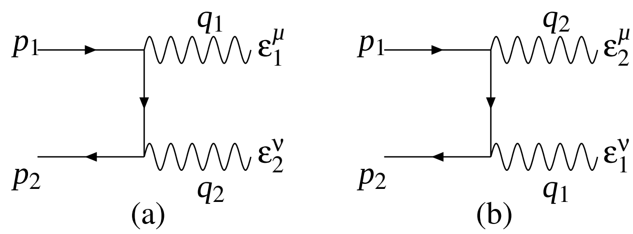
$$\frac{d\sigma}{d\cos\theta} \;=\; \frac{\alpha^2}{s} \left(\frac{(1-\cos^2\theta)}{2} + \frac{(5+2\cos\theta+\cos^2\theta)}{(1-\cos\theta)^2} - \frac{(1+\cos\theta)^2}{(1-\cos\theta)} \right).$$

Note that this differential cross-section diverges in the forward direction as $\theta \to 0$. This is a reflection of the fact that charged particles interact at large distances. The momentum transfer is the Fourier conjugate of the impact parameter b, the perpendicular distance between the incident particles. The divergence of the differential cross-section when this momentum transfer vanishes is equivalent to the statement that the particles interact even when the impact parameter is indefinitely large. In practice this divergence is limited by the physical width of the incident beam.

Application 3:
Electron-positron annihilation into two photons.

$$e^+ e^- \;\to\; \gamma\gamma$$

There are two Feynman graphs:



These are examples of graphs in which there is a fermion propagator between the two vertices between fermions and a photon. Starting at the end of the fermion and working backwards, the contribution from the first graph is

$$\begin{aligned}
\mathcal{M}_{(a)} &= \varepsilon_2^{\mu *}(\lambda_2')\varepsilon_1^{\nu *}(\lambda_1')\overline{v}(\mathbf{p_2},\lambda_2)\left(ie\gamma_\mu\right)\frac{i\left((\not{p}_1-\not{q}_1+m\right)}{\left((p_1-q_1)^2-m^2\right)}\left(ie\gamma_\nu\right)u(\mathbf{p_1},\lambda_1) \\
&= \frac{-ie^2}{(t-m^2)}\overline{v}(\mathbf{p_2},\lambda_2)\not{\varepsilon}_2^*\left(\not{p}_1-\not{q}_1+m\right)\not{\varepsilon}_1^* u(\mathbf{p_1},\lambda_1),
\end{aligned}$$

where $\lambda_{1(2)}$ are the helicities of the fermions and Likewise for the second graph we have

$$\mathcal{M}_{(b)} = \frac{-ie^2}{(u-m^2)}\overline{v}(\mathbf{p_2},\lambda_2)\not{\varepsilon}_1^*\left(\not{p}_1-\not{q}_2+m\right)\not{\varepsilon}_2^* u(\mathbf{p_1},\lambda_1)$$

(note $t=(p_1-q_1)^2,\ u=(p_1`=q_2)^2$)

When we square the matrix-element we get terms such as

$$|\mathcal{M}_{(a)}|^2 = \frac{e^4}{(t-m^2)^2}\overline{v}(\mathbf{p}_2,\lambda_1)\not{\varepsilon}_2^*\left(\not{p}_1-\not{q}_1+m\right)\not{\varepsilon}_1^* u(\mathbf{p}_1,\lambda_1)\overline{u}(\mathbf{p}_1,\lambda_1)\not{\varepsilon}_1\left(\not{p}_1-\not{q}_1+m\right)\not{\varepsilon}_2 v(\mathbf{p_2},\lambda_2)$$

Once again, we average over the incoming fermion helicities to get

$$\frac{1}{4}\sum_\lambda|\mathcal{M}_{(a)}|^2 = \frac{e^4}{4(t-m^2)^2}\mathrm{Tr}\left\{(\not{p}_2-m)\not{\varepsilon}_2^*\left(\not{p}_1-\not{q}_1+m\right)\not{\varepsilon}_1^*(\not{p}_1+m)\not{\varepsilon}_1\left(\not{p}_1-\not{q}_1+m\right)\not{\varepsilon}_2\right\}$$

This looks like a trace of a product of eight γ-matrices, but if we also sum over the helicities of the final-state photons and use (in Feynman gauge)

$$\sum_{\lambda_1'}\varepsilon_1(\lambda_1')^\mu\varepsilon_1(\lambda_1')^{\nu *} = -g^{\mu\nu}$$

$$\sum_{\lambda_2'}\varepsilon_2(\lambda_2')^\mu\varepsilon_2(\lambda_2')^{\nu *} = -g^{\mu\nu}$$

The trace becomes

$$\mathrm{Tr}\left\{(\not{p}_2-m)\gamma^\mu\left(\not{p}_1-\not{q}_1+m\right)\gamma^\nu\left(\not{p}_1+m\right)\gamma_\nu\left(\not{p}_1-\not{q}_1+m\right)\gamma_\mu\right\}$$

Now we can make use of the γ-matrix relations which give

$$\gamma_\mu(\not{p}_2-m)\gamma^\mu = -2\not{p}_2-4m$$

and

$$\gamma^\nu(\not{p}_1+m)\gamma_\nu \;=\; -2\not{p}_1+4m$$

The trace becomes (omitting terms containing strings of an odd number of γ-matrices)

$$4\mathrm{Tr}\{\not{p}_2(\not{p}_1-\not{q}_1)\not{p}_1(\not{p}_1-\not{q}_1)\} \;+\; 4m^2\left(4\mathrm{Tr}\{(\not{p}_1+\not{p}_2-\not{q}_1)\not{q}_1\}-3\mathrm{Tr}\{\not{p}_2\not{p}_1\}\right)-16m^4\mathrm{tr}\{I\}$$

Using $\not{p}_1\not{p}_1 \;=\; p_1\cdot p_1 \;=\; m^2$ and $\not{q}_1\not{q}_1 \;=\; q_1\cdot q_1 \;=\; 0$ This further simplifies to

$$4\mathrm{Tr}\{\not{p}_2\not{q}_1\not{p}_1\not{q}_1)\}+4m^2\left(4\mathrm{tr}\{\not{p}_1\not{q}_1\}+2\mathrm{Tr}\{\not{p}_2\not{q}_1\}-2\mathrm{Tr}\{\not{p}_2\not{p}_1\}\right)-16m^4\mathrm{tr}\{I\}.$$

These traces are readily evaluated using $\mathrm{Tr}\{\not{p}\not{q}\}=4p\cdot q$, and $\not{q}_1\not{p}_1\not{q}_1=2p_1\cdot q_1\not{q}_1$ since $q_1\cdot q_1=0$, yielding

$$32\left(p_1\cdot q_1 p_2\cdot q_1+m^2\left(2p_1\cdot q_1+p_2\cdot q_1-p_1\cdot p_2\right)-2m^4\right)$$

A similar procedure must be applied for $|\mathcal{M}_{(b)}|^2$ and the interferences term between the two graphs, in order to obtain the complete square matrix-element.

15.2 Spin-Projection

In an experiment in which the spin of an incoming or outgoing (or both) spin-$\frac{1}{2}$ particle is measured, we could simply not perform the sum over spins and write out the Dirac spinors explicitly.

It is more convenient, however, to carry out the spin sum and express the square matrix-element in terms of traces, but to insert a spin projection operator next to the spinors which projects out the spin in a particular direction.

Such a spin-projection operator is

$$\Lambda_\pm \;=\frac{1}{2}\left(1\pm\gamma^5\not{n}\right),$$

where $n\cdot n=-1$ and in the rest-frame of the particle is a purely space-like vector unit vector, - $n=(0,\mathbf{n})$ with $\mathbf{n}\cdot\mathbf{n}=1$. For a general frame, where the particle has momentum p, we impose the condition

$$p\cdot n=0.$$

In the rest-frame of the particle and in the Dirac representation, this projection operator takes the form

$$\Lambda_\pm \;=\; \frac{1}{2}\left(1\pm\gamma^5\not{n}\right) \;=\; \begin{pmatrix} \frac{1}{2}(1\pm\sigma\cdot\mathbf{n}) & \\ & \frac{1}{2}(1\mp\sigma\cdot\mathbf{n}) \end{pmatrix}$$

Acting on a positive energy spinor in its rest-frame,

$$u \;=\; \begin{pmatrix} \chi \\ 0 \end{pmatrix},$$

this projects the $\sigma\cdot\mathbf{n} = \pm 1$ component, whereas for a negative energy spinor in its rest frame

$$v = \begin{pmatrix} 0 \\ \chi \end{pmatrix},$$

this projects the $\sigma\cdot\mathbf{n} = \mp 1$ component.

$\Lambda_\pm$ is a projection-operator because

$$\Lambda_\pm\Lambda_\pm = \frac{1}{4}\left(1\pm\gamma^5\not{n}\right)\left(1\pm\gamma^5\not{n}\right) = \frac{1}{4}\left(1\pm 2\gamma^5\not{n}+\gamma^5\not{n}\gamma^5\not{n}\right)$$

$$\gamma^5\not{n}\gamma^5\not{n}\ \not{n}\gamma^5\gamma^5\not{n} = -\not{n}\not{n} = -n\cdot n = 1$$

so that

$$\Lambda_\pm\Lambda_\pm = \Lambda_\pm$$

Note that

$$\left(\frac{\left(1\pm\gamma^5\not{n}\right)}{2}u\right)^\dagger\gamma^0 = u^\dagger\gamma^0\frac{\left(1\pm\gamma^5\not{n}\right)}{2} = \overline{u}\frac{\left(1\pm\gamma^5\not{n}\right)}{2}$$

Furthermore, since $p\cdot n = 0$ $\Lambda_\pm$ commutes with the projection operators $\not{p}\pm m)$.

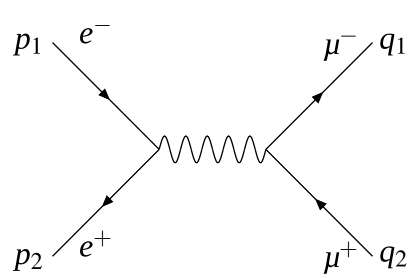
The upshot of this is that we can write

$$\frac{\left(1\pm\gamma^5\not{n}\right)}{2}\left(\not{p}\pm m\right)\frac{\left(1\pm\gamma^5\not{n}\right)}{2} = \frac{\left(1\pm\gamma^5\not{n}\right)}{2}\left(\not{p}\pm m\right),$$

so that in constructing the square matrix element, we only need to insert the spin-projection operator once.

Revisit the process

$$e^+ + e^- \to \mu^+\mu^-$$



In this case, however, we will require that the spin of the outgoing μ^- has component $+\frac{1}{2}$ in the direction n and the incoming electron has spin component $+\frac{1}{2}$ in the direction l.

The matrix element $\mathcal{M}$ is then

$$\mathcal{M} \;=\; i\frac{e^2}{s}\overline{u}_\mu(q_1,\tilde{\lambda}_1)\frac{(1+\gamma^5\not{n})}{2}\gamma^\mu v_\mu(q_2,\tilde{\lambda}_2)\,\overline{v}_e(p_2,\lambda_2)\gamma_\mu\frac{(1+\gamma^5\not{l})}{2}u_e(p_1,\lambda_1)$$

Now we square the matrix-element in the usual way and sum over muon and electron polarisations, keeping the projection operators in place. Note that we now only have a factor of $\frac{1}{2}$ for the average over initial state polarisations as the spin of the electron is determined. Thus we get

$$\begin{aligned}
\frac{1}{2}|\mathcal{M}|^2 \;&=\; \frac{e^4}{2s^2}\mathrm{Tr}\left\{(\not{q}_1+m_\mu)\frac{(1+\gamma^5\not{n})}{2}\gamma^\mu(\not{q}_2-m_\mu)\gamma^\nu\right\}\mathrm{Tr}\left\{(\not{p}_2-m_e)\gamma_\mu\frac{(1+\gamma^5\not{l})}{2}(\not{p}_1+m_e)\gamma_\nu\right\}\\
&=\; \frac{2e^4}{s^2}\left(q_1^\mu q_2^\nu+q_2^\mu q_1^\nu-g^{\mu\nu}(q_1\cdot q_2+m_\mu^2)+im_\mu\varepsilon^{\mu\nu\rho\sigma}(q_1+q_2)^\rho n^\sigma\right)\\
&\quad\left(p_{1\mu}p_{2\nu}+p_{2\mu}p_{1\nu}-g_{\mu\nu}(p_1\cdot p_2+m_e^2)-im_e\varepsilon_{\mu\nu\lambda\tau}(p_1+p_2)^\lambda l^\tau\right)\\
&=\; \frac{4e^4}{s^2}\left(2m_e^2m_\mu^2+p_1\cdot p_2m_\mu^2+q_1\cdot q_2m_e^2+p_1\cdot p_2p_2\cdot q_2+p_1\cdot q_2p_2\cdot q_1\right.\\
&\quad\left.+m_\mu m_e\left((p_1+p_2)\cdot n(q_1+q_2)\cdot l-n\cdot l(p_1+p_2)\cdot(q_1+q_2)\right)\right),
\end{aligned}$$

where we have used

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\lambda\tau} \;=\; 2\left(\delta^\rho_\tau\delta^\sigma_\lambda-\delta^\rho_\tau\lambda^\sigma_\delta\right)$$

The cross-section is now clearly more complicated and depends on the angles between the chosen polarisation directions and the momenta of the particles.

A similar projection is also possible for the case where the polarisation of external photons is measured. However, this cannot be done in Feynman gauge but in a different gauge called the light-like axial gauge. The axial gauge is a gauge in which the component of the photon field $n\cdot A=0$, where n is an external vector and the "light-like" axial gauge is the case where the vector n is light-like, i.e. $n\cdot n$=0. In this case we have the relation

$$\sum_{\lambda=\pm}\varepsilon^\mu(k,\lambda)\varepsilon^{*\nu}(k,\lambda) \;=\; -g^{\mu\nu}+\frac{n^\mu k^\nu+n^\nu k^\mu}{n\cdot k}.$$

We see that this vanishes if contracted with n_μ given $n\cdot n=0$. If, on the other hand, we do *not* sum over the polarisation of the photon then we have

$$\varepsilon^\mu(k,\lambda)\varepsilon^{*\nu}(k,\lambda) \;=\; \frac{1}{2}\left(-g^{\mu\nu}+\frac{n^\mu k^\nu+n^\nu k^\mu+i\lambda\varepsilon^{\mu\nu\rho\sigma}n_\rho k_\sigma}{n\cdot k}\right).$$

We would insert this *before* taking the trace(s) over the γ-matrices - these traces become much more complicated to work out, but with the help of a computer the square matrix-element for the case of polarised photons can be determined. The dependence on the external vector n will cancel out between the contribution from the squares of the different diagrams and the interferences (gauge invariance of physical quantities) but is present in individual contributions.