4 Green Functions - Feynman Propagators

There are two "Green functions" which will turn out to be very useful:

1. The vacuum expectation value of the commutator of two fields

$$i\Delta(x-y) \equiv \langle 0| [\phi(x), \phi(y)] |0\rangle$$

Using the expansion of the fields in terms of creation and annihilation operators this is

$$i\Delta(x-y) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} \frac{d^3\mathbf{p}'}{(2\pi)^3 2E'_p} \left\{ e^{-i(p\cdot x - p'\cdot y)} \langle 0| \left[a(\mathbf{p}), a^\dagger(\mathbf{p}') \right] |0\rangle \right.$$
$$\left. + e^{+i(p\cdot x - p'\cdot y)} \langle 0| \left[a^\dagger(\mathbf{p}), a(\mathbf{p}') \right] |0\rangle \right\},$$

having set the commutators of two creation or two annihilation operators to zero.
Using the commutation relation

$$\left[a((p), a^\dagger(\mathbf{p}') \right] = (2\pi)^3 2E_p \delta^3(\mathbf{p} - \mathbf{p}'),$$

this becomes

$$i\Delta(x-y) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} \left\{ e^{-i(p\cdot(x-y)} - e^{+i(p\cdot(x-y)} \right\}$$

By changing the sign of the three-momentum in the second term, we may rewrite this as

$$\Delta(x-y) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 E_p} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \sin\left(E_p(y_0 - x_0)\right)$$

If $x_0 = y_0$ then this integral vanishes except if **x=y**. More generally, it can be shown that this vanishes if

$$(x-y)^2 < 0,$$

i.e. if the space-time points x and y have space-like separation. This is a statement of causality - it tells us that the operation of creating or annihilation a particle at x must commute with the operation of creating or descrying a particle at y unless it is possible to travel from x to y or from y to x at a speed less than the speed of light so that information can pass from one point to the other.

2. The vacuum expectation of the time-ordered product of two fields:

$$i\Delta_F(x,y) \equiv \langle 0| T\phi(x)\phi(y) |0\rangle.$$

where the time ordering operator T means

$$T\phi(x)\phi(y) = \phi(x)\phi(y), \text{ if } x_0 > y_0$$
$$= \phi(y)\phi(x), \text{ if } y_0 > x_0.$$

In terms of creation and annihilation of particles, it represents the creation of a particle at the point y and its destruction at x if $x_0 > y_0$ and the creation of a particle at the point x and its destruction at y if $x_0 < y_0$. Thus it represents the propagation of a particle from x to y or from y to x.

Expanding in terms of creation and annihilation operators and using the fact that the annihilation operator acting on the vacuum gives zero so that

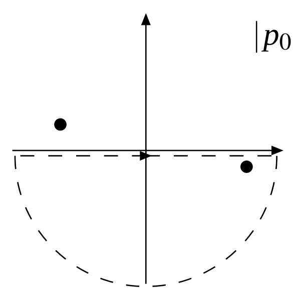
$$\langle 0|a(\mathbf{p})a^\dagger(\mathbf{p}')|0\rangle \;=\; \langle 0|\left[a(\mathbf{p}),a^\dagger(\mathbf{p}')\right]|0\rangle \;=\; (2\pi)^3 2E_P \delta^3(\mathbf{p}-\mathbf{p}'),$$

we have

$$i\Delta_F(x,y) \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left\{\theta(x_0-y_0)e^{-ip\cdot(x-y)} + \theta(y_0-x_0)e^{+ip\cdot(x-y)}\right\}$$

We can cast this into manifestly Lorentz invariant form by considering the integral

$$\lim_{\varepsilon\to 0}\int dp_0 \frac{e^{-ip_0 t}}{p_o^2 - E_P^2 + i\varepsilon}$$



The integrand has poles at

$$p_0 = \pm(E_P - i\varepsilon)$$

If $t > 0$, then we close the contour *below* the real axis, (as shown) so that $e^{ip_0 t} \to 0$ as $|p_0| \to \infty$ and we pick up the pole at $p_0 = E_P - i\varepsilon$, giving the result

$$-\frac{2\pi i}{2E_p}e^{-iE_p t},$$

(the minus sign arising from the fact that the contour is in the *clockwise* direction) whereas if $t < 0$ we need to close the contour in the *upper* plane, thereby picking up the pole at $p_0 = -E_p + i\varepsilon$, giving the result

$$\frac{2\pi i}{-2E_p}e^{+iE_p t},$$

Thus we may write the "Feynman propagator", $\Delta_F(x,y)$ in manifestly Lorentz invariant form as

$$\Delta_F(x,y) \;=\; \lim_{\varepsilon\to 0}\int \frac{d^4 p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{(p^2 - m^2 + i\varepsilon)}$$

Since

$$\Im m\left\{\frac{1}{x+i\varepsilon}\right\} \;=\; -\pi\delta(x),$$

we have the reaction

$$\Delta(x,y) \;=\; 2\Im m\Delta_F(x,y)$$

$\Delta_F(x,y)$ is a Green functions because it obeys the Green function equation

$$\left(\Box+m^2\right)\Delta_F(x,y) = -\delta^4(x-y)$$