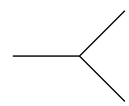
6 Interactions

So far, we have only considered free particles propagating form one space-time point to another - we do not yet have a mechanism for changing the number of particles at some space-time point and thereby accounting for interactions. This is because the Lagrangian for the free field is quadratic in the fields so that each (normal ordered) term contains one creation and one annihilation operator.

In order to incorporate interactions, we must add to the Lagrangian density terms which are of higher degree in the fields - for example we can add a term

$$\mathcal{L}_I = -\frac{g}{3!}\phi^3,$$

(we take real scalar fields for the moment). ϕ^3 contains terms with an additional annihilation operator or one additional creation operator and so it can describe the branching of a particle into two particles



The equation of motion with such a term added is now non-linear

$$\left(\Box + m^2\right)\phi = -\frac{g}{2}\phi^2,$$

and ϕ cannot be expanded in terms of creation and annihilation operators.

However, we can invoke the "adiabatic hypothesis" which tells us that the interactions are switched on at some early time and switched off again at some later time so that as $t \to \pm\infty$ the field ϕ becomes equal to the free field. In fact, we cannot quite achieve this and we have to introduce a proportionality constant, $\sqrt{Z}$, where Z is known as the wavefunction renormalisation constant - we will learn more about this when we consider renormalisation and for the tree-level calculations considered in these lectures it can be set equal to 1, although we keep it now for completeness. Thus we have

$$\phi(x)_{x_0 \to \pm\infty} \to \sqrt{Z}\phi_{\substack{out \\ in}}(x)$$

where $\phi_{\substack{out \\ in}}(x)$ are free fields which can be expanded in terms of creation and annihilation operators $a^\dagger_{in}(\mathbf{p}),\ a_{in}(\mathbf{p})$ for the incoming states and $a^\dagger_{out}(\mathbf{p}),\ a_{out}(\mathbf{p})$ for the outgoing states.

$$\phi_{\substack{in \\ out}}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}\left(a_{\substack{in \\ out}}(\mathbf{p})e^{-ip\cdot x} + a^\dagger_{\substack{in \\ out}}(\mathbf{p})e^{+ip\cdot x},\right)$$

It will be necessary to invert these expressions in order to express $a^\dagger_{in}(\mathbf{p})$ and $a_{in}(\mathbf{p})$ in terms of ϕ_{in} and $\dot\phi_{in}$ (and similarly for the outgoing fields.

$$\dot\phi_{in}(x) = -i\int \frac{d^3\mathbf{p}}{(2\pi)^3 2}\left(a_{in}(\mathbf{p})e^{-ip\cdot x} - a^\dagger_{in}(\mathbf{p})e^{+ip\cdot x},\right)$$

Taking the inverse Fourier transform of the expansions of ϕ and $\dot\phi$ and manipulating, we find

$$a_{in}(\mathbf{p}) = i\int d^3\mathbf{x}e^{ip\cdot x}\left(\dot\phi_{in}(x) - iE_p\phi_{in}(x)\right)$$

We may rewrite this as

$$a_{in}(\mathbf{p}) = i\int d^3\mathbf{x}e^{ip\cdot x}\overleftrightarrow{\partial_0}\phi_{in}(x),$$

where the notation $f(x)\overleftrightarrow{\partial_0}g(x)$ means $f(x)(\partial_0 g(x)) - (\partial_0 f(x))g(x)$.

Likewise

$$a^\dagger_{in}(\mathbf{p}) = -i\int d^3\mathbf{x}e^{-ip\cdot x}\overleftrightarrow{\partial_0}\phi_{in}(x),$$

Similar results can be obtained for the out creation and annihilation operators.

6.1 The S-matrix

The “in” and “out” states are related to each other by a unitary operator called the S-matrix operator, S,

$$|\alpha, in\rangle = S_{\alpha\beta}|\beta, out\rangle$$

and the in and out fields are also related by this operator

$$\phi_{in} = S\phi_{out}S^{-1} = S\phi_{out}S^\dagger$$

The quantum amplitude for an initial state $|\alpha\rangle$ to scatter into a final state $\langle\beta|$ is $\langle\beta, out|\alpha, in\rangle$, so it is this quantity that we need to calculate in order to be able to determine scattering cross-sections (or decay rates). For example, suppose $|\alpha\rangle$ is the two-particle state with momenta p_1 and p_2, and $|\beta\rangle$ is also a two-particle state with momenta q_1 and q_2. The scattering amplitude for this process is

$$S_{\alpha\beta} = \langle q_1, q_2, out|p_1 p_2, in\rangle$$

We can write this as

$$S_{\alpha,\beta} = \langle q_1, q_2, out|a^\dagger(\mathbf{p_1})|p_2, in\rangle.$$

We can express $a^\dagger(\mathbf{p_1})$ in terms of $\phi_{in}(x)$ and if $x_0 \to -\infty$ this can be expressed by the full interacting field (up to a factor of $\sqrt{Z}$), as

$$\langle q_1, q_2, out|p_1 p_2, in\rangle = \lim_{x_1^0\to-\infty}\frac{-i}{\sqrt{Z}}\int d^3\mathbf{x_1}e^{-ip_1\cdot x_1}\overleftrightarrow{\partial_0}\langle q_1, q_2, out|\phi(x_1)|p_2, in\rangle.$$

This can be written as

$$\langle q_1,q_2,out|a^\dagger_{out}(\mathbf{p_1})|p_2,in\rangle + \frac{+i}{\sqrt{Z}}\int d^4x_1\partial_0\left\{e^{-ip_1\cdot x_1}\overleftrightarrow{\partial_0}\langle q_1,q_2,out|\phi(x_1)|p_2,in\rangle\right\},$$

as can be seen by integrating the second term over x_1^0 and cancelling the surface term at $x_1^0 \to +\infty$, which generates the first term.

The first term can be thrown away provided neither $q_1 = p_1$ nor $q_2 = p_1$ (i.e. no forward scattering in which one of the particles goes straight through without changing momentum). †

Using

$$-\partial_0^2 e^{-ip_1\cdot x} = \left(-\nabla^2+m^2\right)e^{-ip_1\cdot x},$$

this becomes

$$\frac{i}{\sqrt{Z}}\int d^4x_1\left\{(-\nabla_1^2+m^2)e^{-ip_1\cdot x_1}\langle q_1,q_2,out|\phi(x_1)|p_2,in\rangle + e^{-ip_1\cdot x_1}\partial_0^2\langle q_1,q_2,out|\phi(x_1)|p_2,in\rangle\right\}.$$

Integrating ∇_1^2 twice by parts this may be written in manifestly Lorentz invariant form as

$$\frac{i}{\sqrt{Z}}\int d^4x_1 e^{-ip_1\cdot x_1}\left(\Box_{x_1}+m^2\right)\langle q_1,q_2,out|\phi(x_1)|p_2,in\rangle.$$

Next, we write this as

$$\begin{aligned}&\frac{i}{\sqrt{Z}}\int d^4x_1 e^{-ip_1\cdot x_1}\left(\Box_{x_1}+m^2\right)\langle q_2,out|a_{out}(\mathbf{q_1})\phi(x_1)|p_2,in\rangle\\ =\ &\lim_{y_1^0\to\infty}\left(\frac{i}{\sqrt{Z}}\right)^2\int d^4x_1 d^3\mathbf{y_1}e^{-ip_1\cdot x_1}e^{iq_1\cdot y_1}\overleftrightarrow{\partial_0}\langle q_2,out|\phi(y_1)\phi(x_1)|p_2,in\rangle\end{aligned}$$

When we go through the same procedure as before and convert the integral over $d^3\mathbf{y_1}$ into and integral d^4y_1, we get a surface term at $y_1^0 \to -\infty$, which contains $a_{in}(\mathbf{q_1})\phi(x_1)|p_2,in\rangle$. We want to be able to reject this on the grounds that $p_2 \neq q_1$, but we could do this *only* if we had the term $\phi(x_1)a_{in}(\mathbf{q_1})|p_2,in\rangle$.. We can achieve this by replacing the term $\langle q_2,out|\phi(y_1)\phi(x_1)|p_2,in\rangle$ by $\langle q_2,out|T\phi(y_1)\phi(x_1)|p_2,in\rangle$, where T is the time ordering operator. This makes no difference in the limit $y_1^0 \to \infty$, and after similar manipulations we get

$$\left(\frac{i}{\sqrt{Z}}\right)^2\int d^4x_1 d^4y_1 e^{-ip_1\cdot x_1}e^{+iq_1\cdot y_1}\left(\Box_{x_1}+m^2)\right)\left(\Box_{y_1}+m^2)\right)\langle q_2,out|T\phi(x_1)\phi(y_1)|p_2,in\rangle$$

Proceeding in the same way for the other particles we end up with

$$\begin{aligned}\langle q_1,q_2,out|p_1,p_2,in\rangle &= \left(\frac{i}{\sqrt{Z}}\right)^4\int d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{-i(p_1\cdot x_1+p_2\cdot x_2-q_1\cdot y_1-q_2\cdot y_2)}\\ &\left(\Box_{x_1}+m^2\right)\left(\Box_{x_2}+m^2\right)\left(\Box_{y_1}+m^2\right)\left(\Box_{y_2}+m^2\right)\langle 0|T\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)|0\rangle\end{aligned}$$

†Note that the operator $a^\dagger$ acting on a bra-state, $\langle\alpha|$, is an annihilation operator.

We recall that the fields ϕ are interacting fields, and so the task now is to calculate the green function

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)|0\rangle$$

for these interacting field.

This "LSZ Reduction formula" (Lehmann, Symanzik, Zimmermann) generalises this to any number of incoming and outgoing particles.

$$\begin{aligned}\langle q_1, q_2 \cdots q_m, out|p_1, p_2 \cdots p_n, in\rangle &= \left(\frac{i}{\sqrt{Z}}\right)^{m+n} \int d^4x_1 \cdots d^4x_n d^4y_1 \cdots d^4y_m \\ &\quad e^{-i\sum_{j=1}^{n} p_j \cdot x_j} e^{+i\sum_{k=1}^{m} q_k \cdot y_k} \left(\Box_{x_1} + m^2\right) \cdots \left(\Box_{x_n} + m^2\right) \left(\Box_{y_1} + m^2\right) \cdots \left(\Box_{y_m} + m^2\right) \\ &\quad \langle 0|T\phi(x_1) \cdots \phi(x_n)\phi(y_1) \cdots \phi(y_m)|0\rangle\end{aligned}$$