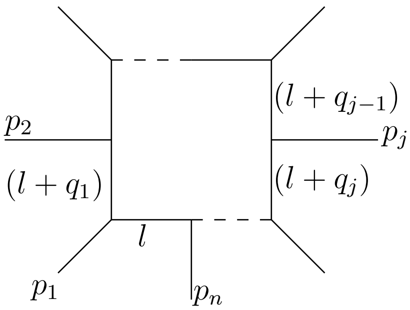
8 Reduction of n-point Scalar Integral to $(n-1)$-point Integrals

This reduction technique was developed by Bern, Dixon and Kosower (BDK).



$$I_n[1] \;\equiv\; -i(\pi)^{\epsilon-2}\int \frac{d^{4-2\epsilon}l}{(l^2-m_0^2)((l+q_1)^2-m_1^2)\cdots((l+q_{n-1})^2-m_{n-1}^2)},$$

$$(q_j = p_1+\cdots p_j, \quad q_0=0)$$

p_i^2 are referred to as "masses". If r of these p_i^2 are non-zero then this is called the r-mass n-point scalar integral.

After Feynman parametrisation and shifting we obtain

$$\begin{aligned} I_n[1] &= -i(\pi)^{\epsilon-2}\Gamma(n)\int_0^1 d\alpha_1 d\alpha_2\cdots d\alpha_n \delta(1-\sum_{i=1}^n \alpha_i)\int \frac{d^{4-2\epsilon}l}{(l^2+A^2)^n} \\ &= (-1)^n\Gamma(n-2+\epsilon)\int_0^1 \frac{d\alpha_1 d\alpha_2\cdots d\alpha_n\delta(1-\sum_i \alpha_i)}{(-A^2)^{n-2+\epsilon}} \end{aligned}$$

where

$$A^2 \;=\; \sum_{i=1}^n (q_{i-1}^2-m_{i-1}^2)\alpha_i - \sum_{i,j=1^n}\alpha_i\alpha_j q_{i-1}\cdot q_{j-1}$$

Consider the integral $I_{n-1}^{(1)}[1]$, which is the $(n-1)$-point integral obtained by taking the n-point integral and "pinching out" the first propagator. This may be written as

$$I_{n-1}^{(1)}[1] \;=\; -i(\pi)^{\epsilon-2}\int d^{4-2\epsilon}l \frac{(l^2-m_0^2)}{(l^2-m_0^2)((l+q_1)^2-m_1^2)\cdots((l+q_n)^2-m_n^2)},$$

We Feynman parametrise and shift as in the case of $I_n[1]$. The numerator shifts to

$$l_2-m_0^2 \;\rightarrow\; l^2+\sum_{i,j=1}^n q_i\cdot q_j\alpha_i\alpha_j - m_0^2 \;=\; l^2+A^2-2A^2+\sum_{i=1}^n (q_{i-1}^2-m_{i-1}^2)\alpha_i,$$

so that

$$
\begin{aligned}
I_{n-1}^{(1)} &= -i(\pi)^{\epsilon-2}\Gamma(n)\int_0^1 d\alpha_1 d\alpha_2\cdots d\alpha_n \delta(1-\sum_i \alpha_i)\int d^{4-2\epsilon}l\left\{\frac{1}{(l^2+A^2)^{n-1}}-\frac{2A^2}{(l^2+A^2)^n}\right\} \\
&\quad +\sum_i (q_{i-1}^2-m_{i-1}^2)I_n[\alpha_i] \\
&= \int_0^1 d\alpha_1 d\alpha_2\cdots d\alpha_n \delta(1-\sum \alpha_i)\frac{(-1)^{n-1}}{(-A^2)^{n-3+\epsilon}}\Gamma(n-3+\epsilon\left\{(n-1)-2(n-3+\epsilon)\right\} \\
&\quad +\ \sum_i (q_{i-1}^2-m_{i-1}^2)I_n[\alpha_i]
\end{aligned}
$$

But

$$
\Gamma(n-3+\epsilon)\frac{(-1)^n}{(-A^2)^{n-3+\epsilon}} \;=\; -i(\pi)^{\epsilon-3}\int\frac{d^{6-2\epsilon}l}{(l^2+A^2)^n},
$$

i.e. the n-point scalar integral calculated in $6-2\epsilon$ dimensions.

So we get

$$
I_{n-1}^{(1)}[1] \;=\; (n-5+2\epsilon)I_n[1]^{d=6-2\epsilon}\;+\;\sum_i (q_{i-1}^2-m_{i-1}^2)I_n[\alpha_i].
$$

Similarly, by multiplying $I_n[1]$ by one of the other propagator denominator factors we obtain

$$
I_{n-1}^{(j)} \;=\; (n-5+2\epsilon)I_n[1]^{d=6-2\epsilon}\;-\;2\sum_i S_{ij}I_n[\alpha_i], \tag{8.1}
$$

where

$$
S_{ij} \;=\; \frac{1}{2}\left(m_{i-1}^2+m_j^2-q_{(i-1),j}^2\right),\quad S_{ii}=0
$$

$(q_{(i-1),j}\equiv p_{i-1}+p_i\cdots p_j)$.

We would like to invert this set of equations to get a set of expressions for $I_n[\alpha_i]$. However, care must be taken for $n\geq 6$ because the fact that only four of the momenta are linearly independent means that the matrix S_{ij} is not invertible (in the case of zero internal masses).

Define parameters $b_i,\ i=1\cdots n$, such that

$$
S_{ij} \;=\; \frac{\rho_{ij}}{b_i b_j}
$$

where ρ_{ij} is a "conveniently chosen" invertible matrix with inverse η_{ij}.

This assignment is *not* unique.

Example 1:
$n=4$, internal masses set to zero ($s=q_{12}^2$, $t=q_{23}^2$).
A possible assignment is

$$
b_1=b_3=\frac{1}{\sqrt{|q_{12}^2|}}
$$

$$b_2 = b_4 = \frac{1}{\sqrt{|q_{23}^2|}}$$

$$\rho = \begin{pmatrix} 0 & \hat{p}_1^2 & 1 & \hat{p}_4^2 \\ \hat{p}_1^2 & 0 & \hat{p}_2^2 & 1 \\ 1 & \hat{p}_2^2 & 0 & \hat{p}_3^2 \\ \hat{p}_4^2 & 1 & \hat{p}_3^2 & 0 \end{pmatrix},$$

$$\hat{p}_i^2 \equiv \frac{p_i^2}{\sqrt{|q_{12}^2\, q_{23}^2|}}$$

On the other hand, if only one of the external masses is non-zero ($p_1^2 \neq 0$) then we can choose an assignment in which the matrix ρ is independent of the kinematics, as follows:

$$b_1 = \sqrt{\left|\frac{q_{23}^2}{q_{12}^2\, p_1^2}\right|}$$

$$b_2 = b_3 = \sqrt{\left|\frac{p_1^2}{q_{12}^2\, q_{23}^2}\right|}$$

$$b_4 = \sqrt{\left|\frac{q_{12}^2}{q_{23}^2\, p_1^2}\right|}$$

and

$$\rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

If two adjacent masses are non-zero ($p_1^2 \neq 0$ and $p_2^2 \neq 0$) then we can also make an assignment for which ρ is independent of the kinematics:

$$b_1 = \sqrt{\left|\frac{p_2^2}{q_{12}^2\, p_1^2}\right|}$$

$$b_2 = \sqrt{\left|\frac{q_{12}^2}{p_1^2\, p_2^2}\right|}$$

$$b_3 = \sqrt{\left|\frac{p_1^2}{q_{12}^2\, p_2^2}\right|}$$

$$b_4 = \sqrt{\left|\frac{p_1^2\, p_2^2}{q_{23}^4\, q_{12}^2}\right|}$$

and

$$\rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Example 2:
$n = 5$, internal and external masses set to zero.
A possible assignment is

$$b_i \;=\; \sqrt{\left|\frac{q^2_{i+1,i+2}\, q^2_{i+2,i+3}}{q^2_{i+3,i+4}\, q^2_{i+4,i+5}\, q^2_{i+5,i+6}}\right|},$$

(all subscripts are MOD 5)

$$\rho \;=\; \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Multiplying both sides of eq.(8.1) by $b_k \sum_j \eta_{kj} b_j$ we get

$$I_n[\alpha_k] \;=\; \frac{1}{2} b_k \sum_j \eta_{kj} b_j I^{(j)}_{n-1}[1] \;+\; \frac{(n-5+2\epsilon)}{2}\; b_k \sum_j \eta_{kj} b_j I_n[1]^{d=6-2\epsilon}$$

Now sum over k using the constraint on the Feynman parameters

$$\sum_{k=1}^{n} \alpha_k \;=\; 1$$

to obtain

$$I_n[1] \;=\; \frac{1}{2}\sum_{k,j=1}^{n} \eta_{kj} b_k b_j I^{(j)}_{n-1} \;+\; \frac{(n-5+2\epsilon)}{2}\left(\sum_{k,j=1}^{n} \eta_{kj} b_k b_j\right) I_n[1]^{d=6-2\epsilon}$$

For zero internal masses it can be shown that

$$\sum_{k,j} \eta_{kj} b_j b_k \;=\; \left(\prod_i b_i^2\right) \frac{\det'(p_i \cdot p_j)}{\det \rho},$$

where $\det'$ means that one of the momenta is omitted because of momentum conservation so this is the determinant of the remaining $(n-1) \times (n-1)$ matrix. This means that in the case where the internal masses are all zero

$$I_n[1] \;=\; \frac{1}{2}\sum_{k,j} \eta_{kj} b_k b_j I^{(j)}_{n-1} \;+\; \frac{(n-5+2\epsilon)}{2\det\rho}\, \det'(p_1 \cdot p_j) I_n^{d=6-2\epsilon}[1]$$

For $n = 5$ the integral $I_n^{d=6-2\epsilon}[1]$ is *neither* UV *nor* IR divergent so it contains no pole and the last term may be set to zero in the four-dimensional limit.

Furthermore for $n \geq 6$ the quantity $\det'(p_1 \cdot p_j)$ vanishes because only four of the momenta can be linearly independent (assuming that the external momenta are defined in four dimensions). We can therefore drop the term involving the $6 - 2\epsilon$ dimensional integral for all $n > 4$.

It is convenient to define "reduced integrals"

$$\hat{I}_n[1] \;\equiv\; \frac{1}{\prod_i b_i} I_n[1]$$

so that the above reduction formula becomes (for $n > 4$)

$$\hat{I}_n[1] \;=\; \frac{1}{2} \sum_{k,j} \eta_{k,j} b_k \hat{I}_{n-1}^{(j)}[1].$$

Example

Pentagon ($n = 5$) integral with all internal and external masses set to zero ($p_i^2 = 0$).

$$\frac{q_{i,i+1}^2}{q_{i+3,i+4}^2} \;=\; \frac{b_{i+3}}{b_{i+2}}$$

$$\eta \;=\; \rho^{-1} \;=\; \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix},$$

so that in terms of the reduced integrals

$$\hat{I}_5[1] \;=\; \frac{1}{2} \sum_j \left(b_{j-2} - b_{j-1} + b_j - b_{j+1} + b_{j+2}\right) \hat{I}_4^{(j)}$$

Collecting the coefficient of b_i this may be written

$$\hat{I}_5[1] \;=\; \frac{1}{2} b_5 \left(\hat{I}_4^{(5)} - \hat{I}_4^{(4)} - \hat{I}_4^{(3)} + \hat{I}_4^{(2)} - \hat{I}_4^{(1)}\right) \;+\; \text{cyclic}$$

The $\hat{I}_4^{(j)}$ are one-mass box ($n = 4$) integrals

$$\hat{I}_4^{(5)}[1] \;\equiv \hat{I}_{1m}(b_1, b_2, b_3, b_4) \;=\; 2c \left(\frac{(b_2 b_3)^\epsilon}{\epsilon^2} + \mathrm{Li}_2\left(1 - \frac{b_1}{b_2}\right) + \mathrm{Li}_2\left(1 - \frac{b_4}{b_3}\right) - \frac{\pi^2}{6} \right)$$

where c is a factor present in all one-loop integrals with zero internal masses

$$c \;\equiv\; \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}.$$

and

$$\mathrm{Li}_2(x) \equiv -\int_0^x \frac{\ln(1-y)}{y} dy.$$

Similarly

$$\hat{I}_4^{(4)} = \hat{I}_{1m}(b_5, b_1, b_2, b_3), \quad \text{etc.}$$

Inserting these expressions we find several cancellations and end up with

$$\hat{I}_5[1] = c(b_5)^{1+2\epsilon} \left(\frac{1}{\epsilon^2} + 2\mathrm{Li}_2\left(1 - \frac{b_4}{b_5}\right) + 2\mathrm{Li}_2\left(1 - \frac{b_1}{b_5}\right) - \frac{\pi^2}{6} \right) + \text{cyclic}.$$

$$\frac{b_4}{b_5} = \frac{q_{15}^2}{q_{23}^2}, \quad \frac{b_1}{b_5} = \frac{q_{34}^2}{q_{12}^2}.$$

$$b_5 = \left(\frac{q_{12}^2\, q_{23}^2}{q_{34}^2\, q_{51}^2} \right) \quad b_1 b_2 b_3 b_4 = \frac{1}{q_{12}^2\, q_{23}^2},$$

so that finally we get

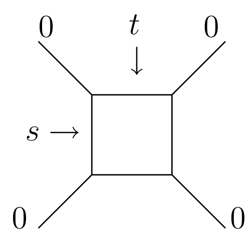
$$I_5[1] = \frac{c}{q_{12}^2\, q_{23}^2} \left(-\frac{q_{12}^2\, q_{23}^2}{q_{34}^2\, q_{45}^2\, q_{51}^2} \right)^{\epsilon} \left\{ \frac{1}{\epsilon^2} + 2\mathrm{Li}_2\left(1 - \frac{q_{15}^2}{q_{23}^2}\right) + 2\mathrm{Li}_2\left(1 - \frac{q_{34}^2}{q_{12}^2}\right) - \frac{\pi^2}{6} \right\} + \text{cyclic}$$

The zero-mass 5-point integral has been expressed in terms of one-mass box integrals.

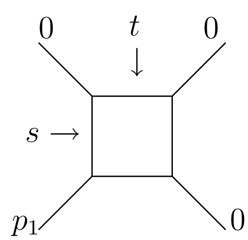
Had we applied the BDK reduction formula twice to a 6-point zero-mass integral, we would have obtained a sum of one-mass and two-mass box integrals, summed over all possible ways that two of the denominators can be "pinched out".

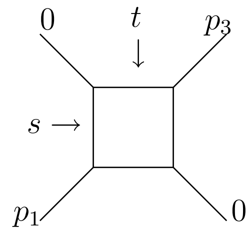
In general, the application of the Veltman-Passarino reductions, followed by the Bern-Dixon-Kosower reductions, reduces any one-loop integral to a linear combination of the following scalar integrals

- zero-mass box integrals ($I_4(s, t, 0, 0, 0, 0)$)

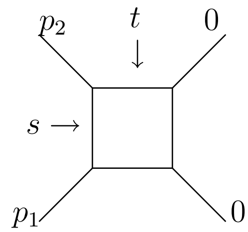


- one-mass box integrals ($I_4(s, t, p_1^2, 0, 0, 0)$)

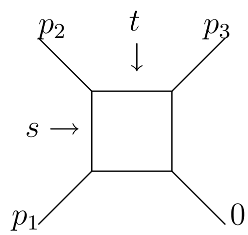




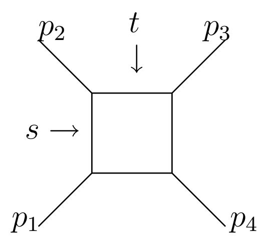
- two-mass-easy box integrals ($I_4(s,t,p_1^2,0,p_3^2,0)$)



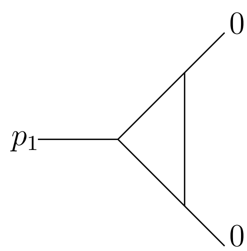
- two-mass-hard box integrals ($I_4(s,t,p_1^2,p_2^2,0,0)$)



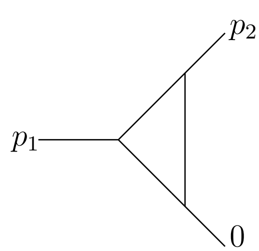
- three-mass box integrals ($I_4(s,t,p_1^2,p_2^2,p_3^2,0)$)



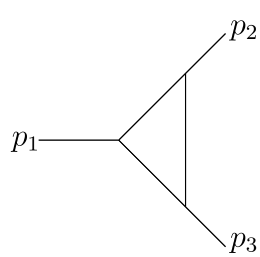
- four-mass box integrals ($I_4(s,t,p_1^2,p_2^2,p_3^2,p_4^2)$)



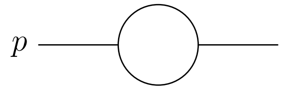
- one-mass triangle integrals ($I_3(p_1^2,0,0)$)



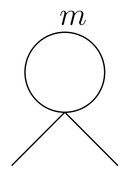
- two-mass triangle integrals ($I_3(p_1^2,p_2^2,0)$)



- three-mass triangle integrals ($I_3(p_1^2,p_2^2,p_3^2)$)



- bubble integral $I_2(p^2)$



- tadpole integral (for non-zero internal mass) $I_1(m^2)$

plus one tensor bubble integral

$$-i(\pi)^{\epsilon-2}\int d^{4-2\epsilon}l\frac{l^\mu l^\nu}{(l^2-m_0^2)((l+p)^2-m_1^2)}.$$

All but the last of these integrals has a unique imaginary part, as a function of the kinematic variables on which the integrals depend, which enables one to reconstruct the loop- amplitude from the coefficients of the cut graphs.