

QUANTUM FIELD THEORY 3

Syllabus

- Higher order perturbative corrections in ϕ^3 theory
- Renormalization
- Renormalization in QED
- The renormalization group - β -functions
- Infrared and collinear singularities
- Causality, unitarity and dispersion relations.

1 Loop corrections in ϕ^3 Theory

Consider the Lagrangian density for a scalar particle of mass m with cubic self-interaction with coupling constant λ

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - m^2 \phi^2 - \frac{\lambda}{3!} \phi^3$$

We wish to calculate the scattering amplitude for two particles of momenta, p_1 and p_2 into two particles with momenta p_3 and p_4 .

The Feynman rules for the n^{th} order perturbative contribution are:

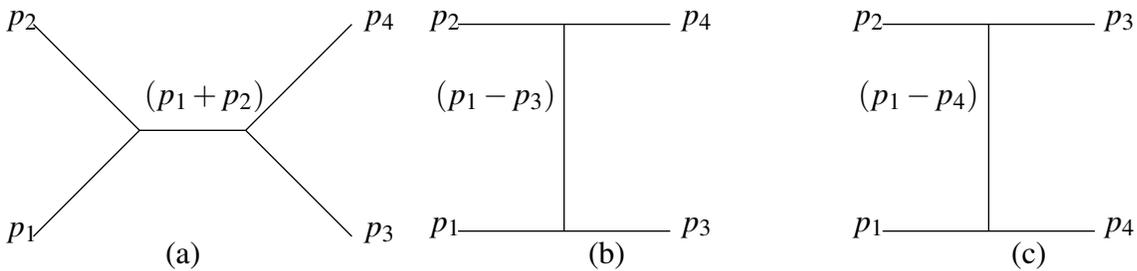
1. Draw all the possible Feynman graphs with n vertices.
2. Write a factor of $1/\sqrt{Z}$ for each external line (this will be explained later).
3. Write a factor of

$$\frac{i}{k^2 - m^2 + i\epsilon}$$

for each internal propagator with momentum k (we take the limit $\epsilon \rightarrow 0$, but we need to keep this term to guarantee the proper time-ordering).

4. Write a factor of $i\lambda$ at each vertex.
5. Introduce an energy-momentum conserving δ -function, $(2\pi)^4 \delta^4(k_1 + k_2 + k_3)$ for a vertex between particles with momenta k_1, k_2 and k_3 .
6. Integrate over $d^4 k_i / (2\pi)^4$ for each internal line of momentum k_i .

At order λ^2 we just have the three tree diagrams



In each diagram, the integration over the internal particle momentum is “soaked up” by one of the energy-momentum conserving δ -functions and we are left with one overall delta function

$$(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4),$$

which multiplies the entire amplitude.

For example, the contribution from tree-graph (a) is

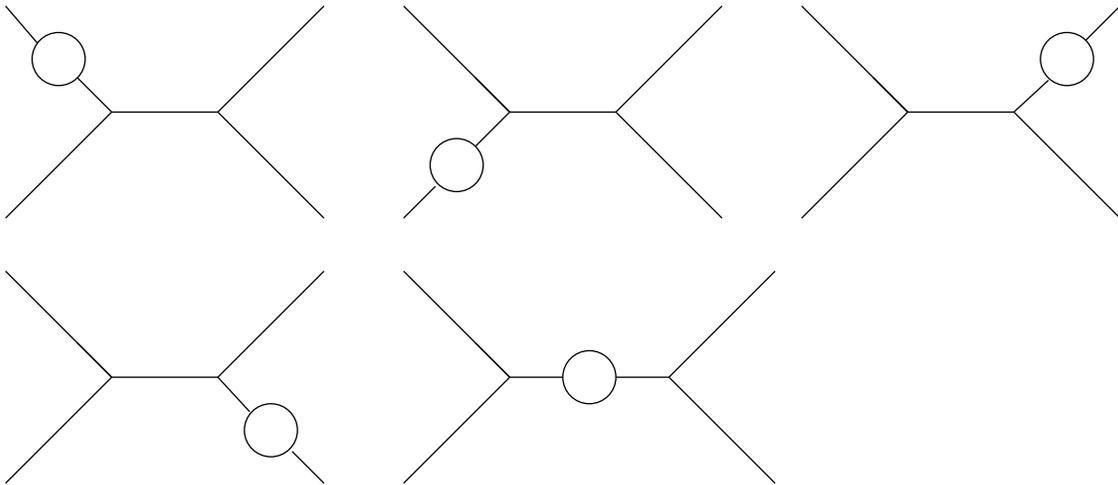
$$-i \frac{\lambda^2}{(s - m^2)}, \quad (1.1)$$

where we have suppressed the overall energy-momentum conserving δ -function and used $s = (p_1 + p_2)^2$.

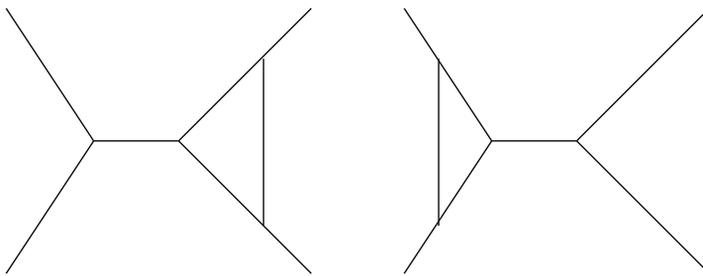
At the next order λ^4 we have graphs which contain one “loop” of internal particles and we will indeed need to integrate over an internal momentum.

For the corrections to the tree-graph (a), we have the following types of one-loop Feynman graphs

- **Self-energy corrections:**



- **Vertex corrections:**



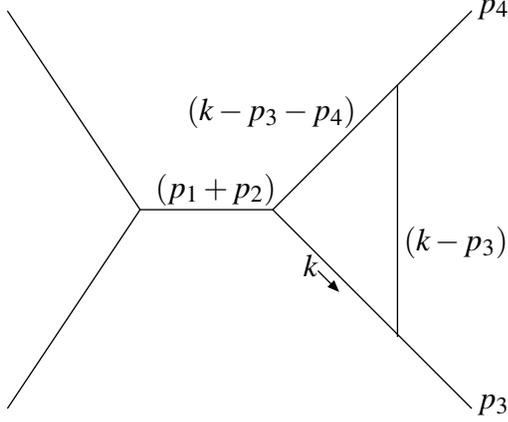
- **Box graphs:** These “box graphs” are generic one-loop graphs and cannot be associated with specific tree-level graphs (unlike the vertex or self-energy correction graphs)



Note that each of these graphs has three more internal lines than the tree-level graph and two more vertices. There is therefore a remaining integral over one of the internal momenta.

1.1 Vertex Corrections:

We will concentrate first on one of the vertex graphs



We have implemented the energy-momentum conserving δ -functions, by ensuring that momentum is conserved at each vertex. There is a remaining internal momentum l over which we need to integrate.

The contribution to the scattering amplitude from this term is

$$\frac{\lambda^4}{(p_1 + p_2)^2 - m^2 + i\epsilon} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\epsilon) ((k - p_3)^2 - m^2 + i\epsilon) ((k - p_3 - p_4)^2 - m^2 + i\epsilon)} \quad (1.2)$$

We have suppressed the overall energy-momentum conserving δ -function and also the factor $1/Z^2$, since Z has a perturbation expansion and is unity at leading order, i.e.

$$Z = 1 + O(\lambda^2),$$

so we do not need it to this order in perturbation theory.

Using the on-shell condition $p_3^2 = m^2$ and $(p_1 + p_2)^2 = (p_3 + p_4)^2 = s$ we may write this as

$$\frac{\lambda^4}{(s - m^2 + i\epsilon)} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\epsilon) (k^2 - 2k \cdot p_3 + i\epsilon) (k^2 - 2k \cdot (p_3 + p_4) + s - m^2 + i\epsilon)} \quad (1.3)$$

The integration over k is implemented using the following steps:

- **Feynman parametrize:**

Here we use the relation

$$\frac{1}{a_1 a_2 \cdots a_n} = (n-1)! \int_0^1 d\alpha_1 d\alpha_2 \cdots d\alpha_n \frac{\delta(1 - \sum_i \alpha_i)}{(a_1 \alpha_1 + a_2 \alpha_2 \cdots + a_n \alpha_n)^n} \quad (1.4)$$

Using this, the integral in eq.(1.3) may be written

$$2 \int \frac{d^4 k}{(2\pi)^4} d\alpha d\beta d\gamma \frac{\delta(1 - \alpha - \beta - \gamma)}{(k^2 - m^2 - 2k \cdot (p_3(\alpha + \beta) + p_4\beta) + s\beta + m^2\alpha)^3}, \quad (1.5)$$

where we have used $\alpha + \beta + \gamma = 1$ in the k^2 and m^2 terms.

- **Shift integration variable:**

$$k^\mu \rightarrow k^\mu + p_3^\mu(\alpha + \beta) + p_4^\mu\beta$$

The integral now becomes

$$2 \int \frac{d^4 k}{(2\pi)^4} d\alpha d\beta d\gamma \frac{\delta(1 - \alpha - \beta - \gamma)}{(k^2 - A^2 + i\epsilon)^3}, \quad (1.6)$$

where

$$A^2 = -s\beta - m^2\alpha + (p_3(\alpha + \beta) + p_4\beta)^2 = -s\beta(1 - \alpha - \beta) + m^2(1 - \alpha(1 - \alpha))$$

(we have used $(p_3 + p_4)^2 = s$ and $p_3^2 = p_4^2 = m^2$)

- **Integration over k :**

This is most easily achieved by rotating k^0 to ik^4 and performing the integral in Euclidean space.

$$2 \int \frac{k^3 dk d\Omega}{(2\pi)^4} d\alpha d\beta d\gamma \frac{\delta(1 - \alpha - \beta - \gamma)}{(k^2 + A^2 + i\epsilon)^3}, \quad (1.7)$$

The integration over Ω gives $2\pi^2$, the area of a three-dimensional spherical surface, and

$$\int \frac{k^3 dk}{(k^2 + A^2)^n} = \frac{(n-3)!}{2(n-1)!(A^2)^{(n-2)},}$$

(provided $n > 2$).

We end up with

$$\frac{i}{16\pi^2} \int_0^1 d\alpha d\beta d\gamma \frac{\delta(1 - \alpha - \beta - \gamma)}{(s(1 - \alpha - \beta)\beta - m^2(1 - \alpha(1 - \alpha)) + i\epsilon)} \quad (1.8)$$

The integration over γ can be done trivially to give

$$\frac{i}{16\pi^2} \int_0^1 d\alpha d\beta \frac{\theta(1 - \alpha - \beta)}{(s(1 - \alpha - \beta)\beta - m^2(1 - \alpha(1 - \alpha)) + i\epsilon)} \quad (1.9)$$

We will leave the result in terms of this integral - whose exact value would be different in the more realistic cases where the masses of the internal particles were not the same. We note, however, that in general the integral has an imaginary part arising from the fact that

$$\Im m \left(\frac{1}{(s(1-\alpha-\beta) - m^2(1-\alpha(1-\alpha)) + i\epsilon)} \right) = -\pi \delta(s(1-\alpha-\beta) - m^2(1-\alpha(1-\alpha)))$$

We write the contribution from this graph to the scattering amplitude as

$$-i \frac{\lambda^4}{(s-m^2)} \Delta F(s) \quad (1.10)$$

What this means is that the right-most coupling λ is replaced by an effective coupling, which depends on the square momentum s coming into the vertex.

$$\lambda \rightarrow \lambda (1 + \lambda^2 \Delta F(s)). \quad (1.11)$$

This means that the coupling is not really constant, but depends on the momenta coming into the vertex.

We now have to give a definition of the coupling in terms of some measurement, which we call the “renormalized coupling constant”. There is some arbitrariness in this definition and we call this arbitrariness “renormalization scheme dependence”.

The coupling parameter that we started off with is called the “bare coupling” and is written λ_0 . It is *not* directly measurable.

Thus, for example, we could define the renormalized coupling to be the coupling in which all momenta coming into the vertex are on shell, i.e. we set ($s = m^2$) and obtain the renormalized coupling as

$$\lambda_R = \frac{\lambda_0}{Z_1}, \quad (1.12)$$

where (to order λ^2)

$$Z_1 = (1 - \lambda^2 \Delta F(m^2)). \quad (1.13)$$

This renormalized coupling can be measured experimentally, and we wish to express the scattering amplitude in terms of this physically measured coupling. To do this we subtract off a “counterterm” corresponding to the conversion of the expression (1.10) into an expression in terms of this renormalized coupling, i.e. adding the contribution (1.10) to

$$-i \frac{\lambda_R^2}{(s-m^2)} (1 + \lambda_R^2 (\Delta F(s) - \Delta F(m^2))). \quad (1.14)$$

(The replacement of λ by λ_R in the correction term does not affect the result at this order in perturbation theory).

The renormalization scheme we have chosen here is called the “on-shell” scheme since it defines the renormalized coupling as the value of the three-point coupling when all three particles are on-shell.

We could have chosen to define λ_R at the coupling at some value $s = \mu^2$, so that eq.(1.13) becomes

$$Z_1 = (1 - \lambda^2 \Delta F(\mu^2)). \quad (1.15)$$

and eq.(1.14) becomes

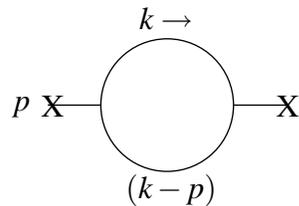
$$-i \frac{\lambda_R^2(\mu^2)}{(s - m^2)} (1 + \lambda_R^2 (\Delta F(s) - \Delta F(\mu^2))). \quad (1.16)$$

The numerical values of eqs.(1.14 and 1.16) are identical (up to order λ^4) - the explicit μ dependence appearing in eq.(1.16) being compensated by the μ^2 dependence of $\lambda_R(\mu^2)$.

We need not have chosen any directly measurable way to define λ_R . For example we could have defined $\lambda_R(\mu)$ as the coupling of the interaction in which all particles are off-shell with square momentum μ^2 . This is often done and it is called the “MOM” scheme. In this scheme the subtraction would again be different and we would get a different expression for the contribution to the scattering amplitude in terms of $\lambda_R^{MOM}(\mu^2)$, but the numerical value would again be the same once we had inserted the corresponding value of the renormalized coupling.

1.2 Self-energy Corrections:

Now we look at the “self-energy” graphs. These are the ones in which the loop has one incoming and one outgoing line (sometimes also called the “two-point function”).

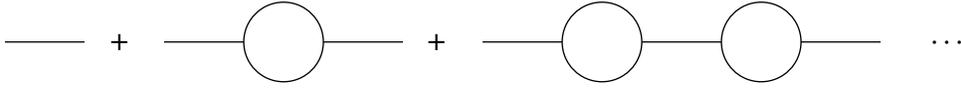


The crosses on the external lines indicate that they have been “truncated” - i.e. the external line propagators are not included in the calculation of the graph.

Define the “self-energy” function, $\Sigma(p^2)$ such that the contribution from the self-energy diagram is $-i\Sigma(p^2)$. This is also a function of the particle mass, m and the coupling λ_R .

Putting back the external propagators, gives (suppressing the $i\epsilon$)

$$\frac{i\Sigma(p^2)}{(p^2 - m^2)^2}$$



It looks as though this has a double pole at $p^2 = m^2$, but if we sum over all the possible numbers of self-energy insertions (including no insertions) we get a geometric series whose sum is

$$\frac{i}{(p^2 - m^2 - \Sigma(p^2))} \quad (1.17)$$

and this is how the propagator is modified by the self-energy insertions.

Expand $\Sigma(p^2)$ about $p^2 = m_R^2$ as

$$\Sigma(p^2) = \frac{1}{Z} ((m_R^2 - m^2) + (Z - 1)(p^2 - m^2) + \Sigma_R(p^2)) \quad (1.18)$$

The quantity $\Sigma_R(p^2)$ vanishes quadratically as $p^2 \rightarrow m_R^2$. (The factor Z in the denominator is unnecessary to this order but would be required for higher order calculations). Inserting this into the expression for the corrected propagator, (1.17) gives

$$\frac{iZ}{(p^2 - m_R^2 - \Sigma_R(p^2))}$$

We see that the pole has moved to m_R . This “renormalized mass” is therefore the physical mass and the parameter, m , used in the Lagrangian is the bare mass and is henceforth written as m_0 . Note that in general $\Sigma(p^2)$ will be complex. For a resonance of an unstable particle, the imaginary part of m_R is the half-width $\Gamma/2$ of the resonance..

In the same way, the field ϕ , which appears in the Lagrangian are “bare fields”,

$$\phi_B = \sqrt{Z}\phi_R$$

These are interacting fields which tend asymptotically (in time) to free in or out free-fields.

$$\phi_B(\mathbf{x}, t) \xrightarrow{t \rightarrow \pm\infty} \phi_{in(out)}(\mathbf{x}, t)$$

It is the propagator of these free fields ϕ_{in} or ϕ_{out} which behave like $i/(p^2 - m_R^2)$

The upshot of this is twofold:

1. The LSZ reduction for an S-matrix element in terms of Green functions (i.e. vacuum expectation values of time-ordered products of fields) should have a factor of $1/\sqrt{Z}$ for each external line. This is the origin of the factor in the Feynman rules mentioned previously.
2. The renormalization of the coupling constant has a factor of \sqrt{Z} for each line coming into the vertex, i.e. a factor of $Z^{3/2}$. So that Eq.(1.12) becomes

$$\lambda_R = \frac{Z^{3/2}}{Z_1} \lambda_0, \quad (1.19)$$

For a self-energy insertion on an internal line the factor of Z is absorbed because a factor of \sqrt{Z} is absorbed into the renormalization of the coupling at either end of the internal propagator. For a self-energy insertion on an external line a factor of \sqrt{Z} is absorbed into the renormalization of the coupling where the external line is attached to the rest of the graph and a factor of \sqrt{Z} cancels against the factor $1/\sqrt{Z}$ in the Feynman rule obtained from the more careful derivation of the LSZ reduction.

Calculation of m_R and Z :

Applying the Feynman rules to the self-energy diagram, we have

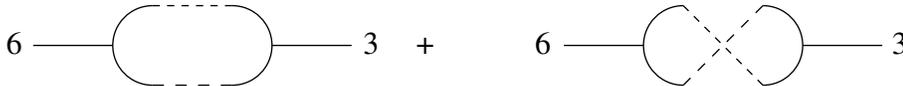
$$\Sigma(p^2) = i \frac{1}{2} \lambda_R^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2)((k-p)^2 - m^2)}. \quad (1.20)$$

λ_R should really be λ_0 , but to this order in perturbation theory we can use the renormalized coupling - (finally we want an expansion in terms of the renormalized coupling since this is related directly to a physically measurable quantity). Moreover m should be taken to mean m_R .

The factor of $\frac{1}{2}$ is a ‘‘combinatorial’’ factor and is determined as follows:

- Expanding the exponential of the interacting part of the action we have, at order λ^2

$$\frac{1}{2!} \left(\frac{\lambda}{3!} \right)^2 \left(\int d^4 x \phi^3(x) \right)^2$$



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There are six ways, to select one of the external lines, three ways to select the other external line (which must be attached to the other vertex) and two ways to join the remaining lines together as internal propagators.

- This gives a total combinatorial factor of

$$6 \times 3 \times 2 \times \frac{1}{2!} \times \left(\frac{1}{3!} \right)^2 = \frac{1}{2}.$$

It is necessary to determine the combinatorial factor for *each* graph. had we done so for the vertex graphs we would have obtained a combinatorial factor of unity.

Write Eq.(1.20) as

$$\Sigma(p^2) = i\frac{1}{2}\lambda_R^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)(k^2 - 2p \cdot k + p^2 - m^2)}. \quad (1.21)$$

Note that we cannot set $p^2 = m^2$ here.

Now introduce the trick of Feynman parametrization

$$\Sigma(p^2) = i\frac{1}{2}\lambda_R^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha d\beta \frac{\delta(1 - \alpha - \beta)}{(k^2 - 2\alpha p \cdot k + p^2\alpha - m^2)^2}, \quad (1.22)$$

(we have used $\alpha + \beta = 1$ in the coefficients of k^2 and m^2).

Shift $k^\mu \rightarrow k^\mu + \alpha p^\mu$ to get

$$\Sigma(p^2) = i\frac{1}{2}\lambda_R^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha \frac{1}{(k^2 + p^2\alpha(1 - \alpha) - m^2)^2}. \quad (1.23)$$

(We have performed the integral over β absorbing the δ -function).

In Euclidean space, after integrating over the angles this is

$$\Sigma(p^2) = -\frac{1}{16\pi^2}\lambda_R^2 \int_0^1 d\alpha \int k^3 dk \frac{1}{(k^2 - p^2\alpha(1 - \alpha) + m^2)^2}. \quad (1.24)$$

This integral is *divergent*. The divergence is called ‘ultraviolet’ as it arises from the $l \rightarrow \infty$ end of the integral.

The modern view of such divergences is that there is some ‘new physics’ at some high scale which serves to regulate these divergences. The most popular such theory is string theory in which what we call point particles are really extended objects with a length of order $1/\Lambda$. The point-like field theory that we use is valid up to a scale of order Λ , above which the string-like properties provide a cutoff for these effective integrals which is of order Λ . Several string theories have been identified which have been shown to be ultraviolet finite when treated correctly - these divergences occurring only when one makes the approximation that the strings can be treated as point-like particles.

What we need to ensure is that physically measurable quantities are independent of the cut-off Λ .

For the self-energy in the ϕ^3 theory introducing the cut-off gives

$$\Sigma(p^2) = -\frac{\lambda_R^2}{32\pi^2} \int_0^1 d\alpha \ln \left(\frac{\Lambda^2}{(m^2 - p^2\alpha(1 - \alpha))} \right) \quad (1.25)$$

The divergence means that the bare mass depends on the cut-off

$$m_0^2 = m_R^2 - \frac{\lambda_R^2}{32\pi^2} \int_0^1 d\alpha \ln \left(\frac{\Lambda^2}{m^2(1-\alpha)(1-\alpha)} \right) \quad (1.26)$$

As we go to higher orders in perturbation theory the bare mass m_0 is adjusted by terms which depend on the cutoff, in such a way that the renormalized mass m_R is the physical mass that is measured. m_0 is not directly observable and so its cut-off dependence is not important.

In most renormalizable theories, such as QED and QCD, the renormalization constants Z and Z_1 are also cut-off dependent (UV divergent). This, in turn, means that the bare coupling is cut-off dependent in such a that the renormalized coupling is related to a physical measurable in a cut-off independent way.

In the ϕ^3 case Z is cut-off independent and is given by

$$(Z-1) = \frac{\partial}{\partial p^2} \Sigma(p^2)|_{p^2=m^2} = -\frac{\lambda_R^2}{32\pi^2} \int_0^1 d\alpha \frac{\alpha(1-\alpha)}{m^2(1-\alpha+\alpha^2)}. \quad (1.27)$$

To calculate the scattering amplitude we also need to consider the box-graphs. These are algebraically very complicated, but in a ϕ^3 theory they do not introduce ultraviolet divergences and are not associated with the renormalization of any of the parameters of the theory.