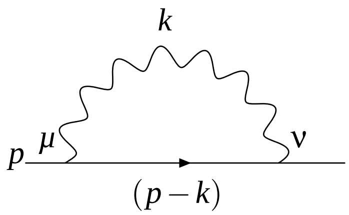
3 One-Loop Counterterms in QED

3.1 Fermion Self-energy



We work in Feynman gauge. Applying the rules of QED we have (in d dimensions)

$$\Sigma(p^2,m) \;=\; i\mu^{(4-d)}\int \frac{d^dk}{(2\pi)^d}(-ie\gamma^\mu)\frac{i(\gamma\cdot(p-k)+m)}{(p-k)^2-m^2}(-ie\gamma^\nu)\frac{-ig_{\mu\nu}}{k^2}, \tag{3.1}$$

where we have displayed explicitly the scale dependence of the coupling outside four dimensions.

Introducing Feynman parameters, we get

$$\Sigma(p^2,m) \;=\; -ie^2\mu^{(4-d)}\int \frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha\int_0^1 d\beta\delta(1-\alpha-\beta)\frac{\gamma^\mu(\gamma\cdot(p-k)+m)\gamma_\mu}{\left(k^2(\alpha+\beta)-2p\cdot k\alpha+(p^2-m^2)\alpha\right)^2} \tag{3.2}$$

Now shift $k\to k+p\alpha$ (and perform the trivial integral over β absorbing the δ-function)

$$\Sigma(p^2,m) \;=\; -ie^2\mu^{(4-d)}\int \frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha\frac{\gamma^\mu(\gamma\cdot p(1-\alpha)+m)\gamma_\mu}{\left(k^2+p^2(\alpha(1-\alpha)-m^2\alpha)\right)^2} \tag{3.3}$$

We have omitted a term

$$-ie^2\mu^{(4-d)}\int \frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha\frac{\gamma^\mu\gamma\cdot k\gamma_\mu}{\left(k^2+p^2(\alpha(1-\alpha)-m^2\alpha)\right)^2},$$

which vanishes by symmetry since the numerator is odd in k, the denominator is even in k and we must integrate over all directions of the vector k.

Setting $d=4-2\varepsilon$, using $\gamma^\mu\gamma\cdot p\gamma_\mu=-2(1-\varepsilon)\gamma\cdot p$, $\gamma^\mu\gamma_\mu=4-2\varepsilon$ and the result from eq.(2.6) we have

$$\Sigma(p^2,m) \;=\; -\frac{e^2}{(16\pi^2)}\Gamma(\varepsilon)\int_0^1 d\alpha\left[2(1-\varepsilon)(1-\alpha)\gamma\cdot p-(4-2\varepsilon)m\right]\left(\frac{4\pi\mu^2}{m^2\alpha-p^2\alpha(1-\alpha)}\right)^\varepsilon \tag{3.4}$$

Expanding in ε and keeping only the terms which do not vanish as $\varepsilon \to 0$, we get

$$\begin{aligned}\Sigma(p^2,m) &= -\frac{e^2}{(16\pi^2)}\left[\int_0^1 d\alpha\,(2(1-\alpha)\gamma\cdot p - 4m)\left(\frac{1}{\varepsilon}+\ln(4\pi)-\gamma_E\right)\right. \\ &\left.+\int_0^1 d\alpha\left(2\gamma\cdot p(1-\alpha)-2m+(2(1-\alpha)\gamma\cdot p-4m)\ln\left(\frac{m^2\alpha-p^2\alpha(1-\alpha)}{\mu^2}\right)\right)\right] \qquad (3.5)\end{aligned}$$

Performing the integral over α except in the last term, this reduces to

$$\begin{aligned}\Sigma(p^2,m) &= -\frac{e^2}{(16\pi^2)}\left[(\gamma\cdot p-4m)\left(\frac{1}{\varepsilon}+\ln(4\pi)-\gamma_E\right)\right. \\ &\left.+\left(\gamma\cdot p-2m+\int_0^1 d\alpha\,(2(1-\alpha)\gamma\cdot p-4m)\ln\left(\frac{m^2\alpha-p^2\alpha(1-\alpha)}{\mu^2}\right)\right)\right] \qquad (3.6)\end{aligned}$$

In order to obtain the (physical) mass subtraction term, δm, and the wavefunction renormalization constant Z_2, we must expand this in a power series in $(\gamma\cdot p-m)$, making use of the relation

$$p^2-m^2 = (\gamma\cdot p-m)(\gamma\cdot p+m) = 2m(\gamma\cdot p-m)+O((\gamma\cdot p-m)^2).$$

This enables us to expand the logarithm about $p^2=m^2$. This gives

$$\begin{aligned}\Sigma(p^2,m) &= \frac{e^2}{(16\pi^2)}\left[3m\left(\frac{1}{\varepsilon}+\ln(4\pi)-\gamma_E\right)-m+2m\int_0^1 d\alpha(1-\alpha)\ln\left(\frac{m^2\alpha^2}{\mu^2}\right)\right] \\ &+\frac{e^2}{(16\pi^2)}\left[1-\frac{1}{\varepsilon}-\ln(4\pi)+\gamma_E+2\int_0^1(1-\alpha)\ln\left(\frac{m^2\alpha^2}{\mu^2}\right)+4\int_0^1 d\alpha\frac{(1-\alpha^2)}{\alpha}\right](\gamma\cdot p-m) \\ &+O((\gamma\cdot p-m)^2) \qquad (3.7)\end{aligned}$$

The terms which are $O((\gamma\cdot p-m)^2$ and higher are finite and independent of the scale μ. They make up the renormalized self-energy $\Sigma_R(p^2,m)$. The last integral over α in eq.(3.7) diverges at $\alpha=0$. This is a new type of divergence caused by the fact that the photon is massless - it is called an “infrared divergence”. For the moment we regularize this infrared divergence by assigning a small mass, λ to the photon wherever necessary (i.e. we only keep terms in λ which are *not* regular as $\lambda\to 0$. When we do this the last integral in eq.(3.7) becomes

$$\int_0^1 d\alpha\frac{\alpha(1-\alpha^2)}{\alpha^2-(1-\alpha)\lambda^2/m^2} = \frac{1}{2}\left(\ln(\frac{m^2}{\lambda^2}-1\right)+O(\lambda^2).$$

Writing (to this order in perturbation theory),

$$\Sigma(p^2,m) = \delta m+(Z_2-1)(\gamma\cdot p-m)+\Sigma_R(p^2,m).$$

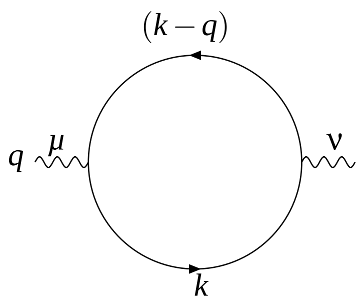
we have for the mass renormalization (introducing the fine-structure constant $\alpha=e^2/(4\pi)$),

$$\begin{aligned}\delta m &= m\frac{\alpha}{4\pi}\left[3\left(\frac{1}{\varepsilon}+\ln(4\pi)-\gamma_E\right)-1-2\int_0^1 d\alpha(1-\alpha)\ln\left(\frac{m^2\alpha^2}{\mu^2}\right)\right] \\ &= 3m\frac{\alpha}{4\pi}\left(\frac{1}{\varepsilon}+\ln(4\pi)-\gamma_E+\frac{4}{3}+\ln\left(\frac{m^2}{\mu^2}\right)\right), \qquad (3.8)\end{aligned}$$

and for the wavefunction renormalization constant,

$$
\begin{aligned}
Z_2 &= 1+\frac{\alpha}{4\pi}\left[1-\frac{1}{\varepsilon}-\ln(4\pi)+\gamma_E+2\int_0^1(1-\alpha)\ln\left(\frac{m^2\alpha^2}{\mu^2}\right)+4\int_0^1 d\alpha\frac{\alpha(1-\alpha^2))}{\alpha^2+(1-\alpha)\lambda^2/m^2}\right] \\
&= 1+\frac{\alpha}{4\pi}\left[-\frac{1}{\varepsilon}-\ln\left(\frac{\mu^2}{m^2}\right)+\gamma_E-\ln(4\pi)-4+2\ln\left(\frac{m^2}{\lambda^2}\right)\right]. \qquad (3.9)
\end{aligned}
$$

3.2 Photon Self-energy (Vacuum polarization)



The photon self-energy $\Pi^{\mu\nu}(q^2)$ is, in general, a two-rank tensor, which is formed from the four-momentum of the photon , q^μ and the (invariant) metric tensor. It must therefore have the form

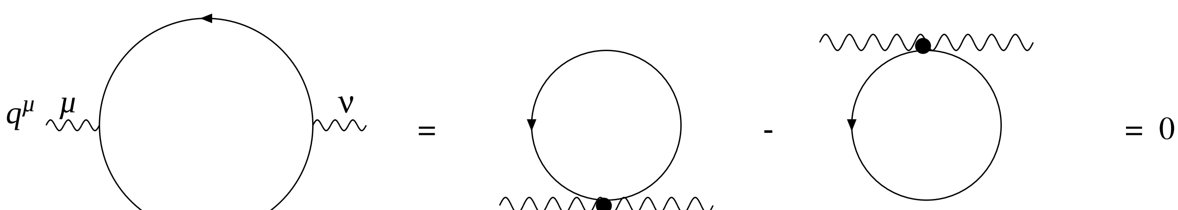
$$\Pi^{\mu\nu}(q^2) \;=\; A(q^2)g^{\mu\nu}+B(q^2)q^\mu q^\nu.$$

On the other hand $\Pi^{\mu\nu}(q^2)$ obeys a Ward identity

$$q_\mu\Pi^{\mu\nu}(q^2) \;=\; 0.$$

This can be seen by writing

$$\frac{1}{(\gamma\cdot k-m)}q^\mu\gamma_\mu\frac{1}{(\gamma\cdot(k-q)-m)}=\frac{1}{(\gamma\cdot(k-q)-m)}-\frac{1}{(\gamma\cdot k-m)}$$



Applying this to the one-loop graph representing the photon self-energy, we get the difference between two graphs in which one of the two internal fermion propagates has been killed. But these two graphs are identical and so the difference is zero.

We may therefore write

$$\Pi^{\mu\nu}(q^2) \;=\; \left(-g^{\mu\nu}q^2+q^{\mu}q^{\nu}\right)\Pi(q^2) \tag{3.10}$$

In other words only the transverse part of the photon propagator acquires a higher order correction.

The photon has no mass and therefore no mass renormalization. There is only a photon wavefunction renormalization constant Z_3.

$$\Pi(q^2) \;=\; \frac{1}{Z_3}\left((Z_3-1)+\Pi_R(q^2)\right), \tag{3.11}$$

where $\Pi_R(q^2)$ is the renormalized (finite) part of the self-energy. At the one-loop level the prefactor $1/Z_3$ in eq.(3.11) may be set to unity.

The fact that only the transverse part of the photon-propagator acquires a higher-order correction means that the gauge parameter, ξ is renormalized. If we write the leading order propagator as

$$-i\frac{\left[\left(g^{\mu\nu}-\frac{q^{\mu}q^{\nu}}{q^2}\right)-(\xi-1)\frac{q^{\mu}q^{\nu}}{q^2}\right]}{q^2}$$

The renormalized propagator is

$$-i\frac{Z_3\left(g^{\mu\nu}-\frac{q^{\mu}q^{\nu}}{q^2}\right)}{q^2\left(1-\Pi_R(q^2)\right)}+i(\xi-1)\frac{q^{\mu}q^{\nu}}{q^4}.$$

The transverse part of the propagator is renormalized but not the longitudinal part. Near $q^2=0$, the renormalized propagator looks like

$$-i\frac{Z_3\left[\left(g^{\mu\nu}-\frac{q^{\mu}q^{\nu}}{q^2}\right)-(\xi_R-1)\frac{q^{\mu}q^{\nu}}{q^2}\right]}{q^2},$$

where

$$(\xi_R-1)=\frac{(\xi-1)}{Z_3}.$$

Now returning to the one-loop graph and inserting the Feynman rules, we get

$$\Pi^{\mu\nu}(q^2) \;=\; -i\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\mathrm{Tr}\left[(-ie\gamma^{\mu})\frac{i(\gamma\cdot(k-q)+m)}{((k-q)^2-m^2)}(-ie\gamma^{\nu})\frac{i(\gamma\cdot k+m)}{k^2-m^2)}\right]. \tag{3.12}$$

An overall minus sign has been introduced for a loop of fermions. This arises form the fact that the Wick contraction required to construct the Feynman graph requires an interchange of two fermion fields, thereby introducing a minus sign.

Feynman parametrization gives

$$\Pi^{\mu\nu}(q^2) \ = \ -ie^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha d\beta\delta(1-\alpha-\beta)\frac{\mathrm{Tr}\left[\gamma^\mu(\gamma\cdot(k-q)+m)\gamma^\nu(\gamma\cdot k+m)\right]}{\left(k^2-2k\cdot q\alpha+q^2\alpha-m^2\right)^2}. \quad (3.13)$$

Performing the trace (and integrating over β) gives

$$\Pi^{\mu\nu}(q^2) \ = \ 4ie^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha\frac{g^{\mu\nu}\left(k\cdot(k-q)-m^2\right)-2k^\mu k^\nu+k^\mu q^\nu+q^\mu k^\nu}{\left(k^2-2k\cdot q\alpha+q^2\alpha-m^2\right)^2}. \quad (3.14)$$

Shifting $k^\mu \to k^\mu + q^\mu\alpha$ we get

$$\Pi^{\mu\nu}(q^2) \ = \ 4ie^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha\frac{g^{\mu\nu}\left(k^2-\alpha(1-\alpha)q^2-m^2\right)-2k^\mu k^\nu+2\alpha(1-\alpha)q^\mu q^\nu}{\left(k^2+q^2\alpha(1-\alpha)-m^2\right)^2}, \quad (3.15)$$

where once again we have omitted terms linear in k, which vanish by symmetric integration.

From eq.(3.10) it is sufficient to extract only the terms in the above integral which are proportional to $g^{\mu\nu}$. Using eqs.(2.6), (2.8) and (2.9) we have (setting $d=4-2\varepsilon$)

$$\begin{aligned}-q^2\Pi(q^2) \ &= \ -\frac{e^2}{4\pi^2}\int_0^1 d\alpha\left(\frac{4\pi\mu^2}{m^2-q^2\alpha(1-\alpha)}\right)^\varepsilon\left[\Gamma(\varepsilon-1)\left(\frac{1}{2}(4-2\varepsilon)-1\right)\left(q^2\alpha(1-\alpha)-m^2\right)\right.\\ &\left.\quad -\Gamma(\varepsilon)\left(q^2\alpha(1-\alpha)+m^2\right)\right] \qquad (3.16)\end{aligned}$$

Using

$$\Gamma(\varepsilon-1) = -\frac{\Gamma(\varepsilon)}{(1-\varepsilon)},$$

it can be seen that the RHS of eq.(3.16) becomes proportional to q^2, so we have

$$\Pi(q^2) \ = \ -\frac{e^2}{2\pi^2}\Gamma(\varepsilon)\int_0^1 d\alpha\alpha(1-\alpha)\left(\frac{4\pi\mu^2}{m^2-q^2\alpha(1-\alpha)}\right)^\varepsilon \quad (3.17)$$

Expanding in ε up to terms which vanish as $\varepsilon\to 0$, and performing the integral over α where appropriate, this gives

$$\Pi(q^2) \ = \ -\frac{e^2}{12\pi^2}\left[\frac{1}{\varepsilon}+\ln(4\pi)-\gamma_E-6\int_0^1 d\alpha\alpha(1-\alpha)\ln\left(\frac{m^2-q^2\alpha(1-\alpha)}{\mu^2}\right)\right] \quad (3.18)$$

We define Z_3 to be $1+\Pi(0)$, so that we have (in terms of the fine-structure constant, α)

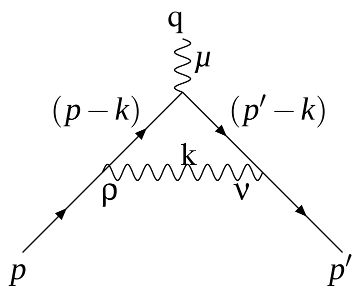
$$Z_3 \ = 1-\frac{\alpha}{3\pi}\left[\frac{1}{\varepsilon}+\ln(4\pi)-\gamma_E-\ln\left(\frac{m^2}{\mu^2}\right)\right], \quad (3.19)$$

and the renormalized photon self energy

$$\Pi_R(q^2) \ = \ \Pi(q^2)-(Z_3-1),$$

is proportional to q^2 so that it vanishes as the photon goes on mass-shell.

3.3 The Vertex Function



The fermions have momenta p and p' and the photon has momentum $q = p' - p$. In general, we will have processes (such as Compton scattering of photons off electrons) in which one of the fermion legs are internal and therefore off-shell. Here we restrict ourselves to fermion scattering in which both the fermion legs are on-shell, i.e. $p^2 = p'^2 = m^2$.

Using the Feynman rules the vertex correction factor, $\Gamma^\mu(p,p')$ is given (in d-dimensions and in Feynman gauge) by

$$\Gamma^\mu(p,p') \;=\; \mu^{4-d}\int\frac{d^dk}{(2\pi)^d}(-ie\gamma^\nu)\frac{i(\gamma\cdot(p'-k)+m)}{((p'-k)^2-m^2)}(\gamma^\mu)\frac{i(\gamma\cdot(p-k)+m)}{((p-k)^2-m^2)}(-ie\gamma^\rho)\frac{-ig_{\nu\rho}}{(k^2-\lambda^2)}, \tag{3.20}$$

where we have introduced a small photon mass λ in anticipation of the fact that we will also have infrared divergences here.

Introducing Feynman parameters (see eq.(1.4)), this may be written

$$\Gamma^\mu(p,p') \;=\; -i2e^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha d\beta d\gamma\frac{\delta(1-\alpha-\beta-\gamma)\mathcal{N}}{(k^2-2k\cdot(p\alpha+p'\beta)-\lambda^2\gamma)^3}, \tag{3.21}$$

where $\mathcal{N}$ is the numerator

$$\mathcal{N} \;=\; \gamma^\nu(\gamma\cdot(p'-k)+m)\gamma^\mu(\gamma\cdot(p-k)+m)\gamma_\nu$$

Shift $k \to k + p\alpha + p'\beta$ (and perform the integral over γ), to get

$$\Gamma^{\mu}(p,p') \;=\; -i2e^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha d\beta d\gamma\frac{\theta(1-\alpha-\beta)(\mathcal{N}_2+\mathcal{N}_0)}{(k^2-\lambda^2(1-\alpha-\beta)-m^2(\alpha+\beta)^2+q^2\alpha\beta)^3}, \tag{3.22}$$

where we have made use the on-shell condition of the fermions and written $p\cdot p' = m^2 - q^2/2$.

$$\mathcal{N}_2 \;=\; k_\rho k_\sigma \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\gamma_\nu,$$

is the part of the integral which will give an ultraviolet divergence. Using eq.(2.8), the contribution to the vertex correction function from this part is

$$\begin{aligned}\Gamma^{\mu div}(p,p') \;&=\; \frac{e^2}{32\pi^2}\Gamma(\varepsilon)\int d\alpha d\beta\theta(1-\alpha-\beta)\gamma^\nu\gamma^\sigma\gamma^\mu\gamma^\sigma\gamma_\nu\left(\frac{4\pi\mu^2}{(m^2(\alpha+\beta)^2-q^2\alpha\beta)}\right)^{\varepsilon}\\ &=\; \frac{e^2}{8\pi^2}\Gamma(\varepsilon)\int d\alpha d\beta\theta(1-\alpha-\beta)\gamma^\mu(1-\varepsilon)^2\left(\frac{4\pi\mu^2}{(m^2(\alpha+\beta)^2-q^2\alpha\beta)}\right)^{\varepsilon} =\\ \frac{e^2}{16\pi^2}\gamma^\mu&\left[\frac{1}{\varepsilon}+\ln(4\pi)-\gamma_E-2-2\int d\alpha d\beta\theta(1-\alpha-\beta)\ln\left(\frac{(m^2(\alpha+\beta)^2-q^2\alpha\beta)}{\mu^2}\right)\right]\end{aligned} \tag{3.23}$$

The $\mathcal{N}_0$ term does not lead to an ultraviolet divergence and may be calculated in four dimensions † . We have

$$\mathcal{N}_0 \;=\; \gamma^\nu(\gamma\cdot p'(1-\beta)-\gamma\cdot p\alpha+m)\gamma^\mu(\gamma\cdot p(1-\alpha)-\gamma\cdot p'\beta+m)\gamma_\nu,$$

which we may write as

$$\mathcal{N}_0 \;=\; -2(\gamma\cdot p(1-\alpha)-\gamma\cdot p'\beta)\gamma^\mu(\gamma\cdot p'(1-\beta)-\gamma\cdot p\alpha)-2m^2\gamma^\mu+4m(1-\alpha-\beta)(p+p')^\mu,$$

where we have used the symmetry under $\alpha\leftrightarrow\beta$.

We can consider $\mathcal{N}_0$ to be sandwiched between fermion spinors $\bar{u}(p',m)$ and $u(p,m)$. We have the identity

$$(p+p')^\mu \;=\; \frac{1}{2}\{\gamma\cdot(p+p'),\gamma^\mu\} \;=\; \gamma\cdot p'\gamma^\mu+\gamma^\mu\gamma\cdot p-\frac{1}{2}q^\nu[\gamma^\nu,\gamma^\mu] \;=\; 2m\gamma^\mu+iq_\nu\sigma^{\mu\nu}$$

where in the last step we have used the fact that $\gamma\cdot p'$ on the left or $\gamma\cdot p$ on the right generates m since they are adjacent to fermion spinors. The matrices $\sigma^{\mu\nu}$ are the generators of Lorentz transformations in the spinor representation

$$\sigma^{\mu\nu} = -\frac{i}{2}[\gamma^\mu,\gamma^\nu]$$

Furthermore, we have

$$\gamma\cdot q\gamma^\mu\gamma\cdot q \;=\; 2q^\mu\gamma\cdot q+q^2\gamma^\mu.$$

† Once again, the numerator term $\mathcal{N}_1$, which is linear in k after shifting, is omitted since it vanishes by symmetric integration.

The first term vanishes when sandwiched between fermion spinors leaving only the term $q^2\gamma^\mu$.

Using these relations the numerator $\mathcal{N}_0$ becomes

$$\begin{aligned}\mathcal{N}_0 &= \left(8(1-\alpha-\beta)-2-2(1-\alpha-\beta)^2)m^2-2(1-\alpha)(1-\beta)q^2\right)\gamma^\mu \\ &\quad +4im\left((1-\alpha-\beta)-(1-\alpha)(1-\beta)\right)q_\nu\sigma^{\mu\nu}.\end{aligned}$$

The counterterm associate wih the vertex is $(Z_1-1)\gamma^\mu$, so we write

$$\Gamma^\mu_R = \Gamma^\mu+(Z_1-1)\gamma^\mu,$$

where Γ^μ_R is the finite renormalized vertex correction.

The usual definition of the renormalized electromagnetic coupling is the coupling at zero momentum transfer. In other words we must choose the renormalization constant Z_1 such that $\Gamma^\mu_R(p,p)=0$, so that we get for

$$\begin{aligned}Z_1 &= 1+\frac{e^2}{16\pi^2}\left[-\frac{1}{\varepsilon}-\ln(4\pi)+\gamma_E-\ln\left(\frac{\mu^2}{m^2}\right)+2\right. \\ &\quad \left.+4\int d\alpha d\beta\theta(1-\alpha-\beta)\left\{\ln(\alpha+\beta)+m^2\frac{2(1-\alpha-\beta)-1-(1-\alpha-\beta)^2}{m^2(\alpha+\beta)^2+\lambda^2(1-\alpha-\beta)^2}\right\}\right]. \qquad (3.24)\end{aligned}$$

The last integral has an infrared divergence as $\lambda\to 0$

The nested integral over α and β is most easily performed by the change of variables

$$\begin{aligned}\alpha &= \rho\omega \\ \beta &= \rho(1-\omega)\end{aligned}$$

The range of ρ and ω are now both from 0 to 1, and there is a factor of ρ from the jacobian. The integrand depends only of ρ so the integral over ω just gives a factor of unity. We now have (in terms of the fine-structure constant, α)

$$\begin{aligned}Z_1 &= 1+\frac{\alpha}{4\pi}\left[-\frac{1}{\varepsilon}-\ln(4\pi)+\gamma_E-\ln\left(\frac{\mu^2}{m^2}\right)+2+4\int_0^1\rho d\rho\left\{\ln(\rho)+\frac{2(1-\rho)-1-(1-\rho)^2}{\rho^2+(1-\rho)\lambda^2/m^2}\right\}\right] \\ &= 1+\frac{\alpha}{4\pi}\left[-\frac{1}{\varepsilon}-\ln(4\pi)+\gamma_E-\ln\left(\frac{\mu^2}{m^2}\right)-4+2\ln\left(\frac{m^2}{\lambda^2}\right)\right] \qquad (3.25)\end{aligned}$$

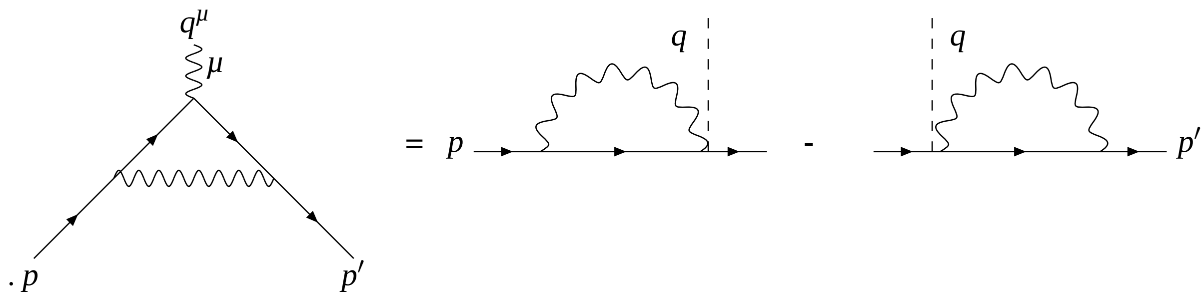
Examination of eq.(3.9) shows that we have

$$Z_1 = Z_2.$$

This is to be expected fropm another Ward identity. As in the case of the photon propagator, we can write

$$\frac{1}{\gamma\cdot(p'-k)-m}\gamma\cdot q\frac{1}{\gamma\cdot(p-k)-m} = \frac{1}{\gamma\cdot(p-k)-m}-\frac{1}{\gamma\cdot(p'-k)-m}$$

which lead diagrammatically to the "Ward identity"



The dashed line just represents the insertion of the momentum q into the fermion line.

Thus we see a relation between the vertex correction and the self-energy of the fermion. One of the consequences of this is that the divergent parts are related such that

$$q_{\mu}\Gamma^{\mu div} = \Sigma^{div}(p) - \Sigma^{div}(p')$$

The LHS is $(Z_1 - 1)q \cdot \gamma$. The mass renormalization cancels out from the two terms on the RHS and we are left with $Z_2(p' \cdot \gamma - p \cdot \gamma)$.

The renormalized vertex function has a term proprtional to γ^{μ} and a term proprtional to $q_{\nu}\sigma^{\mu\nu}$, and may be written as

$$\Gamma^{\mu}_{R} = \gamma^{\mu}F_1(q^2) + \frac{i}{2m}q_{\nu}\sigma^{\mu\nu}F_2(q^2) \tag{3.26}$$

The functions F_1 and F_2 which depend on q^2 for the case of on-shell fermion legs are known as the "electric" and "magnetic" form factors respectively.

The exact expressions for F_1 and F_2 are very complicated, but simplify in the limits $q^2 \gg m^2$ and $q^2 \ll m^2$. For $q^2 \gg m^2$ we have:

$$F_1(q^2) \rightarrow 1 - \frac{\alpha}{2\pi}\left[\left(\ln\left(\frac{-q^2}{m^2}\right) - 1\right)\ln\left(\frac{m^2}{\lambda^2}\right) + \ln\left(\frac{-q^2}{m^2}\right) - 2\right]$$

$$F_2 \sim \frac{q^2}{m^2}$$

For $q^2 \ll m^2$ we have:

$$F_1(q^2) \rightarrow 1 + \frac{\alpha}{3\pi}\frac{q^2}{m^2}\left[\frac{1}{2}\ln\left(\frac{m^2}{\lambda^2}\right) - \frac{3}{8}\right]$$

$$F_2(q^2) \rightarrow \frac{\alpha}{2\pi}$$

The magnetic form-factor in the limit $q^2 \to 0$ acts as a correction to the magnetic moment of the electron, μ. In leading order

$$\mu = g_s \frac{e}{2m},$$

with $g_s = 2$. But in higher order

$$g_s - 2 \;=\; \frac{\alpha}{2\pi} + \cdots.$$

This has now been measured for the muon up to one part in 10^9 and calculated in QED up to five loops. The calculation up to three loops in QED agrees with experiment. There has recently been reported a two standard deviation discrepancy between the experimental observation and the theoretically calculated value. This discrepancy is assumed to be evidence for physics beyond the Standard Model rather than a breakdown of the validity of QED.

3.4 Ward Identities

The fact that $q_\mu \Pi^{\mu\nu}(q^2) = 0$ and $Z_1 = Z_2$ are examples of Ward identities derived from the fact that the interaction Hamiltonian density may be written $eA_\mu(x)\, j^\mu(x)$, where $j^\mu(x)$ is the conserved electromagnetic current, i.e. $\partial_\mu j^\mu = 0$.

The photon propagator, $G^{\mu\nu}(q^2)$, can always be written as the tree-level contribution, $G_0^{\mu\nu}(q^2)$ plus a correction which may be expressed as the tree-level propagator multiplying the vacuum expectation value of the time-ordered product of the electromagnetic current and the photon field. This is because the first interaction of the free photon is always with the electromagnetic current, i.e.

$$G^{\mu\nu}(q^2) \;=\; G_0^{\mu\rho}(q^2)\left[g^{\rho\nu} - i\int d^4x e^{-iq\cdot x}\langle 0|T\, j_\rho(x) A_\nu(0)|0\rangle\right], \tag{3.27}$$

where, in a general gauge, the tree-level propagator is

$$G_0^{\mu\nu}(q^2) \;=\; -i\frac{\left(g^{\mu\nu} - \xi\frac{q^\mu q^\nu}{q^2}\right)}{q^2}$$

$$q_\mu G_0^{\mu\rho}(q^2) \;=\; -i\xi\frac{q^\rho}{q^2}$$

$$\begin{aligned}
-iq^\rho\int d^4x e^{-iq\cdot x}\langle 0|T\, j_\rho(x) A_\nu(0)|0\rangle \;&=\; -\int d^4x e^{-iq\cdot x}\frac{\partial}{\partial x_\rho}\langle 0|T\, j_\rho(x) A_\nu(0)|0\rangle \\
&=\; -\int d^4x e^{-iq\cdot x}\langle 0|\left[j_0(x), A_\nu(0)\right]|0\rangle\delta(x_0) \;=\; 0
\end{aligned}$$

where the last term arises because the derivative w.r.t. x_ρ has to act on the time ordering operator T giving rise to $\delta(x_0)$, as well as acting on the current, giving a term which vanishes by current conservation. The result is zero because the electromagnetic current and the photon field commute.

This then implies that

$$q_\mu G^{\mu\nu}(q^2) \;=\; q_\mu G_0^{\mu\nu}(q^2),$$

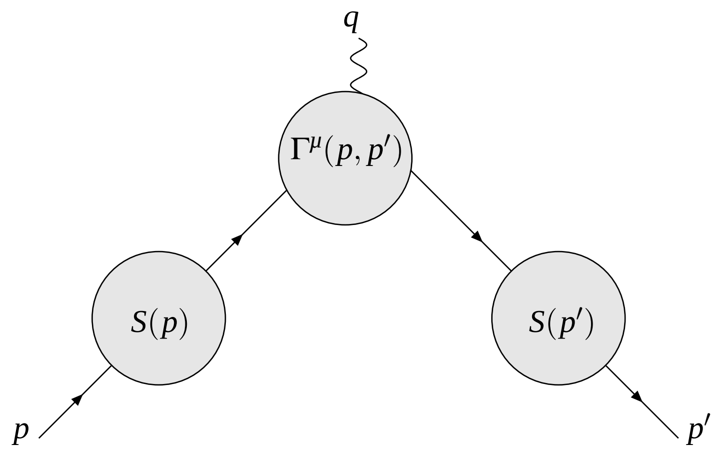
i.e. the longitudinal part of the photon propagator does not acquire higher order corrections to any order in perturbation theory.

We now apply the same technique to the quantity

$$\int d^4x d^4y d^4z e^{i(p'\cdot z - p\cdot y - q\cdot x)}\langle 0|T\, j_\mu(x)\Psi(y)\overline{\Psi}(z)|0\rangle \;=\; S(p')\Gamma^\mu(p,p')S(p)(2\pi)^4\delta^4(p+q-p'),$$

where $S(p)$ is the full electron propagator

$$S^{-1}(p) \;=\; -i(\gamma\cdot p - m - \Sigma(p))$$



Here the quantity denoted as Γ^μ is the one-particle-irreducible vertex calculated to all orders and includes the tree level term $e_R\gamma^\mu$.

Contracting with q_μ,

$$q_\mu S(p')\Gamma^\mu(p,p')S(p)(2\pi)^4\delta^4(p+q-p') \;=\; i\int d^4xd^4yd^4ze^{i(p'\cdot z-p\cdot y-q\cdot x)}\frac{\partial}{\partial x_\mu}\langle 0|T\,j^\mu(x)\Psi(y)\overline{\Psi}(z)|0\rangle$$

Since $\partial_\mu j^\mu = 0$, we only pick up the contributions from the derivative acting on the time-ordering operator so this gives

$$\int d^4xd^4yd^4ze^{i(p'\cdot z-p\cdot y-q\cdot x)}\left\{\langle 0|T\left[j_0(x),\Psi(y)\right]\overline{\Psi}(z)|0\rangle\delta(x_0-y_0)\right.$$
$$\left.-\langle 0|T\left[j_0(x),\overline{\Psi}(z)\right]\Psi(y)|0\rangle\delta(x_0-z_0)\right\},$$

where the relative minus sign arises from commuting two fermion fields. Using

$$j_0(x) \;=\; \Psi^\dagger(x)\Psi(x)$$

and

$$\left\{\Psi(x),\Psi^\dagger(y)\right\}\delta(x_0-y_0) \;=\; \delta^4(x-y)$$

the commutation relations give

$$\left[j_0(x),\Psi(y)\right]\delta(x_0-y_0) \;=\; -\Psi(y)\delta^4(x-y)$$
$$\left[j_0(x),\overline{\Psi}(z)\right]\delta(x_0-z_0) \;=\; -\overline{\Psi}(z)\delta^4(x-y).$$

Integrating over x to absorb the $\delta-$functions, we get

$$\begin{aligned} q_\mu S(p')\Gamma^\mu(p,p')S(p)(2\pi)^4\delta^4(p+q-p') \;&=\\ -i\int d^4yd^4ze^{-i\frac{1}{2}(q-p'+p)\cdot(z+y)}&\left[e^{ip'\cdot(z-y)}\langle 0|T\Psi(y)\overline{\Psi}(z)|0\rangle - e^{ip\cdot(y-z)}\langle 0|T\Psi(y)\overline{\Psi}(z)|0\rangle\right]\\ &=\; i\left(S(p)-S(p')\right)(2\pi)^4\delta^4(p+q-p') \end{aligned} \tag{3.28}$$

(here Γ^μ includes the tree diagram γ^μ as well as all higher order corrections).

Dividing both sides the the external fermion propagators, this gives

$$q_\mu \Gamma^\mu(p,p') \;=\; i\left(S^{-1}(p') - S^{-1}(p)\right) \;=\; \left(\Sigma(p) - \Sigma(p') + \gamma\cdot p - \gamma\cdot p'\right). \tag{3.29}$$

This identity is clearly obeyed in leading order, where it becomes

$$\gamma\cdot q \;=\; \left(\gamma\cdot p' - m\right) - \left(\gamma\cdot p - m\right).$$

We have shown that this works explicitly at the one-loop level. The above derivation establishes the result to all orders in perturbation theory.

For very small momentum transfer $q_\mu \to 0$ the identity reduces to

$$\Gamma^\mu(p,p) \;=\; -\frac{\partial}{\partial p_\mu}\left(\Sigma(p) - \gamma\cdot p\right).$$

Before renormalization, $\Gamma^\mu(p,p)$ (recall that this includes the tree-level contribution) is $Z_1\gamma^\mu$ so we have:

$$\frac{\gamma^\mu}{Z_1} \;=\; \gamma^\mu\left(1 - \frac{(Z_2-1)}{Z_2}\right),$$

where we have written

$$\Sigma(p) = \frac{1}{Z_2}\left(Z_2\delta m + (Z_2-1)(\gamma\cdot p - m)\right) + O((\gamma\cdot p - m)^2).$$

We have thus established the relation

$$Z_1 \;=\; Z_2, \tag{3.30}$$

to all orders in perturbation theory.

Had we calculated Z_1 and Z_2 in a different gauge we would have obtained different values. Z_1 and Z_2 do not themselves correspond to physically measurable quantities and may therefore be gauge dependent, but we always have the relation $Z_1 = Z_2$. Z_3 is gauge invariant. Piecing this together we therefore have the fact that the bare coupling, which is related to the renormalized coupling simply by $e_0 = \sqrt{Z_3}e_R$, is gauge invariant.

3.5 Finite Renormalization

We have defined the renormalized electromagnetic coupling constant to be the value of the coupling of an electron to a zero momentum photon. This is a sensible definition but it is *not* unique.

We could have chosen a different experiment, e.g. e^+e^- scattering at the threshold $s = 4m^2$, to determine the coupling.

Alternatively, we could have chosen a definition which did not directly correspond to a real experiment at all. For example, we could have made $\Gamma^\mu(p, p')$ finite by subtracting the contribution from the Feynman graph at an unphysical point where all three external legs had square momentum $p^2 = -\mu^2$. Furthermore, we could have subtracted the infinities in the electron propagator by defining

$$\begin{aligned}\delta m &= \Sigma(p,m)_{|\gamma\cdot p=i\mu} \\ (Z_2-1) &= \frac{\partial}{\partial(\gamma\cdot p)}\Sigma(p,m)_{|\gamma\cdot p=i\mu}\end{aligned}$$

and for the photon propagator

$$(Z_3-1) = \Pi(-\mu^2).$$

Such definitions would be sufficient to subtract all the infinities rendering the renormalized Green-functions finite.

Such a renormalization scheme has the following consequences:

- The renormalized coupling constant does not correspond to a physical quantity. All such quantities (including zero momentum transfer potential scattering) would have to be calculated in terms of the renormalized coupling e_R defined in this scheme.
- The renormalized fermion self-energy would not be proportional to $(\gamma\cdot p - m)^2$, but would have the form
$$-\Delta m + \Delta Z_2(\gamma\cdot p - m) + O((\gamma\cdot p - m)^2),$$
where Δm and ΔZ_2 are finite. The physical mass (position of the pole of the propagator) would not be at $m_R = m + \delta m$ but at $m_R + \Delta m$.
- Z_2 would not be the same as the Z_2 which appears in the LSZ reduction formula for the S-matrix elements, but would differ from it by a finite amount.

Nevertheless, such unphysical definitions of the counterterms are often useful.

- QED provides a natural definition of the physical coupling. But in other field theories, such as those with massless gauge particles, no such physical definition arises naturally.
- It may not always be possible to perform "physical" renormalizations for all masses and couplings in a given theory without introducing counterterms that violate the internal symmetries of the theory. Spontaneously broken gauge theories in which the gauge-bosons acquire different masses is an example of this. One cannot perform on-shell subtractions for each the gauge bosons, because gauge invariance only allows one wavefunction renormalization constant for *all* of the gauge-bosons.

- A simpler way of defining counterterms can help higher order calculations.
- General renormalizations introduce a subtraction scale μ. The renormalized Green functions depend explicitly on μ, but so do the renormalized parameters e_R and m_R in such a way that the physical S-matrix elements are μ independent. This can be used to obtain information about the behaviour of renormalized Green functions as the momenta are scaled up or down.

Dimensional regularization introduces a scale μ associated with the dimension of the coupling constant outside four dimensions. A simple and practical renormalization prescription is to define the couterterms to be the pole parts of any given graph. This is the "MS" scheme - we can also use the "$\overline{MS}$" scheme in which the countertems consist of the pole part along with the $\ln(4\pi)-\gamma_E$ that always accompanies it. Such a renormalization automatically generates counterterms which generate the (dimensionality independent) symmetries of the theory.

In the $\overline{MS}$ scheme we have, for the fermion propagator

$$\begin{aligned}\delta m &= \frac{3\alpha}{4\pi} m \left[\frac{1}{\varepsilon} + \ln(4\pi) - \gamma_E\right] \\ (Z_2 - 1) &= \frac{\alpha}{4\pi}\left[-\frac{1}{\varepsilon} - \ln(4\pi) + \gamma_E\right]\end{aligned}$$

and the renormalized propagator is

$$\Sigma_R(p,m) = -\frac{\alpha}{4\pi}\left[(\gamma\cdot p - 2m) + \int_0^1 d\alpha\,(2(1-\alpha)\gamma\cdot p - 4m)\ln\left(\frac{m^2\alpha - p^2\alpha(1-\alpha)}{\mu^2}\right)\right]$$

We again have $Z_1 = Z_2$. This is obeyed exactly because the $\overline{MS}$ renormalization scheme preserves the gauge invariance. However, for a general "unphysical" renormalization scheme only the infinite (pole) parts would necessarily obey this relation.

For the photon propagator in the $\overline{MS}$ scheme we have

$$(Z_3 - 1) = -\frac{\alpha}{3\pi}\left[\frac{1}{\varepsilon} + \ln(4\pi) - \gamma_E\right]$$

with the renormalized propagator

$$\Pi_R(q^2) = \frac{2\alpha}{\pi}\left[\int_0^1 d\alpha\alpha(1-\alpha)\ln\left(\frac{m^2 - q^2\alpha(1-\alpha)}{\mu^2}\right)\right]$$

($\Pi_R(q^2)$ does *not* vanish as $q^2 \to 0$ in this scheme and nor does $\Sigma_R(p,m)$ vanish as $\gamma\cdot p \to m$).

In dimensional regularization the dimensionless renormalized coupling, α_R, ($\equiv g_R^2/(4\pi)$) is related to the bare coupling α_0 by the relation

$$\alpha_B = \mu^{2\varepsilon}\alpha_R\left(1 + \beta_0\frac{\alpha_R}{4\pi}\frac{1}{\varepsilon} + \cdots\right), \tag{3.31}$$

so that α_R is an implicit function of the scale μ - despite the fact that it is dimensionless.