6 Unitarity, Causality and Analyticity

The propagator for a scalar particle can be written in terms of a "dispersion relation" sometimes called the "Källén-Lehmann representation"

$$-i\Delta_F(q^2) \equiv \int \frac{d^4x}{(2\pi)^4} e^{iq\cdot x} \langle 0|T\phi(0)\phi(x)|0\rangle = i\int \frac{\rho(\sigma^2)d\sigma^2}{q^2-\sigma^2+i\varepsilon}. \tag{6.1}$$

Taking the imaginary part we have

$$\Im m\left\{\Delta_F(q^2)\right\} = \pi \int \rho(\sigma^2)\delta(q^2-\sigma^2)d\sigma^2 = \rho(q^2).$$

The interpretation of the "spectral function" $\rho(q^2)$ is that it is the probability for a one particle state with square momentum q^2 to decay into all possible (energetically allowed) final states

$$\rho(\sigma^2) = (2\pi)^3 \sum_n \delta^4(p_n-\sigma)|\langle 0|\phi(0)|n\rangle|^2,$$

where p_n is the total momentum of the particles in the state $|n\rangle$ (this is seen by inserting a complete set of states $\sum_n |n\rangle\langle n|$ between the fields in (6.1)).

The vacuum expectation value of the commutator of two fields may also be related to this spectral function

$$\begin{aligned}
\Delta(x) &\equiv \langle 0|\left[\phi(0),\phi(x)\right]|0\rangle = \sum_n |\langle 0|\phi(0)|n\rangle|^2\left(e^{-ip_n\cdot x}-e^{+ip_n\cdot x}\right) \\
&= \frac{1}{(2\pi)^3}\int d^4q(2\pi)^3\sum_n \delta^4(p_n-q)|\langle 0|\phi(0)|n\rangle|^2\left(e^{-ip_n\cdot x}-e^{+ip_n\cdot x}\right) \\
&= \frac{1}{(2\pi)^3}\int d^4q d\sigma^2\rho(\sigma^2)\delta(\sigma^2-q^2)\left(e^{-iq\cdot x}-e^{+iq\cdot x}\right)
\end{aligned} \tag{6.2}$$

Performing the integration over the energy component q_0 this becomes

$$-i\int d\sigma^2\rho(\sigma^2)\int\frac{d^3\mathbf{q}}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}}\frac{\sin(\sqrt{\mathbf{q}^2+\sigma^2}t)}{\sqrt{\mathbf{q}^2+\sigma^2}}.$$

The integral over $\mathbf{q}$ can be performed and a result given in terms of Bessel functions, which can be shown to vanish if $|\mathbf{x}| > t$. Actually we can see by inspection that for $t = 0$ this integral vanishes for any non-zero $|\mathbf{x}|$ and so by Lorentz invariance it must always vanish if the four-vector x is space-like. This result is expected from causality - it tells us that the commutator of two fields vanishes if the arguments of the fields are separated by a space-like quantity.

The above argument can be inverted to show that causality implies that the propagator is analytic in the upper half of the plane in q^2 (this explains the sign of the $i\varepsilon$ term in the denominator of eq.(6.1)).

The argument can be extended to show that causality implies that all scattering amplitudes are analytic in the upper half complex plane for all dynamical variables.

Unitarity further implies that scattering amplitudes are analytic in a plane which is cut along the real axis.

Define the $\mathcal{T}$-matrix from the $\mathcal{S}$-matrix by

$$\mathcal{S}_{ab} \equiv \langle a_{out}|b_{in}\rangle = \delta_{ab} + i\mathcal{T}_{ab}(2\pi)^4\delta^4(p_a - p_b)$$

then the unitarity of the $\mathcal{S}$-matrix, $\mathcal{S}\mathcal{S}^\dagger = 1$ gives

$$\mathcal{T}_{ab} - \mathcal{T}^*_{ba} = i\sum_n (2\pi)^4\delta^4(p_a - p_n)\mathcal{T}_{an}\mathcal{T}^*_{bn} \quad (6.3)$$

The sum over n means that for each possible final state c, consisting of a certain set of final state particles, we must integrate over the whole of the available phase-space. Putting $a = b$ we have an expression for the imaginary part of the forward amplitude, known as the "optical theorem"

$$\Im m\{\mathcal{T}_{aa}\} = \frac{1}{2}\int\sum_c |\mathcal{T}_{ac}|^2 d\{P.S.\} \quad (6.4)$$

The RHS is proportional to the total probability for the state $|a\rangle$ to propagate into some other state $|c\rangle$. If $|a\rangle$ is a two-body state with masses m_1 and m_2 then

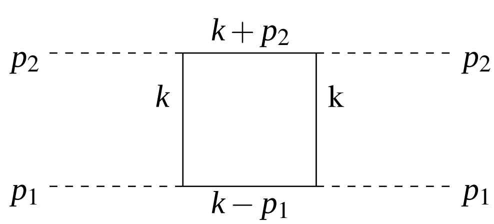
$$\Im m\{\mathcal{T}_{aa}\} = \lambda^{1/2}(s, m_1^2, m_2^2)\,\sigma^{(a)}_{TOT}(s) \quad (6.5)$$

where

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yx.$$

As an example we consider the forward scattering of two massless scalar particles (ϕ) theory interacting via a cubic interaction, $\frac{1}{2}g\phi\chi^2$ with a massive scalar field (χ).

The forward scattering amplitude is calculated from the graph



(the solid line represents the χ particles which have mass m and the dashed lines the massless external particles).

The contribution from this graph (to the $\mathcal{T}$-matrix) is

$$-ig^4\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2 - m^2 + i\varepsilon)^2((p_1 - k)^2 - m^2 + i\varepsilon)((p_2 + k)^2 - m^2 + i\varepsilon)}$$

Feynman parametrizing gives

$$-i6g^4\int\frac{d^4k}{(2\pi)^4}\int d\alpha d\beta d\gamma d\delta\frac{\delta(1-\alpha-\beta-\gamma-\delta)}{(k^2-m^2-2k\cdot(p_1\alpha-p_2\beta)+i\varepsilon)^4}$$

Shift $k\to k+p_1\alpha-p_2\beta$ and make use of the relations $p_1^2=p_2^2=0,\ p_1\cdot p_2=s/2$

$$-i6g^4\int\frac{d^4k}{(2\pi)^4}\int_0^1 d\alpha d\beta d\gamma d\delta\frac{\delta(1-\alpha-\beta-\gamma-\delta)}{(k^2-m^2+s\alpha\beta+i\varepsilon)^4}$$

Now integrate over k to give

$$\frac{g^4}{16\pi^2}\int d\alpha d\beta d\gamma d\delta\frac{\delta(1-\alpha-\beta-\gamma-\delta)}{(m^2-s\alpha\beta-i\varepsilon)^2}$$

Integrating over δ and then over β this gives

$$\frac{g^4}{16\pi^2}\int_0^1 d\alpha d\gamma\frac{\theta(1-\alpha-\gamma)}{s\alpha}\left[\frac{1}{m^2-s\alpha(1-\alpha-\gamma)-i\varepsilon}-\frac{1}{m^2-i\varepsilon}\right] \tag{6.6}$$

The second term in square parenthesis has no imaginary part. The imaginary part of the first term is

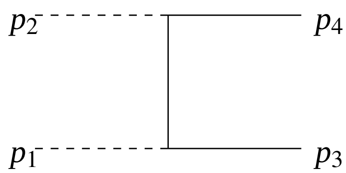
$$\begin{aligned}
&\frac{g^4}{(16\pi)}\int d\alpha d\gamma\frac{1}{s\alpha}\delta(m^2-s\alpha(1-\alpha-\gamma)\theta(1-\alpha-\gamma)\\
=\ &\frac{g^4}{16\pi}\int_0^1\frac{d\rho d\omega}{s\omega}\delta(m^2-s\omega\rho(1-\rho))\\
=\ &\frac{g^4}{16\pi s m^2}\sqrt{1-\frac{4m^2}{s}}
\end{aligned} \tag{6.7}$$

This imaginary part only exists if $s>4m^2$, which is the physical threshold for the production of two $\chi-$ particles in the intermediate state. Note that the maximum value of $\omega\rho(1-\rho)$ is $\frac{1}{4}$, and that $\sqrt{1-4m^2/s}$ is the range in ρ over which we can pick up a zero of the $\delta-$function when integrating over ω.

Now we compare this with the cross-section for the process:

$$\phi+\phi\to\chi+\chi$$

The tree-level amplitude for this process is obtained from the Feynman graph



The amplitude from this graph is

$$\frac{g^2}{(t-m^2)} \quad (t=(p_1-p_3)^2)$$

From this we get the total cross-section to be the phase-space integral

$$\sigma \;=\; \frac{1}{2\lambda^{1/2}(s,0,0)}\int \frac{d^3p_3}{(2\pi)^3\,2E_3}\frac{d^4p_4}{(2\pi)^4}(2\pi)\delta(p_4^2-m^2)\left(\frac{g^2}{(t-m^2)}\right)^2(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)$$

We carry out the phase space integral in the C.M. frame of p_1 and p_2, for which

$$\begin{aligned}
p_1 &= \left(\frac{\sqrt{s}}{2},0,0,\frac{\sqrt{s}}{2}\right)\\
p_2 &= \left(\frac{\sqrt{s}}{2},0,0,-\frac{\sqrt{s}}{2}\right)\\
p_3 &= \left(\frac{\sqrt{s}}{2},\frac{\sqrt{s-4m^2}\sin\theta\cos\phi}{2},\frac{\sqrt{s-4m^2}\sin\theta\sin\phi}{2},\frac{\sqrt{s-4m^2}\cos\theta}{2}\right)\\
p_4 &= \left(\frac{\sqrt{s}}{2},-\frac{\sqrt{s-4m^2}\sin\theta\cos\phi}{2},-\frac{\sqrt{s-4m^2}\sin\theta\sin\phi}{2},-\frac{\sqrt{s-4m^2}\cos\theta}{2}\right)\\
t &= \frac{\left(2m^2-s+\sqrt{s}\sqrt{s-4m^2}\cos\theta\right)}{2}\\
d^3p &= (2\pi)\frac{s}{4}dE_3\,d\cos\theta
\end{aligned}$$

The integral over E_3 is used to absorb the δ-function $\delta(p_4^2-m^2)$ and we have finally

$$\begin{aligned}
\sigma &= \frac{1}{\lambda^{1/2}(s,0,0)}\frac{g^4}{8\pi}2\int_{-1}^{1}d\cos\theta\frac{\sqrt{s}\sqrt{s-4m^2}}{\left(s-\sqrt{s}\sqrt{s-4m^2}\cos\theta\right)^2}\\
&= \frac{1}{\lambda^{1/2}(s,0,0)}\frac{g^4}{16\pi s m^2}\sqrt{1-\frac{4m^2}{s}}. \qquad (6.8)
\end{aligned}$$

Comparing this expression with (6.7) we see that we get agreement with the unitarity condition, (6.5).

6.1 Analytic Structure of scattering amplitudes

In general, we expect a scattering amplitude to be a real analytic function of its dynamical variables (e.g. s and t) except for cuts along the real axis corresponding to a physical region. A real analytic function $f(z)$ of a complex variable z, obeys the relation

$$f(z) = f^*(z^*),$$

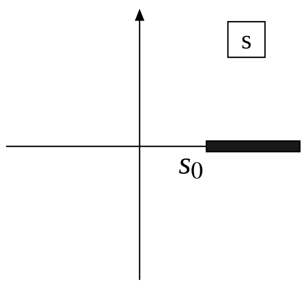
which implies

$$f(z) - f^*(z) = f(z) - f(z^*).$$

Thus the imaginary part of the forward scattering amplitude is one half the discontinuity across the cut in the complex s-plane, i.e.

$$2i\Im m\{F(s,t=0)\} \ = \ F(s+i\varepsilon,0) - F(s-i\varepsilon,0),$$

and by the optical theorem this can be deduced from the total cross-section.



If s is below the threshold for the production of intermediate state particles, s_0, the imaginary part vanishes which implies that the discontinuity vanishes. This means that the cut along the real axis starts at the physical threshold, s_0, which then becomes a branch point. Further cuts open at higher values of s as more and more physical states become energetically allowed. At each such threshold there will be a branch-point singularity.

For a general Feynman graph (for any number of loops) the amplitude, after integrating out the loop momenta, is a function of the momentum invariants, the masses, and a set of Feynman parameters.

$$\mathcal{A} \sim \int d\alpha_1 \cdots d\alpha_n \delta(1-\sum \alpha_i) \frac{1}{(J(\{\alpha_i\},\{p_j \cdot p_k\},\{m_l^2\}) + i\varepsilon}$$

At some points in the space of Feynman parameters, $\alpha_i = \alpha_i^0$, the function J will vanish. We can usually use the $i\varepsilon$ prescription to integrate through such singularities in the integrand. The exceptions are if the J is also at a turning point at the point where it vanishes, *or* any of the the Feynman parameters are at the end-points of the range of integration. At such points the contribution to the amplitude from the Feynman graph has a (branch-point) singularity.

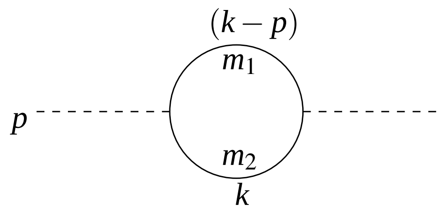
The conditions for a branch point are therefore

$$J \ = \ 0, \ \ \alpha_i = \alpha_i^0$$

$$\text{either } \frac{\partial J}{\partial \alpha_i} = 0, \text{ or } \alpha_i^0 = 0, \textit{ or } \text{ or } \alpha_i^0 = 1 - \sum_{j \neq i} \alpha_j^0.$$

Examples:

1. **Scalar propagator in cubic interaction theory:**
   We will allow the internal particles to have arbitrary masses m_1 and m_2

   The finite part of the self-energy is

   $$-\frac{g^2}{(16\pi^2)} \int_0^1 d\alpha \ln\left(m_1^2\alpha + m_2^2(1-\alpha) - p^2\alpha(1-\alpha)\right).$$

   This gives an imaginary part if the argument of the logarithm becomes negative. The minimum value of p^2 for which this can happen is when

   $$J \equiv m_1^2\alpha + m_2^2(1-\alpha) - p^2\alpha(1-\alpha) = 0$$

   and

   $$\frac{\partial J}{\partial \alpha} = m_1^2 - m_2^2 - p^2(1-2\alpha) = 0 \text{ or } \alpha = 0 \text{ or } \alpha = 1$$

   The solution to this is

   $$p^2 = (m_1 + m_2)^2, \quad \left(\alpha = \frac{m_2}{m_1 + m_2}\right)$$

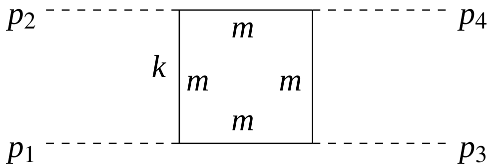
   This is the threshold for the production of two particles with masses m_1 and m_2 in the intermediate state.

   There is also a solution

   $$p^2 = (m_1 - m_2)^2, \quad \left(\alpha = -\frac{m_2}{m_1 + m_2}\right),$$

   but this (“pseudo-threshold”) is outside the range of integration of α and so we discard it.

2. **The one-loop amplitude for the scattering of four massless scalar particles (ϕ) which interact via a cubic interaction term $\frac{1}{2}g\phi\chi^2$**
   We have previously looked at the forward amplitude - we now consider the amplitude for a general square momentum transfer $t = (p_1 - p_3)^2$.



After Feynman parametrization, shifting the loop momentum and integrating over k the amplitude form this graph is

$$\frac{g^4}{(16\pi^2)}\int_0^1 d\alpha d\beta d\gamma \frac{\theta(1-\alpha-\beta-\gamma)}{(s\alpha(1-\alpha-\beta-\gamma)+t\beta\gamma-m^2+i\varepsilon)^2}$$

(we have performed the integral over the Feynman parameter δ absorbing the δ-function). The threshold is at the values of α, β, γ that obey the relations

$$J \equiv s\alpha(1-\alpha-\beta-\gamma)+t\beta\gamma-m^2 = 0$$

and

$$\frac{\partial J}{\partial \alpha} \;=\; s(1-\beta-\gamma-2\alpha)=0, \;\; \text{or } \alpha = 0, \;\; \text{or } \alpha = 1-\beta-\gamma$$

and

$$\frac{\partial J}{\partial \beta} \;=\; -s\alpha+t\gamma=0, \;\; \text{or } \beta = 0, \;\; \text{or } \beta = 1-\alpha-\gamma$$

and

$$\frac{\partial J}{\partial \gamma} \;=\; -s\alpha+t\beta=0, \;\; \text{or } \gamma = 0, \;\; \text{or } \gamma = 1-\beta-\alpha$$

This has a solution within the range of integration at

$$\alpha = \frac{1}{2}, \;\; \beta=\gamma=0, \;\; s=4m^2, \;\; t\leq 0$$

or

$$\alpha = 0, \;\; \beta=\gamma=\frac{1}{2}, \;\; t=4m^2, \;\; s\leq 0$$

The second solution is the physical threshold for the crossed (t-channel) process.

6.2 Cutkosky Rules

The discontinuity across a cut in the variable s of any Feynman graph is written as

$$\mathcal{A}(s+i\varepsilon)-\mathcal{A}(s-i\varepsilon).$$

Since by causality amplitudes are analytic in the upper-half plane we can define

$$\mathcal{S}^+_{ab} \;=\; \delta_{ab}+i(2\pi)^4\delta^4(p_a-p_b)\mathcal{T}^+_{ab} \;=\; \lim_{\varepsilon\to 0}\langle a_{out}|b_{in}\rangle_{|s+i\varepsilon}.$$

and its Hermitian conjugate

$$\mathcal{S}^-_{ab} = \delta_{ab} - i(2\pi)^4\delta^4(p_a - p_b)\mathcal{T}^-_{ab} = \lim_{\varepsilon\to 0} \langle a_{in}|b_{out}\rangle_{|s-i\varepsilon}.$$

$$\left(\mathcal{S}^+_{ab}\right)^* = \left(\mathcal{S}^-_{ba}\right).$$

The quantity $\lim_{\varepsilon\to 0}\langle a_{in}|b_{out}\rangle$ would be calculated (following the steps of the LSZ reduction formula) using the anti-time ordered product (T^*), rather than the time-ordered product in the Green functions. For, example, for the two-point Green function of two scalar fields we have

$$\langle 0|T^*\phi(x)\phi(0)|0\rangle = \int \frac{d^4q}{(2\pi)^4} e^{iq\cdot x} \frac{-i}{q^2 - m^2 - i\varepsilon}$$

This differs form the expression for the time-ordered product by an overall and the sign of the $i\varepsilon$ prescription. The perturbative expansion for the anti-time ordered product also introduces a minus sign for every interaction vertex. Collecting all the signs we find that $\mathcal{T}^-_{ab}$ is obtained from $\mathcal{T}^+_{ab}$ by replacing $i\varepsilon$ everywhere by $-i\varepsilon$. In other words

$$\mathcal{T}^-_{ab} = \left(\mathcal{T}^+_{ab}\right)^*$$

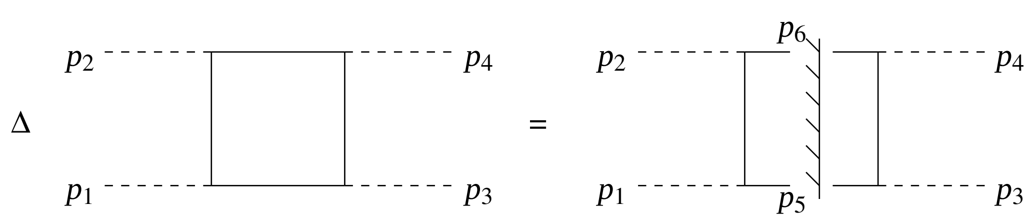
The unitarity of the $\mathcal{S}$-matrix then gives us

$$\Delta\mathcal{T}_{ab} \equiv \mathcal{T}^+_{ab} - \mathcal{T}^-_{ab} = i\sum_n (2\pi)^4\delta^4(p_a - p_n)\mathcal{T}^+_{an}\mathcal{T}^-_{nb}, \tag{6.9}$$

where Δ indicates the discontinuity across the cut . This is the generalization of the optical theorem and it is valid away from the forward direction - for example it refers to the discontinuity in the variable s for a fixed value of t away from zero.

Diagramatically the RHS of eq.(6.9) is interpreted as the sum of all cuts in the channel whose discontinuity is being considered. The part of the diagram on the right of the cut is calculated with the $i\varepsilon$ replaced by $-i\varepsilon$, the cut lines are placed on mass-shell and the phase-space integral for the cut lines is performed (this is implied in the sum σ_n).

For example, the s-channel discontinuity of the one-loop correction to the scattering of two massless scalar particles which interact with massive scalar particles



The amplitude on the left of the cut is

$$\mathcal{T}^+_{an} = \frac{g^2}{((p_1 - p_5)^2 - m^2 + i\varepsilon}$$

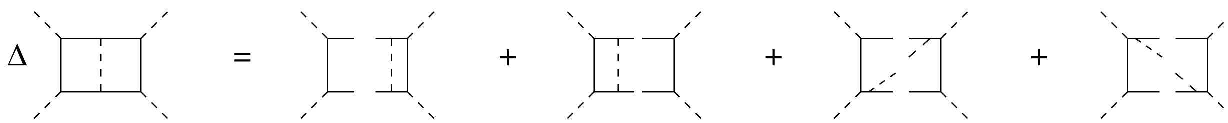
The amplitude on the right of the cut is

$$\mathcal{T}^{-}_{nb} = \frac{g^2}{((p_3 - p_5)^2 - m^2 - i\varepsilon}$$

Multiplying these together and integrating over the phase-space for the intermediate particles with momenta p_5, p_6 we get for the discontinuity across the cut

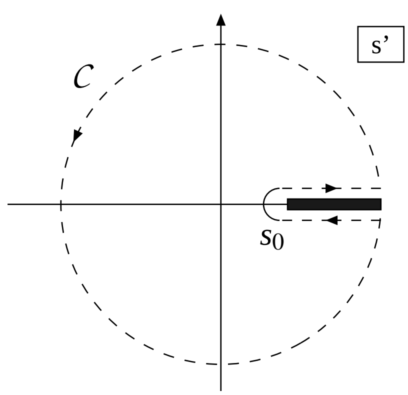
$$\int \frac{d^4 p_5}{(2\pi)^3}\delta(p_5^2 - m^2)\frac{d^4 p_6}{(2\pi)^3}\delta(p_6^2 - m^2)\frac{(2\pi)^4\delta^4(p_5 + p_6 - p_1 - p_2)\,g^4}{((p_1 - p_5)^2 - m^2 + i\varepsilon)((p_3 - p_5)^2 - m^2 - i\varepsilon)}$$

In higher order there are more cut graphs



The first two graphs on the RHS are integrated over two-body phase-space and the last two over three-body phase-space.

6.3 Dispersion Relations



If we consider the integral

$$\oint_C \frac{F(s')}{(s - s' + i\varepsilon)},$$

around the contour shown above, where $F(s)$ is some scattering amplitude (it can also be a function of t and other variables if there are more than two final-state particles), then given that $F(s)$ is analytic inside the contour, the integral is by Cauchy's theorem

$$2\pi i F(s + i\varepsilon).$$

If, furthermore, $F(s)$ goes to zero as $|s| \to \infty$ then the contour integral is just the integral over the discontinuity across the cut and is therefore equal to

$$2i \int_{s_0}^{\infty} ds' \frac{\Im m\{F(s')\}}{(s - s' + i\varepsilon)}$$

repeating this with ε replaced by $-\varepsilon$ and taking th average, we get an expression for the real part of the scattering amplitude i terms of an integral over the imaginary part.

$$\Re e\{F(s)\} = \frac{1}{\pi} \int_{s_0}^{\infty} ds' \frac{\Im m\{F(s')\}}{(s - s')}_{(PV)}, \tag{6.10}$$

where "PV" indicates that the singulaity at $s = s'$ is handled using the Principle Value prescription. This is called a "dispersion relation".

If the above integral over s' does not converge, it is necessary to introduce a subtraction and we have a subtracted dispersion relation, which gives the real part in terms of the real part at some subtraction point s_B,

$$\frac{1}{(s_B - s)} \left(\Re e\{F(s)\} - \Re e\{F(s_B)\}\right) = \frac{1}{\pi} \int_{s_B}^{\infty} ds' \frac{\Im m\{F(s')\}}{(s - s')(s_B - s')} \tag{6.11}$$

A simple example of this is the scalar propagator with equal internal masses m, in the limit $s \gg 4m^2$.

Calculating the one-loop graph we obtain the integral over the Feynman parameter, α as

$$-\frac{g^2}{16\pi^2} \int_0^1 d\alpha \ln\left(m^2 - s\alpha(1 - \alpha)\right) \tag{6.12}$$

The imaginary part is $-\pi$ times the range of α over which the argument of the logarithm is negative, which gives

$$\frac{g^2}{16\pi} \sqrt{1 - \frac{4m^2}{s}} \theta(s - 4m^2).$$

The real part is therefore given by

$$\Re e\left\{\Sigma(s, m^2)\right\} = \frac{g^2}{16\pi^2} \int_{4m^2}^{\infty} ds' \frac{\sqrt{1 - 4m^2/s'}}{(s - s')}$$

This integral diverges (it is the standard ultraviolet divergence) and so we need to subtract the dispersion relation For convenience we choose the subtraction point to be the branch-point $s = 4m^2$, to obtain

$$\begin{aligned}
\Re e\left\{\Sigma(s,m^2)\right\} - \Re e\left\{\Sigma(4m^2,m^2)\right\} &= \frac{g^2}{(16\pi^2)}(s-4m^2)\int_{4m^2}^{\infty} ds' \frac{\sqrt{(1-4m^2/s')}}{(s-s')(4m^2-s')} \\
&= -\frac{g^2}{(16\pi^2)}\sqrt{1-\frac{4m^2}{s}}\ln\left(\frac{1+\sqrt{(1-4m^2/s)}}{1-\sqrt{(1-4m^2/s)}}\right)
\end{aligned}$$

This result could also have been obtained by performing the integral over α in (6.12).

In the case of the forward scattering amplitude for the interacting scalars with equal internal masses m (eq.(6.7)), the real part of the amplitude is given by the integral

$$\begin{aligned}
& \frac{g^4}{16\pi^2 m^2}\int_{4m^2}^{\infty} ds' \frac{1}{(s'-s)s'}\sqrt{1-\frac{4m^2}{s'}} \\
= \; & \frac{g^4}{16\pi^2 s m^2}\left[\sqrt{1-\frac{4m^2}{s}}\ln\left(\frac{1+\sqrt{1-4m^2/s}}{1-\sqrt{1-4m^2/s}}\right)-2\right]
\end{aligned}$$

. This could also have been obtained by performing the integral over α and γ in (6.6).