Advanced Relativity

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1 Vector Spaces.

Special Relativity is concerned with the rules for determining a set of quantities, such as the location and time of an event or the momentum and energy of a particle, measured in one inertial reference frame compared with with the same quantities measured in a different inertial frame. These rules derive from the postulate that there are certain invariant quantities, such as the mass of a particle or the speed of light, which are the same in all reference frames. In order for the laws of physics to be the same in all inertial reference frames, as postulated in Special Relativity, all laws of physics must be expressible in terms of equalities between quantities which transform in the same way under a change in frame of reference

Non-relativistic physics also has rules which determine the transformation of certain quantities measured in different coordinate bases, also derived from the existence of invariants, which remain the same in any coordinate basis. The fact that a physical law cannot depend on the coordinates basis in which the quantities are measured means that the laws of physics must be expressible in terms of equalities between quantities which transform in the same way between different coordinate systems. “Different coordinate systems” might mean simply a rotation between coordinates so that the axes in one frame are rotated relative to another frame, or could mean using a different system for labelling a point, such as using polar coordinates rather than Cartesian coordinates.

An example of this is the statement of Newton’s second law of motion in the case of a body whose mass, m, remains unchanged

$$\mathbf{F} \;=\; m\mathbf{a} \tag{1.1}$$

This is an equation relating two vectors – force and acceleration, both of which transform in the same way under a change of coordinate system. However, the components of force and acceleration are different in different coordinate systems. In system A the three components of force are F_x, F_y, F_z and the components of acceleration are a_x, a_y, a_z, whereas in coordinate system, B which is rotated relative to A’s and may use a different origin the three components of force are F'_x, F'_y, F'_z and the components of acceleration are a'_x, a'_y, a'_z, so that the three components of (1.1) are different in the two coordinate systems although they describe the same physics. The relationship between the coordinates F'_x, F'_y, F'_z and F_x, F_y, F_z is such that the modulus of the force vector $\mathbf{F}$ is the same in both coordinate systems, so that

$$F_x^2 + F_y^2 + F_z^2 \;=\; F'^2 x + F'^2_y + F'^2_z,$$

and similarly for the acceleration vector $\mathbf{a}$.

$$a_x^2 + a_y^2 + a_z^2 \;=\; a'^2_x + a'^2_y + a'^2_z,$$

Since (1.1) (which is actually a set of three equations – one for each component, is valid in both frame A and frame B the relation between (F'_x, F'_y, F'_z) and (F_x, F_y, F_z) is exactly the same as the relation between (a'_x, a'_y, a'_z) and (a_x, a_y, a_z). Both transform as the components of a vector under a rotation, which maintains the magnitude of the vector.

1.1 Basis vectors (axes)

We now wish to generalise the description of vectors in different coordinate systems.

Consider an n-dimensional space with a set of n vectors called "basis vectors" or "axes", $\mathbf{e}_i,\ i = 1 \cdots n$, which are linearly independent, i.e. none of them can be written as a linear sum of the others. In the standard notation the basis vectors are unit vectors and are orthogonal to each other. Such examples are usually easier to deal with, but it is *not* a requirement.

Any point in the vector space can be labelled by a set of coordinates, x^i, such as the vector from the origin to the point is

$$\mathbf{x} \ = \ \sum_{i=1}^{n} x^i \mathbf{e}_i \ \equiv \ x^i \mathbf{e}_i, \tag{1.2}$$

where we have introduced the repeated index notation, namely that an expression with a superscript index i and the same subscript index implies a summation over all values of that index. This notation is used widely and henceforth will be adopted throughout.

The vector $\mathbf{ds}$ between two neighbouring points with coordinates x^i and $x^i + dx^i$ can be written as

$$\mathbf{ds} \ = \ dx^i \, \mathbf{e}_i, \tag{1.3}$$

The same points can be labelled in a different coordinate frame with basis vectors $\mathbf{e}'_j$ by components y^j and $y^j + dy^j$, so that the vector $\mathbf{ds}$ may also be written as

$$\mathbf{ds} \ = \ dy^j \, \mathbf{e}'_j, \tag{1.4}$$

This gives us the relation (locally) between the two sets of basis vectors

$$\mathbf{e}'_j \ = \ \frac{\partial x^i}{\partial y^j}, \mathbf{e}_i \tag{1.5}$$

More generally, a vector $\mathbf{V}$ may be written in the coordinate system with basis vectors $\mathbf{e}_i$ as

$$\mathbf{V} \ = \ V^i \mathbf{e}_i,$$

where V^i are the components of the vector $\mathbf{V}$. Alternatively, in the coordinate system with basis vectors $\mathbf{e}'_j$ we may write the same vector as

$$\mathbf{V} \ = \ V'^j \mathbf{e}'_j,$$

Since this is the same vector, we can use (1.5) to obtain the transformation rule between the components of the vector in the two coordinate systems .

$$V^j \ \xrightarrow{\{x\}\rightarrow\{y\}} \ V'^j \ = \ \frac{\partial y^j}{\partial x^i} V^i. \tag{1.6}$$

We see that the components of a vector transform in the opposite way from the basis vectors. Such vectors are called "contravariant vectors".

In the case where the functions $y^j(x)$ [1] are such that

$$\frac{\partial y^i}{\partial x_j} = \frac{\partial y^j}{\partial x^i}, \quad \text{and} \ \det\left(\frac{\partial y^i}{\partial x^j}\right) = 1,$$

then the transformation of coordinates is a rotation of the axes. The orthogonal rotation matrix, R, is

$$R^j{}_i = \frac{\partial y^j}{\partial x^i},$$

with the properties $R^T = R^{-1}$ and $\det R = 1$.

1.2 Metric

We can also define a "covariant vector" whose components V_i are given by

$$V_i = g_{ij} V^j, \tag{1.7}$$

where

$$g_{ij} \equiv \mathbf{e}_i \cdot \mathbf{e}_j, \tag{1.8}$$

is a tensor called the "metric tensor".

The set of basis vectors possess a dual set $\mathbf{e}^i$ where

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j, \tag{1.9}$$

and the metric has an inverse

$$g^{ij} \equiv \mathbf{e}^i \cdot \mathbf{e}^{\mathbf{j}},$$

such that

$$g^{ik} g_{kj} = \delta^i_j.$$

In the coordinate system with basis vectors $\mathbf{e}'_j$, the metric tensor is

$$g'_{kl} \equiv \mathbf{e}'_k \cdot \mathbf{e}'_l = \frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial y^l} g_{ij} \tag{1.10}$$

The components of a covariant vector transform under a change of coordinate $x_i \to y_j$, the same way that the basis vectors $\mathbf{e}_i$ transform, namely

$$V_j \stackrel{\{x\}\to\{y\}}{\longrightarrow} V'_j = \frac{\partial x_i}{\partial y_j} V_i. \tag{1.11}$$

[1] A function f of position is written $f(x)$. This means a function of all he coordinates $x^1 \cdots x^n$

For any two vectors $\mathbf{V}$ and $\mathbf{W}$, the "scalar product"

$$\mathbf{W} \cdot \mathbf{V} \equiv W_i V^i = W^i V_i = g_{ij} V^i W^j$$

is invariant, i.e. it takes the same value in all coordinate systems.

In particular, the distance between two neighbouring points, is

$$ds^2 = g_{ij} dx^i dx^j. \tag{1.12}$$

In Euclidean space, it is always possible to choose an orthonormal set of basis vectors, $\mathbf{e}_i$ whose α component is $\delta_{i\alpha}$, throughout the space. For this choice of coordinate system, the metric is simply given by $g_{ij} = \delta_{ij}$ and the components of a contravariant and covariant vector are the same. Such coordinates is called "Cartesian coordinates" However, there are other choices of coordinate system, for which this is not the case.

Example:
A point a two-dimensional Euclidean space can be labelled by Cartesian coordinates x, y as

$$x\mathbf{e}_x + y\mathbf{e}_y,$$

where

$$\mathbf{e}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A vector $\mathbf{V}$ has components V^x and V^y. In Cartesian coordinates, the components of the contravariant and covariant vectors are equal.

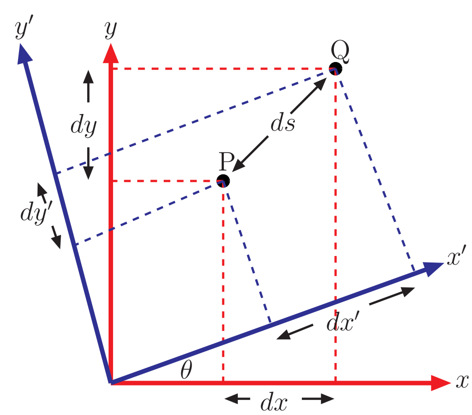

Figure 1: Two neighbouring points P and Q whose x- and y-coordinates differ by dx and dy in one coordinate (red), and by dx' and dy' in another coordinate system (blue), which is rotated by an angle θ. The distance, ds, between the two points is the same in both coordinate systems.

If the axes are rotated through an angle θ we obtain new axes $\mathbf{e}_{x'}$, $\mathbf{e}_{y'}$ where

$$\mathbf{e}_{x'} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \mathbf{e}_{y'} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

and the components of the vector V are transformed to

$$V^x \ \rightarrow \ V^{x'} \ = \ V^x \cos\theta + V^y \sin\theta$$

$$V^y \ \rightarrow \ V^{y'} \ = \ -V^x \sin\theta + V^y \cos\theta.$$

The new coordinates are also Cartesian coordinates and the metric remains unchanged $g_{ij} = \delta_{ij}$.

Alternatively, we can also describe the point in terms of plane-polar coordinates ρ, ϕ where

$$x \ = \ \rho\cos\phi, \quad y \ = \ \rho\sin\phi.$$

Using (1.5), the basis vectors, $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ are given by

$$\mathbf{e}_\rho \ = \ \cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y$$

$$\mathbf{e}_\phi \ = \ -\rho\sin\phi\,\mathbf{e}_x + \rho\cos\phi\,\mathbf{e}_y$$

The metric tensor, g', in this coordinate system, has components

$$g'_{\rho\rho} = 1, \ g'_{\rho\phi} = g'_{\phi\rho} = 0, \quad g'_{\phi\phi} = \rho^2,$$

and the inverse metric has components

$$g'^{\rho\rho} = 1, \ g'^{\rho\phi} = g'^{\phi\rho} = 0, \quad g'^{\phi\phi} = \frac{1}{\rho^2}. \tag{1.13}$$

The components of the contravariant vector $\mathbf{V}'$ are related to V^x, V^y by

$$V^\rho \ = \ \cos\phi\, V^x + \sin\phi\, V^y$$

$$V^\phi \ = \ \frac{1}{\rho}\sin\phi\, V^x + \frac{1}{\rho}\cos\phi\, V^y.$$

There is also a covariant vector whose components are

$$V_\rho \ = \ \cos\phi\, V^x + \sin\phi\, V^y$$

$$V_\phi \ = \ \rho\sin\phi\, V^x + \rho\cos\phi\, V^y.$$

An invariant quantity is a quantity, which takes the same value in all coordinate systems. It is, in general, a function of the point in space at which the quantity is measured, but independent of the coordinate system used to specify that point, i.e. for coordinate systems in which a point is labelled by coordinates x^i or y^i respectively, an invariant quantity, Φ is a function of the point at which it is measured so it takes the same value as a function of $(x^1 \cdots x^n)$ in one coordinate system as it does as a function of and $(y^1 \cdots y^n)$ in another system.

$$\Phi(x) \ = \ \Phi'(y).$$

The functional dependence on the coordinates, Φ and Φ' are different, but if the arguments of these functions are the coordinates of the same point, then the value of the two functions are equal.

An example is the distribution of temperature in a room, which depends on the point at which the temperature is measured, but not on the coordinate system used to label that point.

The gradient of a position dependent invariant quantity, known as a "scalar field", Φ, transforms as the (position dependent) components of a covariant vector, V_i, called a"vector field';,

$$V_j \stackrel{\{x\}\to\{y\}}{\longrightarrow} V'_j \;=\; \frac{\partial}{\partial y^j}\Phi'(y) \;=\; \frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i}\Phi(x) \;=\; \frac{\partial x^i}{\partial y^j}V_i.$$

The set of operators $\frac{\partial}{\partial x^i}$ transform as the components of a covariant vector. Therefore the (covariant) quantity

$$\mathbf{e}^i\frac{\partial \Phi}{\partial x^i}$$

is invariant under coordinate transformation - the covariant transformation of the derivative operators is cancelled by the contravariant transformation of the dual basis vectors.

An example of a vector constructed from the derivatives of a scalar, is an electric field vector, $\mathbf{E}$, which is the gradient of an electric potential, $\Phi(\mathbf{x})$

$$\mathbf{E} \;=\; -\boldsymbol{\nabla}\Phi,$$

or in components

$$E_i = \frac{\partial}{\partial x^i}\Phi(x).$$

The notation ∂_i is often used to denote the derivative operator $\partial/\partial x^i$. Under a change of coordinate systems the components of the electric field transform as

$$E_i(x) \;\to\; E'_i(y) \;=\; \frac{\partial x^j}{\partial y_i}E_j(x) \tag{1.14}$$

Note that not all vectors can be written as the gradients of invariant quantities, but the gradient of an invariant quantity (also called a "scalar" quantity) transforms as (the components of) a covariant vector.

1.3 Volume Element

In Cartesian coordinates an element of volume in n-dimensions is given by

$$dv \;=\; \prod_{i=1}^{n} dx^i$$

In any other coordinate system with metric g, this becomes

$$dv \;=\; |\det g|^{1/2} \prod_{i=1}^{n} dx^i \tag{1.15}$$

$$|\det g|^{1/2} \;= \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_n} \prod_{i=1}^{n} e_{1\alpha_1} e_{2\alpha_2} \cdots e_{n\alpha_n},$$

$e_{i\alpha}$ being the Cartesian coordinates of the basis vector $\mathbf{e}_i$. and $\epsilon^{\alpha_1 \alpha_2 \cdots \alpha_n}$ is the totally antisymmetric Levi-Civita tensor in n-dimensions.

As an example the volume element in two dimensions in plane polar coordinates is

$$dv \;=\; |\det g'|^{1/2}\, d\rho\, d\phi,$$

with the elements of g' given by (1.13), such that

$$dv \;=\; \rho\, d\rho\, d\phi$$

2 Tensors

A rank-r contravariant tensor in an n-dimensional space has $n \times r$ components, which, in a given coordinate system $\{x\}$, can be written

$$T^{i_i i_2 \cdots i_r}.$$

Under a transformation of coordinates to a coordinate system $\{y\}$, these components transform as

$$T^{j_1 j_2 \cdots j_r} \stackrel{\{x\}\to\{y\}}{\longrightarrow} T'^{\,j_1 j_2 \cdots j_r} = \frac{\partial y^{j_1}}{\partial x^{i_1}} \frac{\partial y^{j_2}}{\partial x^{i_2}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}} T^{i_i i_2 \cdots i_r}.$$

We can write the tensor in an invariant form

$$\mathbf{T} \equiv T^{i_i i_2 \cdots i_r} \mathbf{e}_{i_2} \cdots \mathbf{e}_{ir}.$$

This is invariant under change of coordinate system, since the components transform in the opposite way to the basis vectors.

A vector is a tensor of rank one.

An example of a rank-two tensor is the electric quadrupole moment due to a charge distributions $\rho(x)$

$$Q^{ij} \equiv \int d^n x x^i x^j \rho(x)$$

We can also construct corresponding covariant tensors with coefficients

$$T_{j_1 j_2 \cdots j_r} \equiv g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_r j_r} T^{i_i i_2 \cdots i_r},$$

which transform covariantly, namely

$$T_{k_1 k_2 \cdots k_r} \stackrel{\{x\}\to\{y\}}{\longrightarrow} T'_{k_1 k_2 \cdots k_r} = \frac{\partial x^{l_1}}{\partial y^{k_1}} \frac{\partial x^{l_2}}{\partial y^{k_2}} \cdots \frac{\partial x^{l_r}}{\partial y^{k_r}} T_{l_1 l_2 \cdots l_r}$$

We can also have mixed tensors in which some of the components transform contravariantly and others transform covariantly. A tensor with r_1 contravariant indices and r_2 covariant indices is known as as "rank-(r_1, r_2) tensor". For example the tensor with components T_i^{jk} is a rank-(2,1) tensor. A rank-(1,1) tensor is a matrix.

2.1 Irreducible Tensors

For a rank-two tensor we can define the "trace"

$$\mathrm{Tr}(\mathbf{T}) = g_{ij} T^{i,j}$$

This is an invariant quantity - it transforms into itself under a change of coordinates.

The remaining $n^2 - 1$ components,

$$T^{ij} - \frac{1}{n} g^{ij} g_{kl} T^{kl},$$

transform into each other under other a transformation of coordinates. These may be further partitioned into a tensor T_S^{ij} which is symmetric under the interchange of the two indices and a tensor T_A^{ij} which is anti-symmetric under the interchange of the two indices. Symmetric and anti-symmetric tensors retain their symmetry or anti-symmetry properties under a coordinate transformation. Therefore, a rank-two tensor may be written as the sum of three tensors which transform into themselves under a transformation of coordinates:

$$T^{ij} \;=\; \frac{1}{n} g^{ij} \mathrm{Tr}(\mathbf{T}) + T_A^{ij} + T_S^{ij}$$

where the $\frac{1}{2}n(n+1)$ components of T_S are given by

$$T_S^{ij} \;=\; \frac{1}{2}\left(T^{ij} + T^{ji}\right) - \frac{1}{n} g^{ij} \mathrm{Tr}\mathbf{T}$$

the $\frac{1}{2}n(n-1)$ components of T_S are given by

$$T_A^{ij} \;=\; \frac{1}{2}\left(T^{ij} - T^{ji}\right)$$

The components of T_S and T_A transform into linear superpositions of themselves under a coordinate transformation and are known as "irreducible" tensors.

An example of an anti-symmetric (covariant) tensor is a magnetic field defined in terms of the magnetic (covariant) vector potential $\mathbf{A}$

$$B_{ij} \;\equiv\; \frac{\partial}{\partial x^i} A_j - \frac{\partial}{\partial x^j} A_i.$$

We are used to thinking of magnetic field an an axial vector. But this is an accident of the fact that we live in three space dimensions and we can define a vector $\mathbf{B}$, whose components are

$$B^i \;\equiv\; \frac{1}{2} \epsilon^{ijk} B_{jk},$$

where ϵ^{ijk} is the totally anti-symmetric (Levi-Civita) tensor, but it is only a rank-three tensor in three dimensions.

Tensors of higher rank can also be partitioned into irreducible tensors, whose components transform into linear superpositions of themselves under a transformation of coordinates.

2.2 Covariant Derivatives

We have seen above that the gradient of a scalar field generates a vector field. We can generate a rank-(1,1) tensor field by taking the gradient of a vector field $\mathbf{V}$, but we need to be careful when we are in a coordinate system for which the basis vectors (and metric) are also position-dependent. The quantity

$$\mathbf{e}^i \frac{\partial}{\partial x^i} \left(\mathbf{e}_j V^j \right),$$

is unchanged under a change of coordinate system since the contravariant transformation in the dual basis vector $\mathbf{e}^i$ is cancelled by the covariant transformation of the vector operator $\partial/\partial x^i$ and covariant change in the basis vector $\mathbf{e}_j$ is cancelled by the contravariant transformation of the vector V^j. We may write this as

$$\mathbf{e}^i \frac{\partial}{\partial x^i} \left(\mathbf{e}_j V^j \right) = \mathbf{e}^i \mathbf{e}^j T_i^j,$$

where T_i^j are the components of a rank-(1,1) tensor. However $T_i^j \neq \partial V^j / \partial x^i$, but rather it is the "covariant derivative",

$$T_i^j \; = (\nabla_i T))^j \;\; \equiv \;\; (\nabla_i)^j{}_k \, V^k \;\; = \;\; \left(\frac{\partial}{\partial x^i} \delta_k^j + \Gamma_{ik}^j \right) V^k \tag{2.1}$$

where

$$\Gamma_{ik}^j \;\; = \;\; \mathbf{e}^j \cdot \frac{\partial}{\partial x^i} \left(\mathbf{e}_k . \right).$$

The operator ∇ is called the "covariant derivative" operator and (2.1) defines the covariant derivative of a contravariant vector. The quantities Γ_{ik}^j are called the "Christoffel symbols of the second kind". They are symmetric in the lower indices i, j. In Cartesian coordinates for which the basis vectors are independent of position these quantities are zero, but they are non-zero in a general coordinate system.

In the appendix to this section we show that the Christoffel symbols are given in terms of the derivatives of the metric, $g_i(x)$ by

$$\Gamma_{ij}^l \;\; = \;\; \frac{1}{2} g^{kl} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right) \tag{2.2}$$

From (1.9) we see that the partial derivative of the dual basis vector is given by

$$\frac{\partial \mathbf{e}^j}{\partial x^i} \;\; = \;\; -\Gamma_{ik}^j \mathbf{e}^k,$$

and so the covariant derivative of a covariant vector is given by

$$\left(\nabla_i V \right)_j \;\; \equiv \;\; \left(\nabla_i \right)_j{}^k \, V_k \;\; = \;\; \left(\frac{\partial}{\partial x^i} \delta_j^k - \Gamma_{ij}^k \right) V_k. \tag{2.3}$$

We can extend the definition of a covariant derivative to the covariant derivatives of a tensor. For example, the covariant derivative of a rank-(2,0) tensor, T^{jk}, is given by

$$\left(\nabla_i T \right)^{jk} \;\; \equiv \;\; \left(\nabla_i \right)^{jk}{}_{lm} \, T^{lm} \;\; = \;\; \left(\frac{\partial}{\partial x^i} \delta_l^j \delta_m^k + \Gamma_{il}^j \delta_m^k + \Gamma_{im}^k \delta_l^j \right) T^{lm} \tag{2.4}$$

This quantity transforms as a rank-(2,1) tensor

Note that the Christoffel symbols themselves do *not* transform as a rank-(1,2) tensor - i.e. they do not transform into a linear suoerposition of themselves under a transformation of coordinates $\{x\} \to \{y\}$. The transformation generates an in-homogeneous term:

$$\Gamma^i_{jk} \stackrel{\{x\}\to\{y\}}{\longrightarrow} \frac{\partial y^i}{\partial x_l}\frac{\partial x^m}{\partial y_j}\frac{\partial x^m}{\partial y_k}\Gamma^l_{mn} + \frac{\partial^2 x^l}{\partial y^j \partial y^k}\frac{\partial y^i}{\partial x^l} \tag{2.5}$$

For a covariant derivative of a vector, V^i, this in-homogeneous term exactly compensates for the in-homogeneous term in the transformation of the partial derivative $\partial_j V^i$ under coordinate transformations. This implies that one can always choose a coordinate system in such a way that at a given point in space, the metric is Euclidean and at that point (only) the Christoffel symbols vanish.

Example:
We consider a vector in two-dimensional Cartesian coordinates (x, y), $\mathbf{V}$, with components V^x and V^y. We can construct of rank-(1,1) tensor $T^j_i(x)$ with components

$$T^x_x = \frac{\partial}{\partial x}V^x, \quad T^x_y = \frac{\partial}{\partial y}V^x, \quad T^y_x = \frac{\partial}{\partial x}V^y, \quad T^y_y = \frac{\partial}{\partial y}V^y$$

Now change to plane polar coordinates (ρ, ϕ) which has a metric

$$g_{\rho\rho} = 1, \quad g_{\rho\phi} = g_{\phi\rho} = 0, \quad g_{\phi\phi} = \rho^2$$

Using (2.2), the non-zero Christoffel symbols are

$$\Gamma^\phi_{\phi\rho} = \Gamma^\phi_{\rho\phi} = \frac{1}{\rho}, \quad \Gamma^\rho_{\phi\phi} \;=\; -\rho \tag{2.6}$$

The components of the contravariant vector V^ρ, V^ϕ in this coordinate system are related to the Cartesian components by

$$\begin{aligned} V^\rho &= \cos\phi\, V^x + \sin\phi\, V^y \\ V^\phi &= -\frac{1}{\rho}\sin\phi\, V^x + \frac{1}{\rho}\cos\phi\, V^y \end{aligned}$$

The differential operators w.r.t. ρ or ϕ are related to the differential operators w.r.t. x and y by

$$\begin{aligned} \frac{\partial}{\partial \rho} &= \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \phi} &= -\rho\sin\phi\frac{\partial}{\partial x} + \rho\cos\phi\frac{\partial}{\partial y} \end{aligned}$$

Taking the derivatives of the components in plane polar coordinates we find

$$
\begin{aligned}
\frac{\partial}{\partial\rho}V^{\rho} &= \cos^2\phi\, T^x_x + \cos\phi\sin\phi\, T^x_y + \cos\phi\sin\phi\, T^y_x + \sin^2\phi\, T^y_y \\
&= \frac{\partial x}{\partial\rho}\frac{\partial x}{\partial\rho}\, T^x_x + \frac{\partial x}{\partial\rho}\frac{\partial y}{\partial\rho}\, T^x_y + \frac{\partial y}{\partial\rho}\frac{\partial x}{\partial\rho}\, T^y_x + \frac{\partial y}{\partial\rho}\frac{\partial y}{\partial\rho}\, T^y_y \\
\frac{\partial}{\partial\phi}V^{\rho} &= \rho\left(-\sin\phi\cos\phi\, T^x_x - \cos^2\phi\, T^x_y - \sin^2\phi\, T^y_x + \sin\phi\cos\phi\, T^y_y\right) - \sin\phi\, V^x + \cos\phi\, V^y \\
&= \frac{\partial x}{\partial\rho}\frac{\partial x}{\partial\phi}\, T^x_x + \frac{\partial y}{\partial\phi}\frac{\partial x}{\partial\rho}\, T^x_y + \frac{\partial x}{\partial\phi}\frac{\partial y}{\partial\rho}\, T^y_x + \frac{\partial y}{\partial\rho}\frac{\partial y}{\partial\phi}\, T^y_y - \rho V^{\phi} \\
\frac{\partial}{\partial\rho}V^{\phi} &= \frac{1}{\rho}\left(-\sin\phi\cos\phi\, T^x_x - \cos^2\phi\, T^x_y + \sin\phi\, T^y_x + \sin\phi\cos\phi\, T^y_y\right) + \frac{1}{\rho^2}\left(-\sin\phi\, V^x + \cos\phi\, V^y\right) \\
&= \frac{\partial x}{\partial\phi}\frac{\partial x}{\partial\rho}\, T^x_x + \frac{\partial y}{\partial\rho}\frac{\partial x}{\partial\phi}\, T^x_y + \frac{\partial x}{\partial\rho}\frac{\partial y}{\partial\phi}\, T^y_x + \frac{\partial y}{\partial\phi}\frac{\partial y}{\partial\rho}\, T^y_y + \frac{1}{\rho}V^{\phi} \\
\frac{\partial}{\partial\rho}V^{\phi} &= \sin^2\phi\, T^x_x + \cos^2\phi\, T^y_y - \cos\phi\sin\phi T^x_y - \cos\phi\sin\phi T^y_x - \frac{1}{\rho}\left(\cos\phi\, V^x + \sin\phi\, V^y\right) \\
&= \frac{\partial x}{\partial\phi}\frac{\partial x}{\partial\phi}\, T^x_x + \frac{\partial y}{\partial\phi}\frac{\partial x}{\partial\phi}\, T^x_y + \frac{\partial x}{\partial\phi}\frac{\partial y}{\partial\phi}\, T^y_x + \frac{\partial y}{\partial\phi}\frac{\partial y}{\partial\phi}\, T^y_y - \frac{1}{\rho}V^{\rho} \qquad (2.7)
\end{aligned}
$$

see that these derivatives are *not* just linear superpositions of the tensor components in Cartesian coordinates but there are "bits left over". However, we can see that the components,

$$\frac{\partial}{\partial\rho}V^{\rho},\quad \frac{\partial}{\partial\phi}V^{\rho} + \Gamma^{\rho}_{\phi\phi}V^{\phi},\quad \frac{\partial}{\partial\rho}V^{\phi} + \Gamma^{\phi}_{\rho\phi}V^{\phi},\quad \frac{\partial}{\partial\phi}V^{\phi} + \Gamma^{\phi}_{\phi\rho}V^{\rho},$$

with $\Gamma^{\rho}_{\phi\phi}, \Gamma^{\phi}_{\rho\phi},\ \Gamma^{\phi}_{\phi\rho}$ given by (2.6) *are* linear superpositions of the components in Cartesian coordinates and are therefore themselves components of a tensor.

We can show by explicit calculation that all components of the covariant derivative of the metric vanish. We can understand this without going through the messy calculation. In Cartesian coordinates where the metric is constant, the Christoffel symbols vanish and all components of the rank-(0,3) tensor $(\nabla_i g)_{jk}$ are zero. Under a change of coordinates these components transform into linear sums of each other - so that in any coordinate system the components must be zero .

2.3 Appendix: Derivation of Christoffel Symbols

The Christoffel symbol is defined as

$$\frac{\partial \mathbf{e}_j}{\partial x^i} = \Gamma^l_{ij}\mathbf{e}_l$$

and the metric is defined as

$$g_{jk} = \mathbf{e}_j \cdot \mathbf{e}_k \qquad (2.8)$$

Differentiating (2.8) w.r.t. x^i we have

$$\begin{aligned}\frac{\partial}{\partial x^i} g_{jk} &= \Gamma^l_{ij} \mathbf{e}_l \cdot \mathbf{e}_k + \Gamma^l_{ik} \mathbf{e}_l \cdot \mathbf{e}_j \\ &= \Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{lj} \\ &= \Gamma_{kij} + \Gamma jik, \end{aligned} \tag{2.9}$$

where $\Gamma_{kij} = g_{lk}\Gamma^l_{ij}$, are the "Christoffel symbols of the first kind".

Now we can write

$$\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij} = \Gamma_{kij} + \Gamma_{jik} + \Gamma_{ijk} + \Gamma_{kij} - \Gamma_{jki} - \Gamma_{ijk} = 2\Gamma_{kij}. \tag{2.10}$$

(We have used the symmetry property $\Gamma_{jik} = \Gamma_{jki}$). Finally, we contract with the inverse metric g^{kl} to arrive at

$$\Gamma^l_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij} \right)$$

3 Minkowski Space

Special Relativity is concerned with the relationship between the positions and times of events in different reference frames, moving with a constant velocity relative to each other.

In reference frame A, an event is labelled by four coordinates - three space coordinates x_A^1, x_2^A, x_3^A and a time coordinate time t^A. We can describe this using the technology developed above, but with a coordinate system consisting of four coordinates (a four-dimensional coordinate space) with coordinates $x^0, x^1, x^2 x^3$ where $x^0 = ct$. The difference between these coordinates for two neighbouring events, in reference frame A is a four component vector with coordinates[2], $dx_A^\mu,\ \mu = 0 \cdots 3$

$$dx_A^\mu \ = \ \left(dx_A^0, dx_A^1, dx_A^2, dx_A^3\right).$$

In frame B the four coordinates of the vector dx are

$$dx_B^\mu \ = \ \left(dx_B^0, dx_B^1, dx_B^2, dx_B^3\right)$$

Some or all of these coordinates may be different depending on the direction of the velocity of B relative to A. However, we know that the proper time, $d\tau$ is invariant - it takes the same value in *all* inertial frames. The proper time is defined by

$$c^2 d\tau^2 \ = \ (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \ = \eta_{\mu\nu} dx^\mu dx^\nu \tag{3.1}$$

The proper time is the time interval between the two events in the reference frame where the events occur at the same position.

We have written this as the scalar product of the vector dx^μ with itself, using the metric $\eta_{\mu\nu}$ for which[3]

$$\eta_{00} = 1,\ \eta_{11} = \eta_{22} = \eta_{33} = -1, \quad \eta_{\mu\nu} = 0, (\mu \neq \nu) \tag{3.2}$$

This metric is known as the "Minkowski metric". For any two four-vectors, V^μ, W^μ the scalar product

$$V \cdot W \ \equiv \ \eta_{\mu\nu} V^\mu W^\nu, \tag{3.3}$$

is invariant under a general Lorentz transformation between two frames

For a general transformation between frames A and B the components, V^μ of any four-vector are related by

$$V_B^\mu \ = \ \Lambda^\mu{}_\nu V_A^\nu \tag{3.4}$$

where $\Lambda^\mu{}_\nu$ is a rank-(1,1) tensor (i.e. a matrix) with real, space-time independent components which obey the relation

$$\eta_{\rho\mu} \Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma \ = \ \eta_{\nu\sigma}, \tag{3.5}$$

[2]Conventionally we use middle Greek letters $\lambda, \mu, \nu \cdots$ as indices for these four-vectors, and they run over 0,1,2,3.

[3]There are two conventions for the assignment of the signs in the Minkowski metric. Most physicists use the sign convention used in (3.2), and most mathematicians use the opposite sign for each element.

which guarantees the invariance (3.3)

For example, for a Lorentz boost with velocity v in the direction-1 the non-zero components of the tensor Λ are

$$\Lambda^0{}_0 = \Lambda^1{}_1 = \frac{1}{\sqrt{1-v^2/c^2}}, \;\; \Lambda^0{}_1 = \Lambda^1{}_0 = \frac{v/c}{\sqrt{1-v^2/c^2}}, \quad \Lambda^2{}_2 = \Lambda^3{}_3 = 1,$$

Applying the transformation (3.4) to the vector dx, using this value of Λ and using $t = x^0/c$ we obtain the familiar Lorentz transformation

$$dx^i_B = \gamma\left(dx^1_a - v dt_A\right)$$

$$dt_B = \gamma\left(dt_a - v/c^2 dx^1_A\right),$$

with

$$\gamma \;= \frac{1}{\sqrt{1-v^2/c^2}}$$

Note that a pure rotation is a subset of these generalised Lorentz transformation, For example the tensor Λ, whose non-zero components are

$$\Lambda^0{}_0 = \Lambda^3{}_3 = 1, \quad \Lambda^1{}_1 = \Lambda^2{}_2 = \cos\theta, \quad \Lambda^1{}_2 = -\Lambda^2{}_1 = \sin\theta$$

is a rotation about the third axis through angle θ. A general Lorentz transformation consists of a Lorentz boost *and* a rotation.

Note that, in general, a Lorentz boost and a rotation do *not* commute. The Lorentz transformation tensor, Λ for a rotation about the third axis followed by a Lorentz boost along the first axis has non-zero components

$$\Lambda^0{}_0 = \gamma, \;\; \Lambda^0{}_1 = -\gamma v\cos\theta/c, \;\; \Lambda^0{}_2 = -\gamma v\sin\theta/c, \;\; \Lambda^1{}_0 = \gamma v/c, \;\; \Lambda^1{}_1 = \gamma\cos\theta, \;\; \Lambda^1{}_2 = \gamma\sin\theta,$$

$$\Lambda^2{}_1 = -\sin\theta, \quad \Lambda^2{}_2 = \cos\theta, \quad \Lambda^3{}_3 = 1$$

whereas for a Lorentz boost in the first direction followed by a rotation about the third axis we have

$$\Lambda^0{}_0 = \gamma, \quad \Lambda^0{}_1 = -\gamma v/c, \quad \Lambda^1{}_0 = -\gamma v\cos\theta/c, \quad \Lambda^1{}_1 = \gamma\cos\theta, \quad \Lambda^1{}_2 = \sin\theta,$$

$$\Lambda^2{}_0 = v\sin\theta/c, \quad \Lambda^2{}_1 = -\gamma\sin\theta, \quad \Lambda^2{}_2 = \cos\theta, \quad \Lambda^3{}_3 = 1$$

We can think of a general Lorentz transformation as a "rotation" in a four-dimensional Minkowski space in which a rotation between a space-like axis and the time axis signifies a Lorentz boost between two frames. As in the case of rotations in three space-like dimensions, rotations about different axes do not commute,

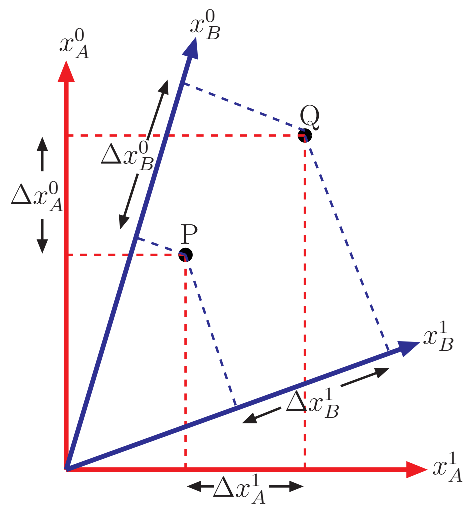

Figure 2: Two events P and Q at locations x_A^1, $x_A^1 + dx_A^1$ in frame A and x_B^1, $x_B^1 + dx_B^1$ in frame B and at times x_A^0/c, $(x_A^0 + dx_A^0)/c$ in frame A and x_B^0/c, $(x_B^0 + dx_B^0)/c$ in frame B, which is moving relative to A in the x^1 direction.

3.1 Minkowski Diagrams

We can represent events in Minkowski space by selecting one of the axes to be the time axis x_0. In Figure 2 we display the x^1 and time (x^0) axes. These are called "Minkowski diagrams" and an event is represented by a point in such a diagram. In Figure 2, P and Q are two events assumed to occur at the same values of x^2 and x^3. We have chosen to plot them with the axes at right-angles in frame A. Under a Lorentz boost in the x^1 direction the x^1 and axes rotate *in opposite senses*, in such a way that the proper time interval between the events P and Q remains unchanged. The angle through which the axes rotate increases with increasing relative velocity. The maximum possible relative velocity is that for which the x^1 and x^0 axes coincide and this corresponds to a relative velocity $v = c$. No body can move with a relative velocity that exceeds the speed of light.

3.2 Energy and momentum

We can construct a four-vector from the energy, E (including its rest energy) of a particle of mass m with three momentum $\mathbf{p} = (p^1, p^2, p^3)$. This 4-vector is

$$p^\mu = \left(p^0, p^1, p^2, p^3\right)$$

The zero component, $p^0 = E/c$ The scalar product of this four-vector, with itself is[4]

$$p^2 \equiv p \cdot p = \eta_{\mu\nu} p^\mu p^\nu = \frac{1}{c^2} E^2 - \sum_{i=1,3} p_i^2 = m^2 c^2. \tag{3.6}$$

[4] The scalar product of a vector V, with itself is often written simply as V^2.

The mass[5] of a particle is invariant under Lorentz transformations, so we were justified in combining the energy and the three-momentum $\mathbf{p}$ into a quantity which transforms as a 4-vector under Lorentz transformations.

The individual component p^μ change between different reference frames by the transformation rules for a 4-vector

$$p^\mu_B \; = \Lambda^\mu{}_\nu p^\nu_A$$

For a Lorentz boost in he x^1 direction this leads to the energy and momentum transformation rules

$$\begin{aligned} E_B &= \gamma\left(E_A - p^1_A c\right) \\ p^1_B &= \gamma\left(p^1_A - vE_A/c^2\right) \end{aligned}$$

In order for the Laws of Physics to be independent of the reference frame, they must be in the form of equations which relate quantities that transform as scalars, vectors, or tensors in Minkowski space.

For example, in the scattering of N_1 incoming particles with four-momenta p^μ_i, $(i = 1 \cdots N_1)$ to N_2 outgoing particles with four-momenta p^μ_j, $(j = 1 \cdots N_2)$, the law of conservation of energy and momentum is combined into one single equation relating all the components of four-vectors

$$\sum_{i=1}^{N_1} p^\mu_i \; = \sum_{j=1}^{N_2} p^\mu_j \tag{3.7}$$

This is clearly obeyed in any frame of reference as both sides of (3.7) transform as a 4-vector in Minkowski space. The $\mu = 0$ component of (3.7) is the law of conservation of energy and the other three components give the law of conservation of the the three (space-like) components of momentum.

3.3 Action for a Relativistic Free Particle

The four coordinates x^μ of a free particle of mass m sweeps out a "world-line" in Minkowski space, $x^\mu(\lambda)$, where λ is a parameter which varies from 0 to 1 so that the coordinates take their initial value at $\lambda = 0$ and their final value at $\lambda = 1$:

$$x^\mu_i = x(0)^\mu, \quad x^\mu_f = x^\mu(1).$$

The action, S, is the mass multiplied by the integral of the proper time, defined by (3.1), over that world-line, i.e.

$$S \; = \; mc^2 \int d\tau \; = \; mc \int_0^1 d\lambda \sqrt{\eta_{\mu\nu}\dot{x}^\mu(\lambda)\dot{x}^\nu(\lambda)}, \tag{3.8}$$

[5]The term "mass" used here is often called the "rest mass" – the inertia of a body increases with increasing momentum, but the rest mass is invariant.

where the four-vector

$$\dot{x}^\mu(\lambda) \;\equiv\; \frac{d}{d\lambda}x^\mu(\lambda),$$

transforms as a contravariant vector.

The canonical momentum vector p^μ is given by

$$p^\mu \;=\; \eta^{\mu\rho}\frac{\delta S}{\delta \dot{x}^\rho} \;=\; m\frac{dx^\mu}{d\lambda}\frac{d\lambda}{d\tau} \;=\; m\frac{dx^\mu}{d\tau} \tag{3.9}$$

where we have used

$$\frac{d\lambda}{d\tau} \;=\; c\left(\eta_{\mu\nu}\dot{x}^\mu(\lambda)\dot{x}^\nu(\lambda)\right)^{-1/2}$$

to obtain the expression for the momentum vector which is a function of proper-time, independent of the parametrisation.

The Lagrange equation of motion for a free particle is

$$\frac{dp^\mu}{d\lambda}\frac{d\lambda}{d\tau} \;=\; \frac{dp^\mu(\tau)}{d\tau} \;=\; 0. \tag{3.10}$$

This is the relativistic generalisation of the conservation of momentum for a free particle.

Using (3.9) we may write this

$$\frac{d^2x^\mu(\tau)}{d\tau^2} \;=\; 0, \tag{3.11}$$

A free particle sweeps out a "world-line" which is a straight line on the Minkowski diagram the world-line with gradient

$$\frac{dx^i}{dt} = \beta^i,$$

where

$$\beta^i = \frac{dx^i}{d\lambda}/\frac{dx^0}{d\lambda},$$

is the constant velocity in units of c. In the rest-frame of the particle itself, the world-line is parallel to the time axis (the space-like components remain constant)

3.4 The Klein-Gordon Equation

The Schrodinger equation for a particle wavefunction $\Psi(x)$, (of a particle with spin-0) is replaced in Special Relativity by the Klein-Gordon Equation which may be written

$$\hbar^2 g^{\mu\nu}\partial_\mu\partial_\nu\Psi(x) \;+\; m^2c^2\Psi(x) \;=\; 0. \tag{3.12}$$

This is the equation (3.6) in which the four momentum, p_μ is replaced by the quantum operator

$$p_\mu \;=\; -i\hbar\partial_\mu.$$

3.5 Three Dimensional Volume element

A three-dimensional volume element in Cartesian coordinates

$$dv \equiv d^3\mathbf{x}$$

is not invariant under Lorentz transformations.

In order to reformulate a physical law involving an integral over volume, so that it is valid in any reference frame, we need to interpret a volume element as the zeroth component of a four vector dS^μ, so that $dv = dS^0$.

Likewise a density ρ (this could mean charge density or mass density or the density of any other quantity) needs to be interpreted as the zeroth component of a four-vector j^μ.

In the case of electromagnetism, the space-like components are $j^i/c,\ i = 1\cdots 3$, where j^i are the components of current density . The invariant total electric charge, Q, in a given volume is

$$Q \;=\; \int j_\mu dS^\mu.$$

In the non-relativistic limit (found by setting $1/c$ to zero) this reduces to the non-relativistic expression

$$Q \approx \int \rho dv.$$

The continuity equation

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0$$

can be written in the manifestly invariant form (dividing throughout by c)

$$\frac{\partial j^0}{\partial x^0} + \partial_i j^i = \partial_\mu j^\mu \;=\; 0. \tag{3.13}$$

Both sides are Lorentz invariant quantities and so this conservation law is valid in any reference frame.

We have introduced the covariant vector derivative operator

$$\partial_\mu \;\equiv\; \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right). \tag{3.14}$$

The corresponding contravariant four-vector operator is (in Minkowski space)

$$\partial^\mu \;\equiv \left(\frac{\partial}{\partial x^0}, -\frac{\partial}{\partial x^1}, -\frac{\partial}{\partial x^2}, -\frac{\partial}{\partial x^3}\right). \tag{3.15}$$

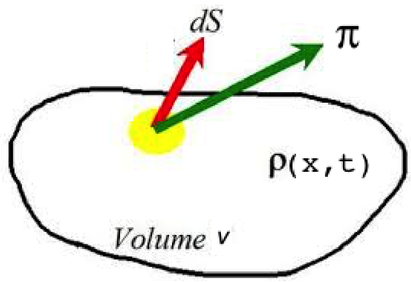

Figure 3: A region of space of volume v in the reference frame in which it is at rest, It contains a mass (energy) distribution, $\rho(\mathbf{x}, t)$ and a momentum density at the surface $\boldsymbol{\pi}$.

3.6 Stress-Energy Tensor

Consider a volume v of a fluid or ensemble of many particles of density $\rho(\mathbf{x})$. In non-relativistic physics in the frame in which the volume v is stationary the rate of change of the mass inside the volume is

$$\frac{d}{dt}\int_v \rho dv.$$

By continuity, the is equal to the ingoing momentum flux integrated over the surface, S, of the volume

$$\frac{d}{dt}\int_v \rho dv \;=\; -\int_S \boldsymbol{\pi}\cdot d\mathbf{S}, \tag{3.16}$$

where $\boldsymbol{\pi}$ is the three-momentum per unit volume. Using Gauss' theorem we may rewrite this as

$$\int_V \left(\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\pi}\right) dv \tag{3.17}$$

To convert this into an equation which is valid in any reference frame we need two steps

1. Replace $c\,\rho$ by the zeroth component of a 4-vector whose space-like components are the components of momentum density, π^i. π^0 is then the energy density (divided by c). Energy means total relativistic energy and in the non-relativistic limit π^0 is approximated by - c multiplied by the mass per unit volume. (3.17) then becomes

$$\int_v \partial_\mu \pi^\mu dv \;= 0 \tag{3.18}$$

2. (3.18) is only valid in the reference frame in which the volume is stationary. In order to formulate this in a format which is valid in all frames we must replace the volume element by the (covariant) 4-vector dS_ν whose zeroth component is dv, i.e. $dv = dS_0$. The vector dS_ν is a three dimensional surface element in four-dimensional Minkowski space– and each component of π^μ is itself the $(\mu, 0)$ component of a tensor $T^{\mu\nu}$, known

as the “stress-energy tensor” or sometimes “energy-momentum tensor”. (3.18) then becomes

$$\int_S \partial_\mu T^{\mu\nu} dS_\nu \ = \ 0 \tag{3.19}$$

This must be true for any three-dimensional surface of Minkowski space, which means that the integrand must vanish

$$\partial_\mu T^{\mu\nu} \ = 0. \tag{3.20}$$

The divergence of the stress-energy tensor is zero[6]

The component T^{00} of the stress-energy tensor is the energy density and the components $T^{0i} = T^{i0}$ are the components of the momentum density. The remaining components are a little more complicated to interpret. T^{ii} is the momentum flux (the rate of change of momentum per unit area) in the direction i . In the case where $T^{11} = T^{22} = T^{33}$, this value is the pressure. The components T^{ij} $i \neq j$ are called the components of “shear stress”.

[6] We have assumed a Minkowski metric whose space-like coordinates are Cartesian. If we use a different coordinate system then the partial derivative needs to be replaced by the covariant derivative.

4 Two Particle Scattering

We look in more detail at the kinematics of a particle of mass m_1 with four-momentum p_1^μ scattering against a particle of mass m_2 with four-momentum p_2^μ and producing two (in general different) final-state particles - one with mass m_3, four-momentum p_3^μ and the other with mass m_4 and four-momentum p_4^μ

The conservation of energy and momentum gives us four equations

$$p_1^\mu + p_2^\mu \;=\; p_3^\mu + p_4^\mu, \quad (\mu = 0, 1, 2, 3) \tag{4.1}$$

From (3.6) we have

$$p_1^2 = m_1^2 c^2, \quad p_2^2 = m_2^2 c^2, \quad p_3^2 = m_3^2 c^2, \quad p_4^2 = m_4^2 c^2, \tag{4.2}$$

4.1 Mandelstam s variable

We define the Mandelstam variable s as the invariant quantity

$$s \;\equiv\; (p_1 + p_2) \cdot (p_1 + p_2) \;=\; (p_3 + p_4) \cdot (p_3 + p_4), \tag{4.3}$$

where we have used energy-momentum conservation (4.1). The interpretation of s is that s is the invariant square total momentum.

Using (4.2) we may write

$$s \;=\; m_1^2 c^2 + m_2^2 c^2 = 2 p_1 \cdot p_2 \;=\; m_1^2 c^2 + m_2^2 c^2 + 2 \left(\frac{E_1 E_2}{c^2} - \mathbf{p}_1 \cdot \mathbf{p}_2 \right), \tag{4.4}$$

where the boldface $\mathbf{p}$ stands for three-momentum. Whereas s is invariant under Lorentz transformations, the quantities E_1, E_2, $\mathbf{p}_1$, $\mathbf{p}_2$ on the RHS of (4.4) are frame-dependent but are always related to each other as given by (4.4).

Similarly we have,

$$s \;=\; m_3^2 c^2 + m_4^2 c^2 = 2 p_3 \cdot p_4 \;=\; m_3^2 c^2 + m_4^2 c^2 + 2 \left(\frac{E_3 E_4}{c^2} - \mathbf{p}_3 \cdot \mathbf{p}_4 \right), \tag{4.5}$$

There are two reference frames which are of particular interest – the centre-of-mass (CM) frame in which the sum of the three-momenta of the participating particles is zero - i.e. their three-momenta are equal and opposite, and the rest-frame of one of the target-particles (e.g. particle 2) which is the frame of reference relevant for fixed target experiments.

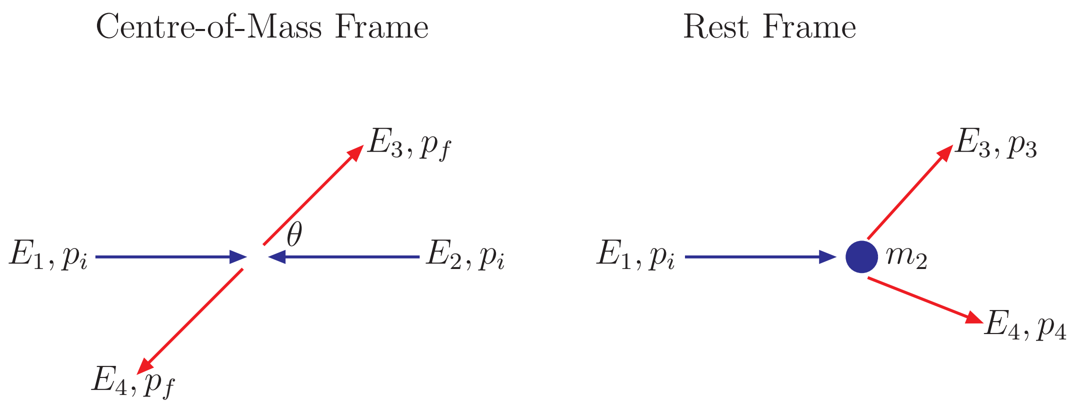

Figure 4: Two-particle to two-particle scattering - in the centre-of-mass frame in which the sum of the three momentum of the two (incoming or outgoing) particles is zero (left) and in the rest-frame of target particle (2) (right).

4.2 Centre-of-Mass frame

In the centre-of-mass frame, the quantity $\sqrt{s}$ is the total relativistic energy of the two incoming (or two outgoing) particles.

Without loss of generality we can assume that the incoming particles are moving along the third axis with magnitude of three-momentum p_i The four momenta of the incoming particles may therefore be written

$$
\begin{aligned}
p_1^\mu &= \left(\frac{E_1}{c}, 0, 0, p_i\right) \\
p_2^\mu &= \left(\frac{E_2}{c}, 0, 0, -p_i\right)
\end{aligned}
$$

where

$$E_1^2 = p_i^2 c^2 + m_1^2 c^4, \quad E_2^2 = p_i^2 c^2 + m_2^2 c^4.$$

Using (4.4) and after some algebra we arrive at

$$
\begin{aligned}
p_1^\mu &= \frac{1}{2\sqrt{s}} \left(s + \left(m_1^2 - m_2^2\right) c^2, 0, 0, \lambda^{1/2}\left(s, m_1^2 c^2, m_2^2 c^2\right)\right) \\
p_2^\mu &= \frac{1}{2\sqrt{s}} \left(s + \left(m_2^2 - m_1^2\right) c^2, 0, 0, -\lambda^{1/2}\left(s, m_1^2 c^2, m_2^2 c^2\right)\right),
\end{aligned} \tag{4.6}
$$

where the Källën function, $\lambda(x, y, z)$,

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$$

This function simplifies in the case $y = z$ (equal masses $m_1 = m_2$) to

$$\lambda(x, y, y) = x(x - 4y)$$

Similarly

$$\begin{aligned} p_3^\mu &= \frac{1}{2\sqrt{s}}\left(s+\left(m_3^2-m_4^2\right)c^2, \lambda^{1/2}\left(s, m_3^2c^2, m_4^2c^2\right)\sin\theta, 0, \lambda^{1/2}\left(s, m_3^2c^2, m_4^2c^2\right)\cos\theta, \right) \\ p_4^\mu &= \frac{1}{2\sqrt{s}}\left(s+\left(m_4^2-m_3^2\right)c^2, -\lambda^{1/2}\left(s, m_3^2c^2, m_4^2c^2\right)\sin\theta, 0, -\lambda^{1/2}\left(s, m_3^2c^2, m_4^2c^2\right)\cos\theta\right), \end{aligned} \tag{4.7}$$

where θ is the scattering angle in the centre-of-mass frame[7]. Note that the scattering angle is frame-dependent - so that the angle used in (4.7) is the centre-of-mass scattering angle.

4.3 Rest frame

In the rest frame of target particle 2 the incoming 4-momenta are

$$\begin{aligned} p_1^\mu &= \left(\frac{E_1}{c}, 0, 0, p_1\right) \\ p_2^\mu &= (m_2c, 0, 0, 0) \end{aligned}$$

The Mandelstam variable s is then

$$s = \left(m_1^2+m_2^2\right)c^2 + 2E_1m_2,$$

E_1 being the energy of the projectile particle in the rest frame of he target particle. The incoming four-momentum of the projectile particle is by

$$p_1^\mu = \frac{1}{2m_2c}\left(\left(s-m_1^2c^2-m_2^2c^2\right), 0, 0, \lambda^{1/2}(s, m_2^2, m_2^2)\right) \tag{4.8}$$

4.4 Mandelstam t variable

A further invariant quantity is t define as

$$t \equiv (p_1-p_3)\cdot(p_1-p_3) = (p_2-p_4)\cdot(p_2-p_4) \tag{4.9}$$

This is the invariant square of the momentum transferred from particle 1 to particle 3 in the scattering process.

We may write this as

$$t = m_1^2c^2 + m_3^2c^2 - 2p_1\cdot p_3$$

In the non-relativistic limit with equal masses $m_1 = m_3$ and

$$E_1 \approx m_1c^2 + \frac{1}{2m_1}|\mathbf{p}_1|^2, \quad E_3 \approx m_3c^2 + \frac{1}{2m_3}|\mathbf{p}_3|^2,$$

[7] We have arbitrarily set the azimuthal angle to zero as kinematics is invariant under azimuthal rotations about the third axis.

t is approximated by

$$-t \approx |\mathbf{p}_1 - \mathbf{p}_3|^2 \,,$$

which is indeed the square of the momentum transferred.

We can use (4.6) and (4.7) to show that there is a relation between s, t, the particle masses and the scattering angle:

$$\begin{aligned} t &= -\frac{1}{2s}\bigl(s^2 - s(m_1^{2\text{‘}} + m_2^2 + m_3^2 + m_4^2)c^2 + (m_1^2 - m_2^2)(m_3^2 - m_4^2)c^4 \\ &\qquad + \lambda^{1/2}(s, m_1^2c^2, m_2^2c^2)\lambda^{1/2}(s, m_3^2c^2, m_4^2c^2)\cos\theta\bigr) \end{aligned} \tag{4.10}$$

In the rest frame, there is a fairly simple relation between t and the energy of particle-4 (the other outgoing particle), since we may write

$$t = (p_2 - p_4)\cdot(p_2 = p_4) = m_2^2c^2 + m_4^2c^2 - 2p_2\cdot p_4 = m_2^2c^2 + m_4^2c^2 - 2m_2E_4 \tag{4.11}$$

4.5 Mandelstam u variable

There is one more invariant quantity u defined by

$$u \equiv (p_1 - p_4)\cdot(p_1 - p_4) = (p_2 - p_3)\cdot(p_2 - p_3)\,. \tag{4.12}$$

However, this is *not* an independent variable.

$$\begin{aligned} s + t + u &= \frac{1}{2}\bigl((p_1+p_2)\cdot(p_1+p_2) + (p_3+p_4)\cdot(p_3+p_4) + (p_1-p_3)\cdot(p_1-p_3) \\ &\quad + (p_2-p_4)\cdot(p_2-p_4) + (p_1-p_4)\cdot(p_1-p_4) + (p_2-p_3)\cdot(p_2-p_3)\bigr) \\ &= \frac{1}{2}(p_1+p_2+p_2+p_4)\cdot(p_1+p_2-p_3-p_4) + p_1^2 + p_2^2 + p_3^2 + p_4^2 \end{aligned} \tag{4.13}$$

But $(p_1 + p_2 - p_3 - p_4)^\mu = 0$ and using the relations $p_i^2 = m_i^2c^2$ for each of the four particles we arrive at

$$s + t + u = \left(m_1^2 + m_2^2 + m_3^3 + m_4^2\right)c^2 \tag{4.14}$$

5 Relativistic Electromagnetism

If we have an electric charge distribution $\rho(\mathbf{x})$ in a frame A in which this charge density is stationary, then in another frame moving with velocity v relative to A we observe both an electric charge density and a current density $\mathbf{j}$. This leads us to combine current density and charge density and current density into a single four-vector j^μ whose zeroth component is $c\,\rho$. We have already seen that conservation of electric charge can be expressed in a manifestly Lorentz invariant format

$$\partial_\mu j^\mu \;=\; 0$$

We also know that if in a particular reference frame we have a static electric field, $\mathbf{E}$, then in a frame moving relative to that frame there will be both an electric fields and a magnetic field $\mathbf{B}$. This means that in order to formulate the laws of electromagnetism in a frame-independent way we need a quantity with six independent components - three of which correspond to the components of the electric field E and three corresponding to the three components of the magnetic field, which transforms in a well-defined way under Lorentz transformations.

Such a quantity is the two-rank anti-symmetric tensor

$$F^{\mu\nu} \;=\; -F^{\nu\mu}$$

This tensor can be constructed from the four-vector A^μ which combines the electrostatic potential Φ and the magnetic vector potential $\mathbf{A}$. The zeroth component of A^μ is Φ/c.

The electromagnetic field tensor is then defined as

$$F^{\mu\nu} \;=\; \partial^\mu A^\nu - \partial^\nu A^\mu \tag{5.1}$$

Recalling that the contravariant derivative operator

$$\partial^\mu \;=\; \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x^1}, -\frac{\partial}{\partial x^2}, -\frac{\partial}{\partial x^3}\right)$$

The components of electric field and magnetic field are given by

$$E^i \;=\; \frac{\partial A^i}{\partial t} - \frac{\partial \phi}{\partial x^i}$$

$$B^i \;=\; \epsilon^{ijk}\frac{\partial A_j}{\partial k},$$

where ϵ^{ijk} is the completely anti-symmetric Levi-Civita tensor in three dimensions.

The components of the contravariant electromagnetic field tensor are therefore

$$F^{0i} \;=\; -F^{i0} \;=\; -\frac{E^i}{c} \tag{5.2}$$

$$F^{ij} = -F^{ji} = -\epsilon^{ijk} B_k \tag{5.3}$$

For the covariant electromagnetic field tensor

$$F_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma} F^{\rho\sigma}$$

the sign of the electric field components is reversed but the sign of the magnetic field components is unchanged.

The sress-energy tensor for an electromagnetic field is

$$T_{\mu\nu} = -\frac{1}{\mu_0}\left(g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \tag{5.4}$$

This has components (in terms of electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$),

$$\begin{aligned}
T_{00} &= \frac{1}{2}\left(\epsilon_0 |\mathbf{E}|^2 + \frac{1}{\mu_0} |\mathbf{B}|^2 \right) \\
T_{ii} &= \epsilon_0 \left(\frac{1}{2}|\mathbf{E}|^2 - \mathbf{E}_i^2 \right) + \frac{1}{\mu_0}\left(\frac{1}{2}|\mathbf{B}|^2 - \mathbf{B}_i^2 \right) \\
T_{0i} &= -\sqrt{\frac{\epsilon_0}{\mu_0}} \left(\mathbf{E} \times \mathbf{B} \right)_i \\
T_{ij} &= \epsilon_0 \mathbf{E}_i \mathbf{E}_j - \frac{1}{\mu_0} \mathbf{B}_i \mathbf{B}_j,
\end{aligned} \tag{5.5}$$

where we have made use of the relation $c^2 = (\epsilon_0 \mu_0)^{-1}$.

The components T_{0i} are proportional to the Poynting vector and represent the momentum density.

Note that the trace of the stress-energy tensor, $g^{\mu\nu} T_{\mu\nu}$, vanishes.

5.1 Transformations of Electric and Magnetic Field

The transformation of the components of electric and magnetic fields under a Lorentz transformation can be obtained from the transformation of the electromagnetic field tensor

$$F^{\mu\nu} \longrightarrow F'^{\mu\nu} = \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}_{\sigma} F^{\rho\sigma} \tag{5.6}$$

For a Lorentz boost of velocity $\mathbf{v}$, the components of electric and magnetic fields in the direction of $\mathbf{v}$ (the longitudinal components) remain unchanged, whereas the transverse components of the fields transform into each other

$$\mathbf{E}_T \rightarrow \mathbf{E}'_T = \gamma \left(\mathbf{E}_T - \mathbf{v} \times \mathbf{B}_T \right) \tag{5.7}$$

$$\mathbf{B}_T \rightarrow \mathbf{B}'_T = \gamma \left(\mathbf{B}_T + \frac{1}{c^2} \mathbf{v} \times \mathbf{E}_T \right) \tag{5.8}$$

In 1905 Einstein did not now about tensors. They were introduced to him by Marcel Grossman in 1912 and were used extensively in General Relativity. In his 1905 paper Einstein derived these results the hard way. From the Lorentz transformations of $\mathbf{x}$ am and time, t, he deduced the Lorentz transformations of the operators $\boldsymbol{\nabla}$ and the time derivative. He then deduced the transformations of the components of electric and magnetic fields from the fact that Maxwell's equation (in a vacuum) had to be valid on all reference frames. The algebra is much simpler if we use tensors in Minkowski space, although Einstein's long-winded method provides more physical insight.

5.2 Gauge Transformations

The components of the four-vector A^μ are not unique.

We see that if we add to A^μ any the derivative of any Lorentz invariant (scalar) function $\Omega(x)$

$$A^\mu \ \rightarrow A^\mu + \partial^\mu \Omega \tag{5.9}$$

then we can see immediately for (5.1) that the components the terms proportional to Ω cancel and of the electromagnetic field tensor remain the same.

The transformation (5.9) is called a "gauge transformation"" and the fact that such a transformation does not affect the components of the electric or magnetic field tells us that the components of A^μ are *not* physical observables. Only the components of the electric and magnetic field are physically measurable quantities.

We are free to choose a gauge at our convenience, by imposing a condition on the four-vector potential. There are three popular gauges which are often used:

- **Landau Gauge:**

  $$\partial_\mu A^\mu = 0$$

  This is useful when looking at the wave equation obeyed by A^μ.

- **Coulomb Gauge:**

  $$\boldsymbol{\nabla} \cdot \mathbf{A} \ = 0$$

  This is useful for electrostatic problems but it explicitly breaks Lorentz invariance and so it is rarely used in relativistic problems.

- **Axial gauge**

  $$n \cdot A \ = \ 0$$

  where n is some suitably chosen four-vector. This also breaks manifest Lorentz invariance and also rotation invariance, but it is nevertheless sometimes useful.

5.3 Relativistic Form of Maxwell's Equations

Two of Maxwell's equations of electromagnetism relate derivatives of electric or magnetic fields to components of current density or charge density. These can be combined into a single equation relating four-vectors

$$\partial_\mu F^{\nu\mu} \;=\; \mu_0 j^\nu, \tag{5.10}$$

where μ_0 is the permeability of the vacuum, related to c and the permittivity of the vacuum, ϵ_0

$$c^2 \;=\; \frac{1}{\mu_0\epsilon_0}$$

For $\nu = 0$, (5.10) becomes

$$\boldsymbol{\nabla}\cdot\mathbf{E} \;=\; \frac{\rho}{\epsilon_0}$$

and for $\nu = i$ the equations are the components of

$$\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \;=\; \mu_0\mathbf{j}.$$

(5.10) is an equation between two quantities which transform as a four-vector under Lorentz transformations. It is therefore valid in any reference frame.

If we use (5.1) to write $F^{\mu\nu}$ in terms of the vector potential A^μ then (5.10) becomes

$$\partial_\nu\partial^\nu A^\mu - \partial^\mu\left(\partial\cdot A\right) \;=\; \mu_0 j^\mu \tag{5.11}$$

we can fix the gauge such that $\partial\cdot A = 0$ and this becomes the standard (in-homogeneous) wave equation for each remaining component of A^μ,

$$\Box A^\mu \;\equiv\; g^{\alpha\beta}\partial_\alpha\partial_\beta\, A^\mu \;=\; \mu_0 j^\mu \tag{5.12}$$

The other two Maxwell's equations require the totally anti-symmetric (covariant) Levi-Civita tensor in Minkowski space $\epsilon^{\mu\nu\rho\sigma}$. The components of this tensor vanish if any two of the indices are equal, take the value 1 if μ,ν,ρ,σ is an even permutation of (1,2,3,4) and -1 for sn odd permutation. The contravariant tensor $\epsilon_{\mu\nu\rho\sigma}$ (in Minkowski space) has the opposite sign.

The tensor $\partial^\rho\partial^\mu A^\nu$ is symmetric under the interchange of indices $\rho \leftrightarrow \mu$ and likewise the tensor $\partial^\rho\partial^\nu A^\mu$ is symmetric under the interchange of indices $\rho \leftrightarrow \nu$. Since $\epsilon_{\mu\nu\rho\sigma}$ is antisymmetric under the interchange of any pair of indices, it follows that

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} \;=\; \epsilon_{\mu\nu\rho\sigma}\partial^\nu\partial^\rho A^\sigma - \epsilon_{\mu\nu\rho\sigma}\partial^\nu\partial^\sigma A^\rho \;=\; 0 \tag{5.13}$$

This can be rewritten as a Bianchi identity

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} \;=\; 0 \tag{5.14}$$

For $\mu = 0$ (5.13) becomes

$$\boldsymbol{\nabla} \cdot \mathbf{B} \;=\; 0$$

and for $\mu = i$ we get the components of

$$\boldsymbol{\nabla} \times \mathbf{E} \;=\; -\frac{\partial \mathbf{B}}{\partial t}.$$

Again as this is an equation involving a quantity which transforms as a covariant vector under Lorentz transformations, it is valid in any reference frame.

5.4 Particle Moving in Electromagnetic Field

The action for a particle of mass m and charge q in an electromagnetic field derived from an electromagnetic four-vector potential, A^μ, in Minkowski space, is obtained by adding the interaction term

$$q\,c\int d\lambda\,\eta_{\mu\nu}\dot{x}^\mu A^\nu$$

($\cdot$ meaning differentiating w.r.t the parameter λ)

$$S \;=\; c\int d\lambda\,\left\{m\,(\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2} + q\,\eta_{\mu\nu}\dot{x}^\mu A^\nu\right\}, \tag{5.15}$$

The Lagrange equation of motion is

$$\frac{d}{d\lambda}\left\{\left(\eta_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right)^{-1/2}\dot{x}^\nu + \frac{q}{m}\dot{A}^\nu\right\} \;=\; \frac{q}{m}\eta^{\mu\nu}\eta_{\rho\sigma}\dot{x}^\rho\frac{\partial A^\sigma}{\partial x^\mu} \tag{5.16}$$

Using

$$\frac{d\lambda}{d\tau} \;=\; c\left(\eta_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right)^{-1/2},$$

and expressing $\dot{A}^\nu$ as

$$\dot{A}^\nu \;=\; \eta^{\mu\nu}\frac{\partial A_\mu}{\partial x^\rho}\,\dot{x}^\rho$$

we can rewrite (5.16) as

$$\frac{d}{d\lambda}\left(\frac{dx^\nu}{d\tau}\right) \;=\; \frac{q}{m}\eta^{\mu\nu}\eta_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{\partial A^\sigma}{\partial x^\mu}\frac{d\tau}{d\lambda} - \frac{q}{m}\eta^{\mu\nu}\frac{\partial A_\mu}{\partial x^\rho}\frac{dx^\rho}{d\tau}\frac{d\tau}{d\lambda}$$

and finally

$$\frac{d^2x^\nu}{d\tau^2} \;=\; \frac{q}{m}\eta^{\mu\nu}F_{\mu\rho}\frac{dx^\rho}{d\tau} \tag{5.17}$$

Both sides of (5.17) transform as a four-vector and so the equation of motion is valid in any reference frame.

In the non-relativistic limit, we may set $d\tau \approx dt$ and for space-like indices $\nu = i$ of (5.17) gives the non-relativistic law for the force, $\mathbf{F}_{em}$, acting on a charged particle moving in an electric and magnetic field

$$\mathbf{F}_{em} \;=\; m\frac{d^2\mathbf{x}}{dt^2} \;=\; q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

6 Principle of Equivalence

So far we have been considering Special Relativity, in which gravity is neglected and inertial frames move with a constant velocity relative to other inertial frames.

6.1 Weak Equivalence Principle

The weak equivalence principle states that any point particle follows the same path in a gravitational field - i.e its gravitational mass m_G, which determines its coupling to gravity is identical to its inertial mass m_I, which determined how it responds to the application of a force. In Newtonian mechanics, this means that if the force acting on a particle in a gravitational field, $\mathbf{G}$ is given by

$$\mathbf{F} = m_G \, \mathbf{G}$$

and the acceleration $\mathbf{a}$ resulting from the application of that force is given by

$$\mathbf{F} = m_I \mathbf{a}$$

then

$$m_G = m_I$$

This was tested extensively by Eötvös who carried out experiments between 1885 and 1909, using a torsion balance. Galileo suspected that this principle was true three centuries earlier when he performed his experiment of dropping objects of different mass but the same volume from the tower of Pisa and showed that they took the same time to fall to the ground .

6.2 Strong Equivalence Principle

The weak equivalence principle refers to gravitational interactions. The strong equivalence principle, which is an axiom of General Relativity, extends this to *all* laws of physics. It states that one cannot perform an experiment which can distinguish between a non-accelerating frame of reference in a uniform gravitational field and a frame in which gravity is absent, but which is accelerating uniformly.

If we are in a windowless laboratory on Earth and we drop an object, it accelerates towards the centre of the Earth at a rate of $9.81\,\mathrm{m\,s^{-2}}$. If we were in a windowless laboratory in outer space where gravity is negligible then this would not happen and we would experience weightlessness. However, if the laboratory in outer space were to accelerate upwards at a rate of $9.81\,\mathrm{m\,s^{-2}}$, the object would fall to the ground in the same way. The strong principle of equivalence tells us that this is the case for *any* physics experiment that one can devise[8].

[8]For the Earth's gravitational field this is not completely correct, because the gravitational field is not uniform as it is pointed radially towards the centre of the Earth, giving rise to ";l effects" which will be discussed later.

A reference frame moving with constant velocity on Earth is not really an inertial frame as it is equivalent to an accelerating frame. An inertial frame is therefore redefined as a frame which is in free fall in a gravitational field and is only the same as a non-accelerating frame in the absence of a gravitational field. The first postulate of Special Relativity is then amended to " the laws of physics are the same in any frame which is in free fall in a gravitational field" .

6.3 Light Bent by Gravity

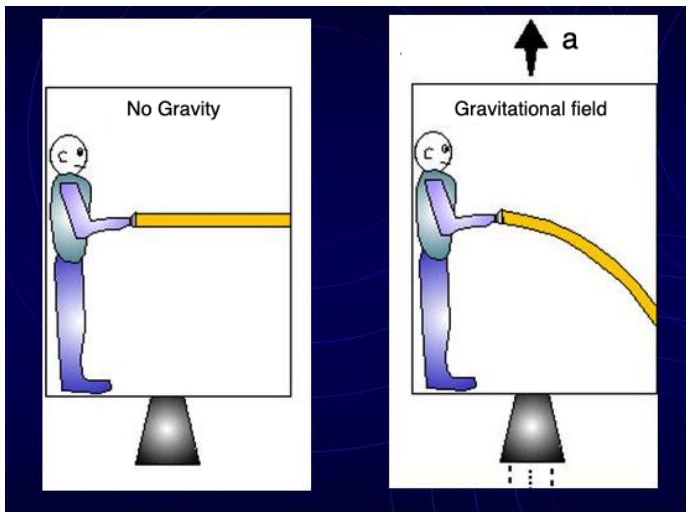

Figure 5: Path of a light beam in no gravitational field (left) and in an upward accelerating frame equivalent to a downward gravitational field (right).

An immediate consequence of this is that light is bent by gravity as shown in Figure 5 in which the right-hand diagram shows an upwardly accelerated frame. Between the time that the light is emitted from the source and the time it hits the far wall, the upward velocity of the module, in which the experiment is being conducted, has increased and so the light beam lands at a place which is lower down in the module. Since upward acceleration is equivalent to a downward gravitational field, this means that light is bent by gravity.

The bending of light by gravity was demonstrated by two observations of a solar eclipse in 1919 – one by a team led by Arthur Eddington on the island of Principe off the coast of West Africa and another by a team led by Andrew Crommelin in Sobral, Brazil. During this eclipse Eddington was able to observe a group of stars known as Hyades, which was directly behind the sun, but could nevertheless be observed owing to the bending of the light from Hyades caused by the gravitational field of the sun.

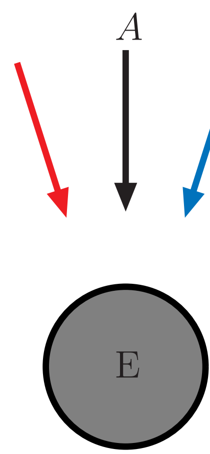

Figure 6: Two objects, either side of observer A. All three are in free fall. The two objects are seen to approach each other as they fall.

6.4 Tidal Effects

The Principle of Equivalence refers to reference frames in a *uniform* gravitational field, i.e. everywhere of the same magnitude and in the same direction. The gravitational field of the Earth points radially towards the centre - and therefore not always in the same direction. If an observer, A, is in free fall in the Earth's gravitational field and there are objects either side of the observer which are also in free fall, then, as seen by a free-falling observer, B, in outer space, where the Earth's gravitational field is negligible, those two objects will have a very small component of acceleration perpendicular to the direction of the acceleration of observer A. The two objects therefore are accelerating relative to observer A and their motion towards each other can be observed by A. This is called the "tidal effect".

The presence of tidal effects provides a restriction to the validity of the principle of equivalence. An observer, E, on Earth (not accelerating but in a gravitational field, will observe B and C to be accelerating towards each other. But this would *not* be the case if the Earth were absent but E was accelerating in th direction of A - th principle of equivalence is only valid "locally" i.e. over distances for which variations in th gravitational field may be neglected.

6.5 Curved Space-Time

The bending of light in a gravitational field implies an re-interpretation of the law of rectilinear propagation, which tells us that light travelling in a vacuum, moves in straight lines

Consider two inertial frames A and B in free-fall in the Earth's gravitational field, but where B is closer to the Earth than A so that the gravitational field is stronger. B is accelerating relative to A - but they are both inertial frames. If B shines a light beam across, perpendicular to the direction of acceleration of her space-module, then she will observe it to travel in a straight line across the module. However, as observed by A the light path is bent due to the acceleration of B relative to A.

This apparent dilemma is removed if “straight line” is interpreted as “the shortest distance between two points”. This can appear to be other than a straight line if space-time is curved. In a space with curvature, a straight line is defined as the shortest distance between two points and is known as a “geodesic’. The interpretation of gravity in General Relativity is that in a gravitational field space-time is curved but there is no “force” in the curved space-time - so that an object travelling in a gravitational field travels also along a geodesic in curved space-time. Conversely, matter (which generates a gravitational field) induces the curvature of space-time in a manner described by the equation of General Relativity.

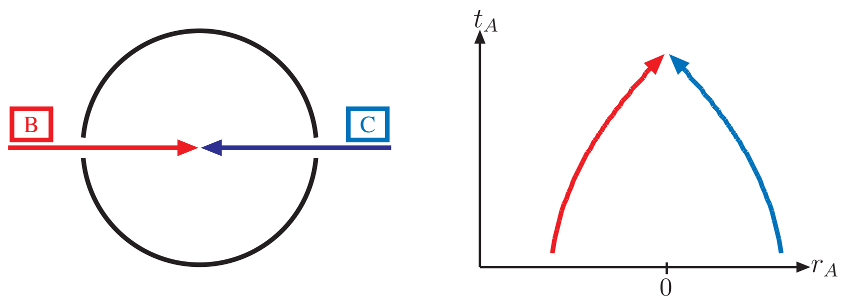

Figure 7: Two observers B and C at rest relative to two objects falling to Earth from opposite sides, as seen from an observer A in outer space. In either frame B or C the world-lines are parallel to the time axis, but in frame A they curve towards each other (as shown on the right) and collide at the centre of the Earth (marked as ’O’).

A further demonstration of the curvature of space-time induced by a gravitating object can be seen by imagining two objects, released at the same time from the same distance from the centre of the Earth, but from opposite sides. We will also imagine that there is a hole through the Earth so that these objects can eventually collide.

In the inertial (free-falling) frames, B and C, of each of these objects, they are stationary. In other words, their world-lines are straight lines running parallel to the time axis. However, as seen by an observer A in outer space where the Earth’s gravitational field is negligible, the world-lines of B and C start off parallel to the time axis, but then turn towards each other, as shown on the right of Figure 7, so that they meet at the Earth’s centre. So we see that a straight line in one inertial frame of reference appears as a curve in another inertial frame. In General Relativity the effect of the gravitational field is not to generate a force acting on a falling object but rather to induce a curvature on space-time so that the curved world-line obtained by plotting distance (in a given direction) against time is actually a geodesic path in the curved space-time. The falling object is *not* subjected to a force, but moves along a geodesic in the curved space.

7 Two Dimensional Curved Space

The geometry in flat space – Euclidean geometry – is a set of theorems concerning the properties of shapes drawn on a flat sheet of paper, such as the sum of the angles of a triangle, the circumference of a circle of a given radius, Pythagoras' theorem etc.

The study of geometry in curved space is called "Riemannian geometry".

We cannot imagine curved space in three dimensions, let alone a curved four-dimensional space-time, but we can consider geometry in a two-dimensional sub-space which is curved, such as the surface of a balloon or a globe. On such a curved surface, many of the theorems of Euclidean geometry do not hold.

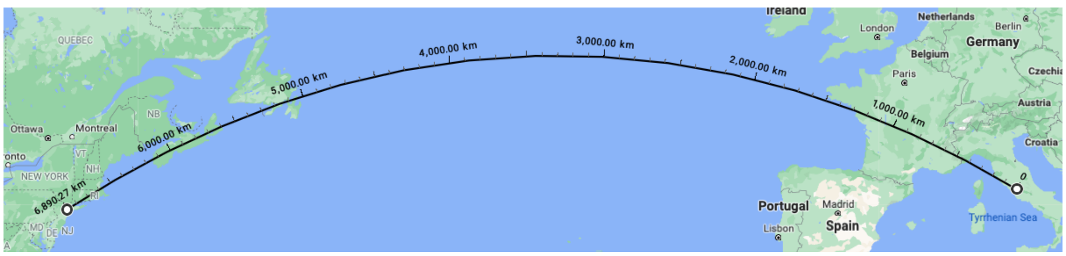

Figure 8: Shortest path between Rome and New York. It is *not* due West, but has a starting direction 55° North

The cities of New York and Rome are approximately at the same latitude, but distance between these cities along a path pointing East-West is 7130 km long, whereas the shortest distance between the cities is only 6891 km. The path (geodesic on the Earth's curved surface) starts in a direction which is approximately North-West from Rome to New York.

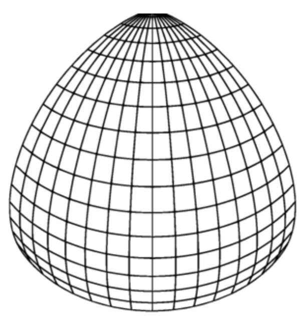

Figure 9: Octant of a sphere whose boundary is a triangle with three right-angles

The sum of angles in a triangle drawn on the Earth's surface depends on the size of the triangle. If the triangle is much smaller than the radius of the Earth then it is almost indistinguishable from a triangle drawn on a flat surface, so the sum of their angles is very close to 180°, but for larger triangles we get a larger angle sum. For example, start at the North pole and draw a line down the Greenwhich meridian (longitude 0°) as far as the equator, turn through a right-angle and go due East until you get to the meridian at longitude 90°, turn again through a right-angle and return along the 90° meridian to the North pole. You approach the North pole at 90° to the Greenwich meridian. This triangle is a boundary of one octant of the sphere as shown in Figure 9 This means that you have drawn a triangle with three right-angles - a sum of 270°.

The circumference of a circle of radius R is less than $2\pi R$ and depends on the size of the circle. Again, for small circles whose radius is much smaller than the Earth's surface the curvature, has a negligible effect. On the other hand, suppose we draw a circle whose radius, R, is $\frac{1}{4}$ of the circumference of the Earth, with its centre at the North pole. Such a circle is, in fact, the equator and the circumference is therefore equal to the circumference of the Earth which is $4R$ - somewhat less than $2\pi R$ for a circle of the same radius drawn in flat space.

Consider an observer A on the equator in a village (village (A)) in Gabon, at longitude 9° East. He sets the origin of his coordinate system at the centre of the village with the x axis pointing East and the y-axis pointing North. He drives to a neighbouring village 12 km due North and 16 km due East. Since these distances are extremely small compared with the radius of the Earth, the direct distance between the villages is given by Pythagoras' theorem - 20 km. Furthermore it makes no difference if A travels north and then east or East and then North.

A now travels to observer B located at village (B) which is 53° North and 83° East. . He arrives there by travelling 8000 km due East and then 6000 km due North. Pythagoras theorem would suggest that the distance between village(A) and village (B) is 10000 km, whereas the distance along the geodesic path linking the two villages is only 9600 km. Furthermore, if A travels 6000 km North followed by 8000 km East (rather than the other way around) he would end up 1300 km East of village (B). On a curved surface, the order in which A travels over a series of paths of given length affects the destination.

Observer B sets her coordinate system up in the same way as observer A - namely she places her origin at the centre of her village at 53° North and 83° East. with the x-axis pointing East and the y-axis pointing North. But if observer A looks at her coordinate system he sees her x-axis pointing in the same direction as his, whereas he sees her y-axis as pointing at an angle of 125° to the x-direction rather than at right-angles. Moreover, the inclination of (North-pointing) y-axis in observer B's frame increases as one goes North and this means that observer A's view of the y-axis in observer B's frame is actually curved (see Figure 10).

All this can be encoded by amending the expression for the distance, Δs, between two neighbouring points. This is achieved by generalising the expressions for the components of the metric. The distance ds between two neighbouring points (in a two-dimensional space) is given (see (1.12)) by

$$ds^2 = g_{xx}dx^2 + g_{yy}dy^2, \tag{7.1}$$

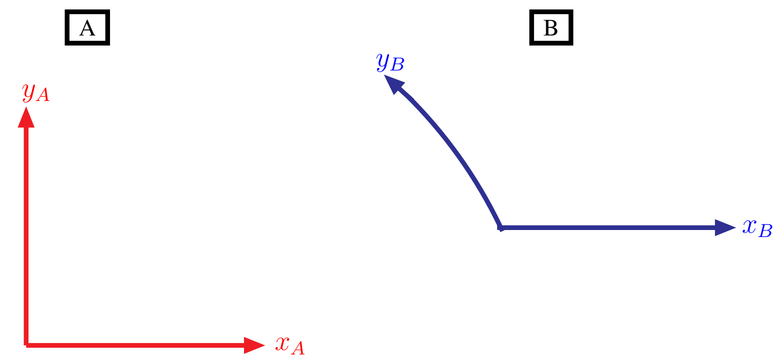

Figure 10: Observer A's view of observer (from village (A) at 0° N, 9° E) of B's axes which are rectilinear in observer frame located at village (B) which is 53° North, 85° East. Note that B's y-axis is seen to be inclining to the West and curved. We can picture the y-axes in A's and B's frames as two meridian lines at different longitudes.

but now the components, g_{xx}, g_{yy} of the metric g_{ij} vary from position to position on the spherical surface – they are themselves functions of x and y, unlike the case of flat Euclidean space where $g_{xx} = g_{yy} = 1$. For small x and y - where we are near the origin, they are both very close to one and flat-space (Euclidean) geometry is a very good approximation.

On the equator the lines of longitude are parallel and perpendicular to the lines of latitude, so that the metric is approximately Euclidean near any one point. We could try to stick a postage stamp on a point. The postage stamp would have to be "scrunched up" to be stuck over the surface but if the stamp was sufficiently small compared with the radius of the sphere this would be a negligible effect[9]. On the other hand, we can choose the "equator" to run through any point. For geographical reasons we choose the Earth's to be in the plane normal to its axis of rotation, but in general the equator could be any great circle on the sphere, passing through any point. This means that we can always choose a coordinate system such that in the neighbourhood of a given point on the sphere the metric is Euclidean, its derivatives with respect to the coordinates vanish and the Christoffel symbols vanish. In the above example we could choose a coordinate system which was Euclidean at the location of observer B. She would then see the longitude lines of observer A as being curved. In four-dimensional space-time this is extended to mean that for any curved space-time there exists a coordinate system such that for a sufficiently small region surrounding any space-time point the metric is Minkowskian up to any given approximation.

In the case of a spherical surface of radius r, g_{xx} and g_{yy} are

$$g_{xx} = \sqrt{1 - \frac{y^2}{r^2}}$$

$$g_{yy} = 1,$$

[9]Strictly a flat postage stamp only touches the sphere at one point. The Euclidean axes on the stamp are axes in the "tangent space" at the point of contact.

so we see that if y is much smaller than the radius R the effect of the curvature is negligible. But for points with larger values of y, the x-axis "shrinks". On a globe this is equivalent to the fact that the distance between places with a given difference in longitude, decreases as one moves away from the equator. The distance between two places on the equator whose longitudes differ by 1° is 111 km, whereas the distance between Pisa and Florence which are almost at the same latitude – 44° North – and approximately 1° apart in longitude, is only 80 km.

We can, of course, have other types of curves surfaces for which g_{xx} has a different dependence of position (x, y), and g_{yy} differs from one (and we can also have a metric with a non-zero off-diagonal component, g_{xy}.

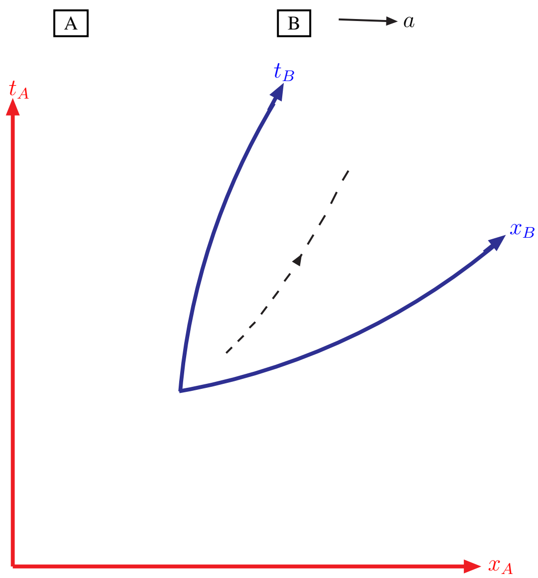

Figure 11: Observer A's view of observer B's space-time coordinate system. B is at a point of lower gravitational potential and is therefore accelerating relative to B As seen by A, the x- and time coordinates in observer B's frame are curved. The angular rotation of the time and x-coordinates for two coordinates systems (Minkowski diagrams) which are separated in space and time is interpreted as a Lorentz transformation between the two coordinate systems. The curved dashed line (green) is the path of a light beam (along a geodesic) as seen by observer A.

A similar thing happens in curved space-time. Return to the example of observers A and B released from starting points a long way above the surface of the Earth, but where B is released earlier and from greater height than A. As observer B passes A she is moving relative to A in the radial direction with some velocity. At that instant A and B are both accelerating at the same rate and we can apply the laws of Special Relativity - both observers are moving in a four-dimensional space-time with a Minkowski metric, and momentarily B is moving with uniform velocity relative to A. At that moment, $A's$ observation of B's frame will show the radial and time axes oriented towards each other as shown in Figure 2, but nevertheless the axes appear as straight lines

However, at some later time B will be closer to the Earth than A and therefore will be in a stronger gravitational field. They are both frames in free-fall. However, A's observation of B's Minkowski frame will show that not only are the axes oriented towards each other but this angle of orientation increases as one moves away from the origin (in the positive direction) and therefore the axes are curved as shown in Figure 11

8 Motion in Curved Space-Time

For Newtonian gravity a particle of mass m in a gravitational field $\mathbf{G}$ is subject to a force $m\mathbf{G}$ which causes the particle to accelerate. In General Relativity there is *no* force due to gravity, so that in the absence of any other force (such as that produced by an electromagnetic field) the particle is treated as a free particle and as such the path of the particle is a "straight line", but in a curved space-time. The curvature is encoded by replacing the Minkowski metric $\eta_{\mu\nu}$ by a metric tensor $g_{\mu\nu}$ whose components are, in general, functions of the space-time coordinates, x^μ.

8.1 Free Particles in a Gravitational Field

A free particle moving in a space with metric $g_{\mu\nu}$ sweeps out a world-line, whose coordinates x^μ are functions of some parameter, λ.

The action, S, for such a particle as it moves from initial coordinates $x^\mu(0) = x^\mu_i$ to final coordinates $x^\mu(1) = x^\mu_f$, is given by an integral over the parameter λ

$$S \ = m \int_0^1 d\lambda \sqrt{g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}, \tag{8.1}$$

where

$$\dot{x}^\mu \ \equiv \ \frac{dx^\mu}{d\lambda}.$$

The geodesic path followed by the free particle is the path that minimizes this action, given by the Lagrange equation

$$\frac{d}{d\lambda}\left(\frac{g_{\mu\nu}\dot{x}^\nu}{2\sqrt{g_{\sigma\tau}\dot{x}^\sigma\dot{x}^\tau}}\right) \ = \ \frac{1}{2\sqrt{g_{\sigma\tau}\dot{x}^\sigma\dot{x}^\tau}}\frac{\partial}{\partial x^\mu} g_{\nu\rho}\dot{x}^\nu\dot{x}^\rho \tag{8.2}$$

Using

$$\frac{dg_{\mu\nu}}{d\lambda} \ = \ \frac{\partial g_{\mu\nu}}{\partial x^\rho}\dot{x}^\rho,$$

and some algebra, we find that the geodesic equation for the coordinate x^μ is

$$\frac{d^2 x^\mu}{d^2\lambda} + \Gamma^\mu_{\nu\rho}\frac{\partial x^\nu}{d\lambda}\frac{\partial x^\rho}{d\lambda} \ = \ 0 \tag{8.3}$$

where $\Gamma^{mu}_{\nu\rho}$ are the Christoffel symbols given by (2.2).

Example 1: Polar coordinates in 2-Dimensions

In this space the metric is $g_{\rho\rho},\ g_{\phi\phi} = \rho^2,\ \ g_{\rho\phi} = g_{\phi\rho} = 0$. The non-zero Christoffel symbols are

$$\Gamma^\rho_{\phi\phi} = -\rho,\ \Gamma^\phi_{\rho\phi} = \Gamma^\phi_{\phi\rho} = \rho^{-1}. \tag{8.4}$$

The geodesic equation for ϕ is

$$\ddot{\phi} + 2\frac{\dot{\rho}\dot{\phi}}{\rho} \;=\; 0, \tag{8.5}$$

which has solution

$$\dot{\phi} = \frac{1}{\rho^2}, \tag{8.6}$$

(the overall constant can be absorbed into the re-scaling of the parameter λ).

The geodesic equation for ρ is

$$\ddot{\rho} - \rho\dot{\phi}^2 \;=\; 0. \tag{8.7}$$

Using (8.6) this may be written

$$\ddot{\rho} - \frac{1}{\rho^3} \;=\; 0. \tag{8.8}$$

which has solution

$$\dot{\rho} \;=\; \sqrt{c^2 - \frac{1}{\rho^2}} \tag{8.9}$$

Dividing (8.6) by (8.9), we get

$$\frac{d\phi}{d\rho} \;=\; \frac{1}{\rho\sqrt{c^2\rho^2 - 1}}$$

This has solution

$$\rho\cos(\phi - \alpha) = -\frac{1}{c}.$$

This is the equation for a straight line with gradient α and an intercept at $x = -1/(c\cos\alpha)$

Example 2: Surface of a 2-Sphere

The metric on the surface of a sphere (of unit radius) is

$$g_{\theta\theta} = 1,\; g_{\phi\phi} = \sin^2\theta,\; g_{\theta\phi} = g_{\phi\theta} = 0,$$

where θ is the polar angle and ϕ is the azimuthal angle. The non-zero Christoffel symbols are

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\,\cos\theta,\; \Gamma^{\phi}\phi\theta = \Gamma^{\phi}_{\theta\phi} = \cot\theta \tag{8.10}$$

The geodesic equation for ϕ is

$$\ddot{\phi} + 2\cot\theta\,\dot{\theta}\ddot{\phi} \;= 0 \tag{8.11}$$

with solution

$$\dot{\phi} \;=\; \frac{a}{\sin^2\theta} \tag{8.12}$$

The geodesic equation for θ is

$$\ddot{\theta} - \sin\theta\,\cos\theta\,\dot{\phi}^2 \;=\; 0, \tag{8.13}$$

Using (8.12) we may write this as

$$\ddot{\theta} - a^2 \frac{\cos\theta}{\sin^3\theta}. \tag{8.14}$$

This has a solution

$$\dot{\theta} \;=\; \sqrt{1 - \frac{a^2}{\sin^2\theta}}, \tag{8.15}$$

Using (8.12 this gives the dependence of ϕ on θ

$$\frac{d\phi}{d\theta} \;=\; \frac{a}{\sin\theta\left(\sin^2\theta - a^2\right)}$$

This has solution

$$\cos\left(\theta - \theta_0)\right) \;=\; \lambda\cot\phi, \tag{8.16}$$

($\lambda = a/\sqrt{1-a^2}$). The constants θ_0 and λ are obtained by inserting the initial and final values of θ and ϕ.

This path is clearly not a straight line. If the two points lie on the equator then the shortest path between them is also along the equator. For the path between any other two points which are not on the equator we can rotate the globe until the two points lie on the new equator. The geodesic is therefore lies on the great circle (largest possible circle) around the sphere (i.e. the "rotated equator" which passes through the two points).

Example 3: One dimensional Static Gravitational Field

Consider a metric[10]

$$g_{00} = 1 + \frac{2}{c^2}\Phi(\mathbf{x}),\; g_{11} = -\left(1 + \frac{2\Phi}{c^2}\right)^{-1},\; g_{22} = g_{33} = -\delta_{ij},$$

where $\Phi(\mathbf{x})$ is the gravitational potential.

This is a fictitious metric but it is a good approximation to the metric which corresponds to a weak static gravitational field (in the reference frame for which the gravitational field is static).

The non zero components of the Christoffel symbols are

$$\Gamma^i_{00} = \left(1 + \frac{2\Phi}{c^2}\right)\frac{\Phi'}{c^2},\quad \Gamma^0_{10} \;=\; -\frac{\Phi'}{c^2}\left(1 + \frac{2}{c^2}\Phi\right)^{-1}\frac{\Phi'}{c^2},\quad \Gamma^1_{11} \;=\; -\frac{\Phi'}{c^2}\left(1 + \frac{2}{c^2}\Phi\right)^{-1}, \tag{8.17}$$

($\Phi' \equiv \partial_x\Phi$).

The geodesic equation for x is

$$\ddot{x} + \Phi'\left((1 + 2\Phi/c^2)\dot{t}^2 - \frac{\dot{x}^2}{1 + 2\Phi/c^2}\right) \;=\; 0 \tag{8.18}$$

[10]This metric does *not* obey Einstein's equation of General Relativity, xcept in the case of a constant gravitational field ($\Phi = -ax$.)

($\cdot$ meaning differentiating w.r.t. proper time) The geodesic equation for time ($t = x^0/c$) is

$$\ddot{t} + \frac{2}{c}\left(\boldsymbol{\nabla}\Phi\right) \cdot \dot{\mathbf{x}}\,\dot{t}\left(1 + \frac{2}{c^2}\Phi\right)^{-1} = 0 \tag{8.19}$$

For a particle of mass m and energy E, (in the observer's frame) (8.19) has a solution

$$\dot{t} \;=\; \frac{E}{mc^2}\left(1 + \frac{2}{c^2}\Phi\right)^{-1}, \tag{8.20}$$

This expression is consistent with the definition for the energy as measured by an observer whose four-velocity is v^μ

$$E(v) \;=\; mc\, g_{\mu\nu} v^\mu \dot{x}^\nu$$

In the observer's rest frame $v^\mu = (c, 0, 0, 0)$ so that

$$E \;=\; mc^2\left(1 + \frac{2\Phi}{c^2}\right)\dot{t}$$

The expression for the metric can be written as

$$\left(1 + \frac{2\Phi}{c^2}\right) m^2 c^4 \;=\; E^2 - m^2 c^2 \left|\dot{\mathbf{x}}\right|^2$$

If we write

$$E \;=\; mc^2 + T + m\Phi,$$

then to leading order in the kinetic energy T and the potential energy $m\,\Phi$ we find

$$T \;=\; \frac{1}{2} m \left|\dot{\mathbf{x}}\right|^2$$

which is the result from Newtonian mechanics valid on tthe non-relativistic limit, but acquiring realtivistic correction for large kinetic energy and/or large potential energy.

Inserting (8.20) into the (8.18) and performing some algebraic manipulations, we get

$$\ddot{x} \;=\; -\Phi' \tag{8.21}$$

This is Newton's expression for the acceleration of a particle in a potential Φ, but with the acceleration defined at the second derivative of the position vector w.r.t. *proper*-time. Prior to final derivation of Einstein's equation for gravity, he showed that if the acceleration is taken to be the second derivation w.r.t. (the observer's) time, then the relation between acceleration and the derivative of the potential could not be consistent with the postulate that the speed of light was universal.

8.2 Laws of Physics in a Gravitational Field

For the laws of physics to be valid in any reference frame in curved space (i.e. in a gravitational field) they have to be expressed in terms of relations between scalars, vectors or tensors, which transform in a specified way under a change of coordinate system. Such a change in coordinate system could be a general Lorentz transformation, but could also be a displacement in space-time to a point for which the metric takes a different value owing to a change in gravitational field. For this to happen derivatives of vectors or tensors must be replaced by covariant derivatives.

- Conservation of electric current (3.13) becomes

$$(\nabla_\mu j)^\mu \ = \ \partial_\mu j^\mu + \Gamma^\mu_{\mu\nu} j^\nu \ = \ 0 \tag{8.22}$$

- The conservation of the stress-energy tensor (3.20) becomes

$$(\nabla_\mu T)^{\mu\nu} \ = \ \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\sigma} T^{\sigma\nu} + \Gamma^\nu_{\mu\sigma} T^{\mu\sigma} \ = \ 0 \tag{8.23}$$

- The electromagnetic field tensor, $F^{\mu\nu}$ is constructed from the covariant derivative of the electromagnetic vector potential, A^μ. However, if we look at the covariant tensor $F_{\mu\nu}$ the terms involving the Christoffel symbols cancel out.

$$\begin{aligned} F_{\mu\nu} &= (\nabla_\mu A)_\nu - (\nabla_\nu A)_\mu \\ &= \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho - \partial_\nu A^\mu + \Gamma^\rho_{\mu\nu} A_\rho \\ &= \partial_\mu A_\nu - \partial_\nu A^\mu \end{aligned} \tag{8.24}$$

  (the Christoffel symbols are symmetric in the lower the indices).

  The contravariant tensor is formed using the inverse metric

$$F^{\rho\sigma} \ = \ g^{\rho\mu} g^{\sigma\nu} F_{\mu\nu}$$

- The first relativistic Maxwell's equation (5.10) becomes

$$(\nabla_\mu F)^{\mu\nu} \ = \ \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} + \Gamma^\nu_{\mu\rho} F^{\mu\rho} = \mu_0 j^\nu . \tag{8.25}$$

- The second Maxwell equation in terms of the covariant tensor $F_{\mu\nu}$ (5.14) (a Bianchi identity) remains unchanged

The Klein-Gordon equation in curved space is amended to

$$\hbar^2 g^{\mu\nu} \nabla_\mu \partial_\nu \Psi(x) \, + \, m^2 c^2 \Psi(x) \ = \ \hbar^2 g^{\mu\nu} \left(\partial_\mu \partial_\nu + \Gamma^\mu_{\mu\rho} \partial^\rho \right) \Psi(x) \, + \, m^2 c^2 \Psi(x) \ = \ 0. \tag{8.26}$$

8.3 Curvature

There are some spaces for which the geodesic is simply the equation for a straight line but expressed in terms of a transformed set of coordinates,

$$x^{\mu} \rightarrow y^{\mu}(x), \tag{8.27}$$

(x^{μ} are the coordinates in Minkowski space.) The metric in the transformed coordinates is

$$g_{\rho\sigma}(y) = \eta_{\mu\nu}\frac{\partial x^{\mu}}{\partial y^{\rho}}\frac{\partial x^{\nu}}{\partial y^{\sigma}} \tag{8.28}$$

For example the case of cylindrical polar coordinates (example 1 above) the transformation of coordinates is

$$y^2 = (x^1)^2 + (x^2)^2, \quad y^{\mu} = x^{\mu}, \ (\mu \neq 2)$$

Such coordinate transformations do *not* induce curvature and the spaces described by such metrics are flat.

However, other metrics which cannot be written in the form of (8.28) have curvature. Spaces described by such metrics have curvature.

We define the rank-(1,3) Riemann tensor, $R^{\rho}_{\sigma\mu\nu}$

$$R^{\rho}_{\sigma\mu\nu} \equiv \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{8.29}$$

This transforms as a tensor under coordinate transformations. In a flat space *all* of the components of the Riemann tensor are zero, but in a curved space some of the components are non-zero. Under a coordinate transformation the in-homogeneous terms in the transformation of the Christoffel symbols shown on (2.5) cancel in the construction of the Riemann tensor so that (8.29) does indeed transform as a tensor, despite the fact that the Christoffel symbols do not.

The Riemann tensor has the following symmetries

1. $$R^{\rho}_{\sigma\mu\nu} = -R^{\rho}_{\sigma\nu\mu} \tag{8.30}$$

2. $$g_{\tau\rho}R^{\rho}_{\sigma\mu\nu} = -g_{\sigma\rho}R^{\rho}_{\tau\mu\nu} \tag{8.31}$$

3. $$g_{\tau\rho}R^{\rho}_{\sigma\mu\nu} = g_{\mu\tau}R^{\tau}_{\nu\rho\sigma} \tag{8.32}$$

4. The Bianchi identity
$$R^{\rho}_{\sigma\mu\nu} + R^{\rho}_{\nu\sigma\mu} + R^{\rho}_{\mu\nu\sigma} = 0 \tag{8.33}$$

5. The differential Bianchi identity

$$(\nabla_\lambda R)^\rho_{\sigma\mu\nu} + (\nabla_\mu R)^\rho_{\sigma\nu\lambda} + (\nabla_\nu R)^\rho_{\sigma\lambda\mu} \;=\; 0 \tag{8.34}$$

We can always select a coordinate system in space-time such that at a given space-time point the metric is Minkowskian and the first derivatives of the metric w.r.t. the coordinates vanishes leading to vanishing Christoffel symbols. On the other hand the components of the Riemann tensor involve the *second* derivatives of the metric which do not vanish. The Riemann tensor is a property of the space under consideration and cannot be gauged away by a transformation of coordinates.

The metric tensor is symmetric and so in four dimensions has 10 independent components. The Bianchi identities give four constraints for the metric. They reflect the fact that the Riemann tensor is invariant under general coordinate transformations. This means that the number of physically relevant components of the metric (i.e. components that affect the curvature) is reduced to six.

We can also define a rank-2 covariant (symmetric) Ricci tensor, $R_{\mu\nu}$ by

$$R_{\mu\nu} \;\equiv\; R^\rho_{\mu\rho\nu} \tag{8.35}$$

It is possible for a space to be "Ricci flat" meaning that all components of the Ricci tensor vanish but some components of the Riemann tensor are non-zero.

Finally, for a space which is not Ricci flat we have a non-zero curvature scalar, R defined by

$$R \;\equiv g^{\mu\nu} R_{\mu\nu} \tag{8.36}$$

Contracting the indices μ and ρ in the differential Bianchi identity (8.34), we find

$$(\nabla_\lambda R)_{\sigma\nu} + (\nabla_\mu R)^\mu_{\sigma\nu\lambda} - (\nabla_\nu R)_{\sigma\lambda} \;=\; 0 \tag{8.37}$$

where in the last term we have used (8.30).

Using the fact that the covariant derivative of the metric vanishes - we can commute the metric through the covariant derivative operator, ∇_μ, we contract this with the inverse metric $g^{\nu\sigma}$ to yield

$$\nabla_\lambda R - (\nabla_\mu R)^\mu_\lambda - (\nabla^\sigma R)_{\sigma\lambda}\,,$$

where again we have used (8.30) in the middle term. Now since

$$(\nabla_\mu R)^\mu_\lambda \;=\; (\nabla^\sigma R)_{\sigma\lambda}$$

we get the relation

$$\nabla^\sigma \left(R_{\sigma\lambda} - \frac{1}{2} g_{\sigma\lambda} R \right) \;= 0 \tag{8.38}$$

Example 1: Polar coordinates in 2-Dimensions
Inserting the non-zero components of the Christoffel symbols, given in (8.4) into (8.29), and using

$$g_{\rho\rho} = 1, \quad g_{\phi\phi} = \left(g^{\phi\phi}\right)^{-1} = \rho^2,$$

we find

$$R^{\rho}_{\phi\rho\phi} = -\frac{d}{d\rho}\rho + \rho\rho^{-1} = 0$$

From the symmetry properties (8.30), (8.31), (8.32), all the other components of the Riemann tensor vanish. This is an expected result since polar coordinates are simply an alternative method from Cartesian coordinates, of labelling point in flat space. There is no curvature.

Example 2: Surface 2-Dimensional sphere
Inserting the non-zero components of the Christoffel symbols, given in (8.10) into (8.29), and using

$$g_{\theta\theta} = 1, \quad g_{\phi\phi} = \left(g^{\phi\phi}\right)^{-1} = \sin^2\theta,$$

we find

$$R^{\theta}_{\phi\theta\phi} = \sin^2\theta, \quad R^{\phi}_{\theta\theta\phi} = 1$$

with all other components obtained from the symmetry relations (8.30) and (8.31). The Ricci tensor is then found to be

$$R_{\theta\theta} = 1, \quad R_{\phi\phi} = \sin^2\theta, \quad R_{\theta\phi} = 0$$

and the curvature scalar is

$$R = 1 + \frac{1}{\sin^2\theta}\sin^2\theta = 2$$

In this case the curvature is non-zero, which is expected since this coordinate system labels points on the surface of a sphere.

Example 3: Weak Static Gravitational Field in x direction
Inserting the non-zero components of the Christoffel symbols, given in (8.17) into (8.29), and working only to first order in the gravitational potential Φ, we find

$$R^{0}_{i0j} = -\frac{1}{c^2}\partial_i\partial_j\Phi, \quad R^{i}_{0j0} = -\frac{1}{c^2}\partial^i\partial_j\Phi.$$

9 Curvature Induced by Matter/Energy

In Newtonian gravity, the gravitational potential, Φ is related to the energy density ρ by

$$\nabla^2\Phi \;=\; -4\pi G\rho, \tag{9.1}$$

where G is the gravitational coupling constant, $G = 6.67\times10^{-11}\,\mathrm{m}^3\,\mathrm{kg.\,s}^{-2}$.

$\rho\,c^2$ is the component T_{00} of the stress-energy tensor, and as we have seen we can obtain the motion of a particle moving in a weak gravitational field by adding the gravitational potential (multiplied by a factor $2/c^2$) to the component g_{00} of the metric, so that (9.1) can be rewritten

$$\nabla^2 g_{00} \;=\; 2\pi G\,T_{00},. \tag{9.2}$$

In order to amend this equation so that it is valid in any frame of reference in a general curved space, we need to be able to express this as an equation between tensors. T_{00} is a component of the stress-energy tensor $T_{\mu\nu}$ and so the equation of general relativity, which tells us what the metric is for a space in which the (covariant) stress-energy tensor has elements $T_{\mu\nu}$. In other words we need to find a rank-(0,2) tensor, $G_{\mu\nu}$ constructed solely out of the metric, which is proportional to $T_{\mu\nu}$.

$$G_{\mu\nu} \;=\; \kappa T_{\mu\nu}. \tag{9.3}$$

The constant of proportionality, κ is chosen so that in limit of a weak gravitational field in the frame in which the energy distribution is static, we recover the Newtonian expression (9.1).

There are two rank-(0,2) tensors that are constructed from the metric – one is the Ricci tensor and the other is the metric itself, $g_{\mu\nu}$. The "Einstein tensor", $G_{\mu\nu}$ must be a linear sum of these two tensors. The Ricci tensor has (space)-dimension 2 and so we need to multiply the metric by a scalar of dimension 2 to match the dimensions and the only such scalar constructed out of the metic is the curvature scalar R. Furthermore, the conservation of energy-momentum requires that the covariant divergence of the stress energy vanishes. This therefore implies that the divergence of $G_{\mu\nu}$ must also vanish. From (8.38) this gives

$$G_{\mu\nu} \;=\; R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{9.4}$$

The constant κ is determined by expanding the metric about the Minkowski metric

$$g_{\mu\nu} \;=\; \eta_{\mu\nu} + h_{\mu\nu}$$

and making the assumption that only h_{00} is non-zero. Working to first order in h_{00} we find the component G_{00} of the Einstein tensor is given by

$$G_{00} \;=\; \frac{1}{4}\nabla^2 g_{00}$$

Comparing this with (9.2) we find $\kappa = -4$, so finally the equation of general relativity is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \;=\; -\frac{8\pi G}{c^2}T_{\mu\nu} \tag{9.5}$$

9.1 Schwarzschild Metric

Schwarzschild has identified a static spherically symmetric metric, which in spherical polar coordinates (ct, r, θ, ϕ) is

$$g_{00} = \left(1 - \frac{r_S}{r}\right), \quad g_{rr} = -\left(1 - \frac{r_S}{r}\right)^{-1}, \quad g_{\theta\theta} = -r^2, \quad g_{\phi\phi} = -r^2 \sin^2\theta. \quad (g_{\mu\nu} = 0, \; \mu \neq \nu) \tag{9.6}$$

This metric generates non-zero components for the Riemann tensor, but the Ricci tensor vanishes so this metric is a solution to the equation of general relativity in free space.

If the source of the metric is a body of mass M centred at the origin $r = 0$), then the parameter r_S takes the value

$$r_S = \frac{2GM}{c^2} \tag{9.7}$$

The metric element g_{rr} has a singularity at the "Schwarzschild radius" $r = r_S$. However, this metric is only valid in free space, i,e, outside the surface of the body of the gravitating body of radius r_0. for most bodies (other than black holes) $r_0 \gg r_S$, so that we never get close to the Schwarzschild radius. For example for the sun with mas $M_\odot = 2 \times 10^{30}$ kg. and radius $r_\odot = 7 \times 10^8$ m.,

$$\frac{r_\odot}{r_S} \approx 20000.$$

The effect of general relativity become significant if the gravitational potential Φ is approaching c^2.

We can check that in the non-relativistic limit the Schwarzschild metric reproduces Newton's law of motion for the gravitational field of a spherically symmetric body of mass M. We consider only motion in the radial direction, so we set $d\theta = d\phi = 0$. The relevant Christoffel symbols are:

$$\Gamma^r_{00} = \frac{1}{2}\left(1 - \frac{r_S}{r}\right)\frac{r_S}{r^2}, \quad \Gamma^0_{r0} = \frac{1}{2}\left(1 - \frac{r_S}{r}\right)^{-1}\frac{r_S}{r^2}, \quad \Gamma^r_{rr} = -\left(1 - \frac{r_S}{r}\right)^{-1}\frac{r_S}{r^2}$$

The geodesic for time, $t = x^0/c$ is

$$\ddot{t} + \left(1 - \frac{r_S}{r}\right)^{-1}\frac{r_S}{e^2}\dot{r}\dot{t} = 0,$$

with solution

$$\left(1 - \frac{r_S}{r}\right)\dot{t} = 1, \tag{9.8}$$

where on the RHS the arbitrary constant has been set to unity by appropriate scaling of the parameter λ. The geodesic equation for the radial distance r is

$$\ddot{r} + \frac{1}{2}\left(1 - \frac{r_S}{r}\right)\frac{r_S}{r^2}c^2\dot{t}^2 - \left(1 - \frac{r_S}{r}\right)^{-1}\frac{r_S}{r^2}\dot{r}^2 = 0 \tag{9.9}$$

r_S is $\mathcal{O}(1/c^2)$ and making use of (9.8) we find that (9.9) can be written as

$$\frac{d^2r}{dt^2} + \frac{c^2 r_S}{2}\frac{1}{r^2} = \mathcal{O}\left(\frac{1}{c^2}\right) \tag{9.10}$$

Inserting (9.7) we see thst in the non-relativistic limit ($1/c^2 \to 0$) we reproduce Newton's law for a particle moving radially in the gravitational field of a spherically symmetric body of mass M:

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} \tag{9.11}$$

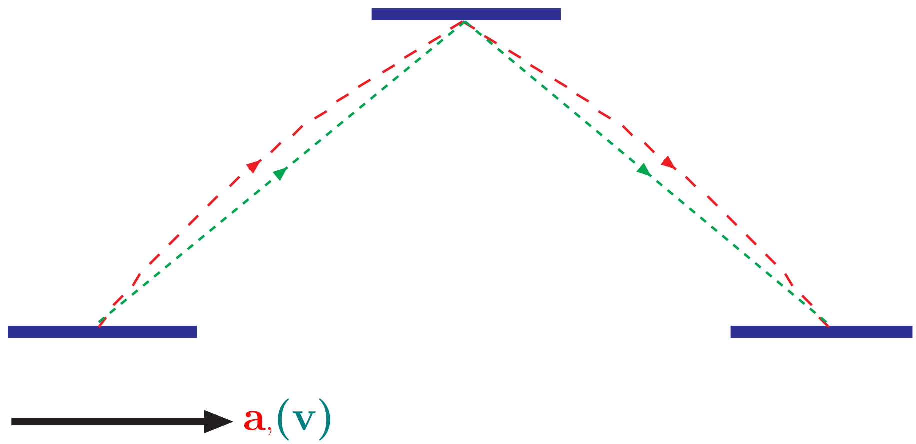

Figure 12: Path of light between two mirrors of a light clock in a frame which is accelerating relative to the observer (red) compared with moving with constant velocity relative to the observer (green).

10 Gravitational Redshift

The curvature of space-time means that there is a further contribution to time dilation arising from the presence of a gravitational field. Observers A and B are both inertial frames in free fall in the Earth's gravitational field, B being closer to the Earth than A and therefore in lower potential is accelerating relative to A. At any one instant B will be moving relative to A with velocity v relative to A and holds a light-clock. Observer A's observation of the light path between the mirrors will be a curve (even though it is seen by B to be a straight line - see Figure 12). Since light travels with the same speed in all frames it will take longer, according to A, to travel the curved distance between the mirrors of the light clock. This is in addition to the time dilation due to the velocity of B relative to A.

The Schwarzschild metric (9.6) is the metric which corresponds to a static spherically gravitational field. If momentarily B has zero velocity relative to A then at that instant we can set dr, $d\theta$, $d\phi$ to zero in (9.6) and we have the relation between proper time τ (the time measured by the clock of observer B at her origin) and time t in the frame of observer A

$$d\tau \; = \; \sqrt{1 - \frac{2GM}{c^2 r}} dt$$

Although for the Earth's gravitational field this is a tiny effect, it has actually been observed in atomic clocks held at different altitudes. Experiments carried out on a space station travelling at 7800 metres per second over a period of six months produced an observable time dilation of 7 milliseconds. The effect has to be accounted for in modern GPS devices.

This time dilation means that a light-source at a distance r from the centre of a gravitating body, of wavelength emitting light with wavelength λ_r, has a period

$$d\tau \; = \; \lambda_r / c$$

Since c is the same in all frames, then to an observer at an infinite distance from the gravitating body the observed wavelength is λ_∞, where

$$\frac{\lambda_\infty}{\lambda_r} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \equiv 1 + z, \tag{10.1}$$

where the quantity z is known simply as the "redshift". If r is sufficiently large that we may approximate the redshift to leading order in G then the energy of the photon, E_r emitted at radius r exceeds the energy of the photon, E_∞ observed at an infinite distance by

$$E_r - E_\infty \approx \frac{GME_r}{c^2 r}. \tag{10.2}$$

The RHS of (10.2) is the gravitational potenntial energy of a particle with initial energy E_r.

The gravitational red-shift was first observed in 1925 by Walter Adams from observations of the white dwarf star Sirius B, whose radius is 5.8×10^8 m. Adams measured one of the spectral lines of hydrogen (the H_α line). It was found to have a wavelength which is larger than that observed from a terrestrial measurement of the same spectral line by 0.025%. This means that the redshift $z = 2.5 \times 10^{-4}$. Inserting this and the radius of Sirius B into (10.1) we can calculate the mass of Sirius B to be 2×10^{30} kg (about $1\,M_\odot$.)

10.1 The Rebka-Pound experiment

In 1959, Robert Pound and Glen Rebka detected this gravitational shift using a Nuclear Physics technique developed by the Mössbauer. A radioactive isotope which emits γ-rays can also absorb the γ-rays with exactly the same frequency. This is called "resonance absorption". But it only works if the absorber is stationary relative to the source. If source and absorber move relative to each other then the frequency of the emitted γ-rays, as measured by the absorber is shifted due to the Doppler shift and the absorption does not take place. This technique can detect shifts in γ-ray frequencies of one part in 10^{14}. On the other hand, if the γ-ray frequency is shifted by a tiny amount due to the gravitational red-shift, then by adjusting the velocity of the absorber until the red-shift is cancelled by the Doppler effect due to the moving absorber, resonance absorption is recovered, The velocity needed to recover this resonance absorption can then be used to determine the gravitational shift in the frequency of the γ-rays.

They placed a source sample of the iron isotope ^{56}Fe on the top of the tower of the Jefferson Laboratory at Harvard University, 22.6 metres above ground-level, and an absorber sample at the bottom of the tower, with a scintillation counter below it. The velocity at which the source must move relative to the absorber to recover resonance absorption, was 0.00074 mm per second. This very small velocity, at which absorption was observed, was measured by placing the sample at the top of the tower in the cone of a loudspeaker to which they applied a pure signal with frequency that ranged between 10 and 50 Hertz. Absorption occurs once every cycle when the velocity of the loudspeaker membrane is exactly equal to the resonant velocity. The determination of the precise phase of the oscillation at which this absorption is observed allowed them to calculate the velocity of the membrane of the loudspeaker at which absorption occurs. In order to improve the accuracy, the experiment was conducted both with the upper sample as the source and the lower sample as the absorber and the other way around.

They obtained a result which was within 10% of the theoretical result. This accuracy was later improved to 1%.

11 The Orbit of Mercury

Kepler's laws of planetary motion were originally derived from astronomical observations but were later derived by Newton using Newtonian mechanics and Newton's law of gravitation.

One of these laws states that planets revolve around the sun in fixed elliptic orbits. The orbit of the Earth around the sun is almost circular – the Earth is 3% close to the sun in January than in July. But the orbit of Mercury is a much more eccentric ellipse. At its point of closest approach (called its "perihelion") it is two-thirds of the distance when it is furthest away (called its "aphelion').

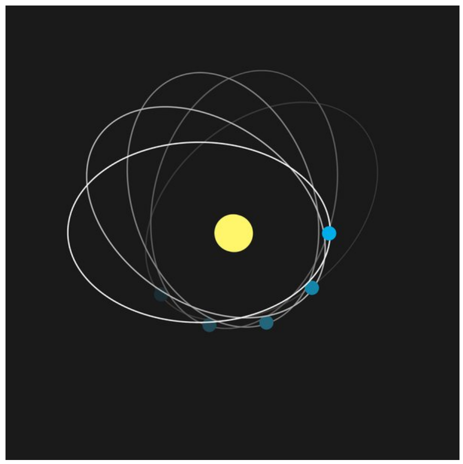

Figure 13: Precession of the orbit of Mercury

Although Kepler's laws predict a fixed elliptic orbit, the orbit of Mercury has been observed to precess very slowly at a rate of 575″ of arc every century. This was known in the nineteenth century and it was assumed that this was caused by the gravitational pull of other nearby planets in the solar system (such perturbations are not accounted for in the derivation of Kepler's laws). Much work was done in calculating the effect of such gravitational interaction with neighbouring planets, but the result yielded a precession rate of only 532″ of arc every century.

To examine the motion of planets around the sun (a static spherically symmetric gravitational field), we start with the Schwarzschild metric (9.6). We can restrict ourselves to the two-dimensional space-like plane $\theta = \frac{1}{2}\pi$, $d\theta = 0$. The expression for proper-time τ in the Schwarzschild metric with fixed polar angle ($\sin\theta = 1$, $d\theta = 0$) is

$$d\tau^2 = \left(1 - \frac{r_S}{r}\right) dt^2 - \frac{1}{c^2}\left(\left(1 - \frac{r_S}{r}\right)^{-1} dr^2 + r^2 d\phi^2\right) \tag{11.1}$$

The non-zero Christoffel symbols are then

$$\Gamma^r_{00} = \frac{1}{2}\left(1 - \frac{r_S}{r}\right)\frac{r_S}{r^2} \tag{11.2}$$

$$\Gamma^r_{rr} = -\frac{1}{2}\frac{1}{(1-r_S/r)}\frac{r_S}{r^2} \tag{11.3}$$

$$\Gamma^r_{\phi\phi} = -\left(1-\frac{r_S}{r}\right)r \tag{11.4}$$

$$\Gamma^0_{r0} = \frac{1}{2}\frac{1}{(1-r_S/r)}\frac{r_S}{r^2} \tag{11.5}$$

$$\Gamma^\phi_{\phi r} = \frac{1}{r} \tag{11.6}$$

Geodesic for t ($= x^0/c$):

$$\frac{d^2t}{d\lambda^2} + \frac{r_S}{(1-r_S/r)\,r^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \tag{11.7}$$

Solution:

$$\left(1-\frac{r_S}{r}\right)\frac{dt}{d\lambda} = b \tag{11.8}$$

For massive particles

$$\left(1-\frac{r_S}{r}\right)\frac{dt}{d\lambda} = \frac{E}{mc^2}, \tag{11.9}$$

where, in the parametrisation that identifies λ with proper time, E is the energy of the planet at infinite distance from the sun.

Geodesic for ϕ

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} = 0 \tag{11.10}$$

Solution:

$$r^2\frac{d\phi}{d\lambda} = \frac{L}{c} \tag{11.11}$$

Geodesic for r:

$$\frac{d^2r}{d\lambda^2} - \left(1-\frac{r_S}{r}\right)r\left(\frac{d\phi}{d\lambda}\right)^2 - \frac{1}{2\,(1-r_S/r)}\frac{r_S}{r^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{r_S}{2r^2\,(1-r_S/r)}\frac{E^2}{m^2c^4} = 0, \tag{11.12}$$

where we have used (11.9)

Identifying λ with $c\tau$, the expression for the metric (11.1) gives

$$1 = \left(1-\frac{r_S}{r}\right)c^2\left(\frac{dt}{d\lambda}\right)^2 - \frac{1}{(1-r_S/r)}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 \tag{11.13}$$

From (11.11) and (11.9) multiplying throughout by $(1-r_S/r)$ this becomes

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{(E^2-m^2c^4)}{m^2c^4} + \frac{r_S}{r} - \frac{L^2}{c^2r^2} + \frac{L^2r_S}{c^2r^3} \tag{11.14}$$

The last term is the correction due to General Relativity.

Again using (11.11) we can write an expression for the derivative of r w.r.t. azimuthal angle ϕ as

$$\frac{L^2}{r^4}\left(\frac{dr}{d\phi}\right)^2 = \frac{(E^2-m^2c^4)}{m^2c^2}+\frac{r_Sc^2}{r}-\frac{L^2}{r^2}+\frac{L^2r_S}{r^3} \tag{11.15}$$

Now define

$$u \equiv \frac{1}{r}$$

In terms of u we have

$$L^2\left(u'\right)^2 = \frac{(E^2-m^2c^4)}{m^2c^2}+r_Sc^2u-L^2u^2+L^2r_Su^3, \tag{11.16}$$

where $'$ indicates differentiation w.r.t. ϕ.

Differentiating w.r.t. ϕ and dividing throughout by $2L^2u'$, we have

$$u''+u = \frac{r_Sc^2}{2L^2}+\frac{3}{2}r_Su^2 \tag{11.17}$$

In leading order (i.e. neglecting the last term) the equation reads

$$u_0''+u_0 = a \tag{11.18}$$

$$\text{where } a = \frac{r_Sc^2}{2L^2}.$$

This has a solution:

$$u_0(\phi) = a+b\cos\phi, \tag{11.19}$$

$$\text{where } b = \sqrt{\frac{r_S^2c^4}{4L^4}-\frac{2\epsilon}{L^2m}}.$$

We have set $\phi=0$ to be the angle at which the orbit is at perihelion, introduced the binding energy, ϵ,

$$E = mc^2-\epsilon,$$

and neglected terms of order ϵ^2. This solution gives Kepler's laws for planetary motion, derived using Newtonian mechanics (after identifying $r_S=2GM/c^2$).

From (11.19), the maximum and minimum distances of the planet from the centre of the sun obey the relations

$$\frac{1}{r_{max}}+\frac{1}{r_{min}} = 2a = \frac{r_Sc^2}{L^2} = 2\frac{M_\odot}{L^2} \tag{11.20}$$

$$\frac{1}{r_{min}}-\frac{1}{r_{max}} = 2b = \sqrt{\frac{r_S^2c^4}{L^4}-\frac{8\epsilon}{L^2m}} = \sqrt{\frac{4M_\odot^2}{L^4}-\frac{8\epsilon}{L^2m}} \tag{11.21}$$

Now return to (11.17)

$$u''+u = a+3r_Su^2, \tag{11.22}$$

To first order in the ratio
$$\frac{r_S^2 c^2}{L^2},$$
this differential equation has a solution

$$\begin{aligned} u(\phi) &= \frac{L^2}{2r_S} - 3\frac{r_S}{L^2}\frac{\epsilon}{mc^2} + \frac{9r_S^3}{8L^4} + \sqrt{\frac{r_S^2}{4L^4} - \frac{2\epsilon}{L^2 mc^2}}\cos\left(\left(1-\frac{3r_S^2}{2L^2}\right)\phi\right) \\ &\quad + \left(\frac{r_S^2}{L^2}\frac{\epsilon}{mc^2} - \frac{r_S^3}{8L^4}\right)\cos\left(2\left(1-\frac{3r_S^2}{2L^2}\right)\phi\right) \end{aligned} \tag{11.23}$$

The trigonometric functions have a period of

$$T = \frac{2\pi}{(1 - 3r_S^2/2L^2)}$$

This differs from 2π indicating that the perihelion moves through

$$\frac{3\pi r_S^2}{2L^2} = \frac{3\pi}{2}\left(\frac{1}{r_{max}} + \frac{1}{r_{min}}\right)\frac{GM_\odot}{c^2}$$

every orbit.

Inserting numerical values

$$\begin{aligned} M_\odot &= 1.99\times 10^{30}\,\mathrm{kg}. \\ G &= 6.67\times 10^{-11}\,\mathrm{J\,m\,kg^{-1}} \\ c &= 3\times 10^{8}\,\mathrm{m\,s^{-1}} \\ r_{min} &= 4.6\times 10^{10}\,\mathrm{m} \\ r_{max} &= 6.9\times 10^{10}\,\mathrm{m} \end{aligned}$$

we get a precession of 5×10^{-7} radians per orbit. Mercury performs 415 orbits per century so the precession of the perihelion is $43''$ per century. This accounts exactly for the discrepancy between the observed rate of precession of the perihelion of mercury and the value calculated from the gravitational interaction with other planets.

12 The deflection of light by the sun

The metric corresponding to the gravitational field of the sun is the Schwarzschild metric discussed in the previous section and the Christoffel symbols ar given by (11.2)-(11.6).

The geodesic equation for the azimuthal angle, ϕ leads to (11.11) which for the case of light that passes the edge of the sun at a distance $r_\odot$, we my write as

$$\frac{d\phi}{d\lambda} = \frac{r_\odot}{cr^2} \tag{12.1}$$

The geodesic equation for t is given in (11.7). For a massless particle for which $d\tau^2 = 0$, we cannot identify the parameter, λ, with proper time, but we can scale it such that the solution of (11.7) is

$$\frac{dt}{d\lambda} = \left(1 - \frac{r_s}{r}\right)^{-1}. \tag{12.2}$$

From the metric with $d\tau^2 = 0$ we have

$$\left(1 - \frac{r_s}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 = \left(\frac{dr}{d\lambda}\right)^2 \left(1 - \frac{r_s}{r}\right)^{-1} + \frac{r_\odot^2}{r^2} \tag{12.3}$$

Using (12.2) this may be written

$$\left(\frac{dr}{d\lambda}\right)^2 = 1 + \frac{r_\odot^2}{r^2}\left(1 + \frac{r_s}{r}\right) \tag{12.4}$$

and using 12.1) and defining

$$u \equiv= \frac{1}{r}$$

we get

$$u'^2 + u^2 = \frac{1}{r_\odot^2} + r_s u^3 \tag{12.5}$$

where $'$ means derivative w.r.t ϕ.

Write

$$u(\phi) = \frac{\sin\phi}{r_\odot} + \frac{r_s}{r_\odot^2}\delta(\phi)$$

To first order in r_s, δ obeys the differential equation

$$\cos\phi\delta' + \sin\phi\delta = \frac{r_s}{2r_\odot}\sin^3\phi \tag{12.6}$$

with solution

$$\delta = \frac{r_s}{r_\odot}\left(1 - \frac{\sin^2\phi}{2}\right),$$

yielding

$$r_\odot u(\phi) \;=\; \sin\phi + \frac{r_s}{r_\odot}\left(1 - \frac{\sin^2\phi}{2}\right) + \mathcal{O}\left(\frac{r_s}{r_\odot}\right)^2 \tag{12.7}$$

The initial and final angles,$\phi_\pm$, of a light beam deflected by the sun are the values of ϕ for $u = 0$ (corresponding to infinite distance). Solving (12.7) for $u = 0$ we find

$$\sin\phi_\pm \;=\; -\frac{r_s}{r_\odot} + \mathcal{O}\left(\frac{r_s}{r_\odot}\right)^2 \tag{12.8}$$

The angle of deflection, $\Delta\phi$ is given (to first order in $r_s/r_\odot$), by

$$\Delta\phi \;=\; \phi^+ - \phi_- - \pi \;=\; \frac{2r_s}{r_\odot} \;=\; \frac{4GM_\odot}{r_\odot c^2} \tag{12.9}$$

inserting numerical values

$$\begin{aligned} M_\odot &= 1.99\times 10^{30}\,\mathrm{kg.} \\ G &= 6.67\times 10^{-11}\,\mathrm{J\,m\,kg^{-1}} \\ c &= 3\times 10^{8}\,\mathrm{m\,s^{-1}} \\ r_\odot &= 7.0\times 10^{8}\,\mathrm{m} \end{aligned} \tag{12.10}$$

we find a deflection of 8.48×10^{-6} radians $\equiv 1.75''$.

Originally Einstein made an error in this calculation by a factor of 2 predicted a deflection to be only $0.87''$. The observed deflection measured by Eddington at Principe was $1.6''\pm 0.3''$, and the deflection measured by Crommelin at Sobral was $1.98''\pm 0.12''$ – in good agreement with the (corrected) theoretical prediction.

13 The Unruh Effect

In the 1970's Fulling, Davies and Unruh independently predicted that if an observer, A, in an inertial frame is surrounded by a vacuum, then an observer B who is undergoing acceleration relative to A, will experience a bath of blackbody radiation corresponding to a (very small) temperature which is proportional to the acceleration.

Note that a quantum state, $|E_A\rangle$ which is an eigenstate of the Hamiltonian operator $\hat{H}$ in an inertial frame A, is also an energy eigenstate, $|E_B\rangle$ in another inertial frame B moving with constant velocity v relative to A, with E_B and E_A related by a Lorentz transformation whose operator $\hat{L}(v)$ acts on the energy eigenstates as

$$|E_B\rangle \;=\; \hat{L}(v)|E_A\rangle.$$

However, if the frame of observer i B s not an inertial frame, because B is accelerating relative to A. then the wavefunction in the frame of observer B is *not* an eigenstate of the Hamiltonian, i.e. the state $\hat{L}(v(t))|E_A\rangle$ is not an eigenstate of the Hamiltonian, when the paramter v is time dependent. It is in this way that if observer A is in a vacuum then the same quantum state as seen by B can contain particles (photons) with non-zero energy.

13.1 Accelerated frame Coordinates

The coordinates in the inertial frame A are x^μ and in the accelerated frame B are y^μ. We assume the relative motion is along the x^1 direction so we have

$$x^2 \;=\; y^2, \;\; x^3 \;=\; y^3$$

and henceforth confine ourselves to the two dimensions 0 and 1. The two components are related by

$$\begin{aligned} x^1 &= \left(\frac{c^2}{a}+y_1\right)\cosh\left(\frac{ay^0}{c^2}\right) \\ x^0 &= \left(\frac{c^2}{a}+y_1\right)\sinh\left(\frac{ay^0}{c^2}\right). \end{aligned} \tag{13.1}$$

In the frame of observer A, the value of $x^1(y^1=0), y^0)$, is the position of the origin of B as a function of proper-time y^0/c. For small y^0, where relativistic effects are negligible, we get the non-relativistic result

$$\begin{aligned} x^0 &\approx y^0 \\ x^1 &\approx \frac{c^2}{a}+\frac{a}{2c^2}(y^0)^2 \end{aligned}$$

For a more general point, the vector $(0, y^1)$ in frame B which is the separation of two events with coordinates (y^0, y^1) and $(y^0, 0)$ in frame A, is transformed using a Lorentz transformation with (instantaneous) relative velocity, v where

$$\tanh\left(\frac{v}{c}\right)) \;=\; \frac{ay^0}{c^2}$$

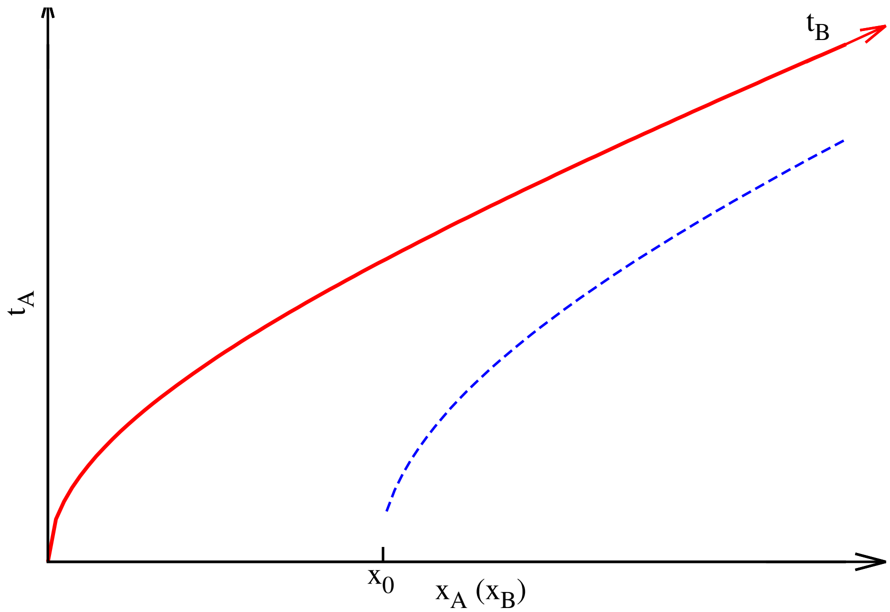

Figure 14: The horizontal axis is the x-axis in both the stationary frame A and the accelerated frame B. he vertical line is the time axis in frame A and the red (hyperbolic) line is the time axis in frame B. The dashed line is the world-line of a point in frame B some distance, x_0, from B's origin.

The (Minkowski) metric, $g_{\mu\nu}$ in frame A is

$$g_{00} = 1, \quad g_{11} \ = -1 \tag{13.2}$$

whereas the metric $g'_{\mu\nu}$ in frame B is[11]

$$g'_{00} \ = \ \left(1 + \frac{ay^1}{c^2}\right)^2, \quad g'_{11} \ = \ -1 \tag{13.3}$$

The domain of the y^1 coordinate is

$$-\frac{c^2}{a} \ \leq \ y^1 \ < \ \infty.$$

No signal from $y^1 < -c^2/a$ will ever reach observer B. this value of y^1 therefore acts as a horizon.

13.2 Wave solutions

In frame A, the (one-dimensional) wave equation is

$$g^{\mu\nu}\nabla_\mu \partial_\nu \phi(x) \ = \ \partial^\mu \partial_\mu \phi(x) \ = \ 0, \tag{13.4}$$

[11]Note that this metric is only equal to the metric corresponding to a uniform gravitational field for small values of y, reflecting the fact that the principle of equivalence is only valid locally.

(all $\Gamma^{\mu}_{\nu\rho}$ vanish in the Minkowski metric) This has normalized plane-wave solutions of frequency ω

$$\phi^{\pm}_{\omega}(x) \;=\; \frac{1}{\sqrt{2\pi c}}\exp\left(-i\frac{\omega}{c}\left(x^0 \mp x^1\right)\right), \tag{13.5}$$

with normalisation

$$\int_{\infty}^{\infty} dx^1 \phi^{\pm *}_{\omega}(x)\phi^{\pm}_{\omega'}(x) \;=\; \delta\left(\omega-\omega'\right) \tag{13.6}$$

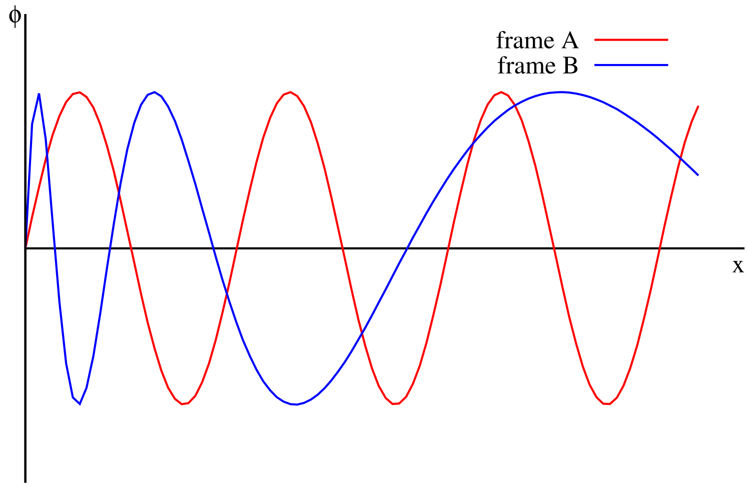

Figure 15: Plane-wave solution in stationary frame A (red) and in the accelerated frame B (blue). Note that in the accelerated frame the wavelength increases with increasing distance for the horizon (this is a manifestation of the equivalent redshift experienced by a photon moving ina gravitational field).

In the accelerated frame, B, for which the non-zero Christoffel symbols are

$$\Gamma^{1}_{00} \;=\; -\Gamma^{0}_{10} \;=\; -\frac{a}{c^2}\left(1+\frac{ay^1}{c^2}\right)$$

the wave equation is

$$g'^{\mu\nu}\nabla_{\mu}\partial_{\nu}\overline{\phi}(y) \;=\; \overline{\phi}''(y) + \frac{a}{c^2}\left(1+\frac{ay^1}{c^2}\right)\overline{\phi}'(y) - \frac{1}{(1+ay^1/c^2)}\ddot{\overline{\phi}}(y) \;=0, \tag{13.7}$$

where $'$ means ∂_i and $\dot{}$ means ∂_0.

This has solutions with frequency Ω

$$\overline{\phi}^{\pm}_{\Omega}(y) \;=\; \frac{1}{\sqrt{2\pi c}}e^{-i\Omega y^0/c}\left(1+\frac{ay^1}{c^2}\right)^{\pm i\Omega c/a} \tag{13.8}$$

with normalisation

$$\int_{-c^2/a}^{\infty} \frac{dy^1}{(1+ay^1/c^2)} \overline{\phi}_{\Omega}^{\pm *}(y) \overline{\phi}_{\Omega'}^{\pm}(y) \;=\; \delta\left(\Omega - \Omega'\right) \tag{13.9}$$

(note that the integral over the space-like coordinate goes from the horizon to infinity – the wavefunction is zero for $y^1 < -c^2/a$.)

The remainder of this section requires some knowledge of quantum field theory. The main result is that the temperature of the blackbody radiation seen by an observer who is accelerating with an acceleration a, relative to an observer in a vacuum is given by (13.21).

13.3 Expansion of Electromagnetic Field Operator

The explanation of the Unruh effect – the thermal bath of blackbody radiation seen by an observer who is accelerating relative to an observer in a vacuum – arises from the expansion of the (free) electromagnetic field in terms of creation and annihilation operators of photons with a given frequency, ω, with coefficients which are the solutions of the wave equation with frequency ω. The salient point is that whereas for inertial observer A these solutions are plane-wave solutions, the coordinates in the frame of observer B lead to a different wave equation with a different set of solutions. This means that the creation (annihilation) operators, $b_\Omega^\dagger$ (b_Ω) in the coordinate frame of observer B is a linear superposition of *both* creation and annihilation operators $a_\omega^\dagger$ and a_ω in the coordinate system of observer A. This, in turn, implies that the photon density operator of photons with frequency Ω,

$$N_\Omega^{(B)} \;\propto b_\Omega^\dagger b_\Omega$$

will contain terms proportional to $a_\omega a_\omega^\dagger$ and will therefore *not* vanish when acting on a vacuum state of observer A.

The expansion, in the frame of observer A, of $\epsilon_\lambda \cdot A(x)$ where $A_\mu(x)$ is the electromagnetic field operator and ϵ_λ^μ is the polarisation vector for a photon with helicity λ.

We restrict ourselves to one space-like dimension, x^1, Furthermore, It is convenient to write this out as a sum of right-moving waves $\epsilon_\lambda \cdot A^+(x)$ and right-moving waves $\epsilon_\lambda \cdot A^-(x)$. Expanding in terms of creation operators $a^\dagger(\omega,\lambda)$ and annihilation operators $a(\omega,\lambda)$, we have

$$\begin{aligned}
\epsilon_\lambda \cdot A^+(x) \;&=\; \int_0^\infty \frac{d\omega}{2\omega\sqrt{2\pi}} \left(a(\omega,\lambda)\phi_\omega^+(x) + a^\dagger(\omega,\lambda)\phi_\omega^{+*}(x) \right) \\
&=\; \int_0^\infty \frac{d\omega}{2\omega 2\pi\sqrt{c}} \left(a(\omega,\lambda)e^{-i\omega(x^0-x^1)/c} + a^\dagger(\omega,\lambda)e^{i\omega(x^0-x^1)/c} \right)
\end{aligned} \tag{13.10}$$

and $\epsilon_\lambda \cdot A^-(x)$ obtained by reversing the sign of x^1.

In the frame of the accelerated observer we expand in terms of creation operators $b^\dagger(\Omega,\lambda)$

and annihilation operators $b(\Omega,\lambda)$, we have

$$\begin{aligned}\epsilon\cdot A^+(y) &= \int_0^\infty \frac{d\Omega}{2\Omega\sqrt{2\pi}}\left(b(\Omega,\lambda)\overline{\phi}^+_\Omega(y)+b^\dagger(\Omega,\lambda)\overline{\phi}^{+*}_\Omega(y)\right)\\ &= \int_0^\infty \frac{d\Omega}{2\Omega 2\pi\sqrt{c}}\left(b(\Omega,\lambda)e^{-i\Omega y^0/c}\left(1+\frac{ay^1}{c^2}\right)^{i\Omega c/a}\right.\\ &\qquad\left.+b^\dagger(\Omega,\lambda)e^{i\Omega y^0/c}\left(1+\frac{ay^1}{c^2}\right)^{-i\Omega c/a}\right) \qquad (13.11)\end{aligned}$$

13.4 Bogoliobov Transformations

Since $\epsilon_\lambda\cdot A^+$ is an invariant quantity it must have the same value in both the frames. This means that we can equate the RHS of (13.10) and (13.11). There is therefore a relation between the creation and annihilation operators in the two frames

$$\frac{b(\Omega,\lambda)}{\sqrt{2\pi}2\Omega} = \int_0^\infty d\omega\left(B(\Omega,\omega)a(\omega,\lambda)+B(\Omega,-\omega)a^\dagger(\omega,\lambda)\right) \qquad (13.12)$$

with a similar expression for $b^\dagger(\Omega,\lambda)$ obtained by taking the Hermitian conjugate of (13.12). These are known as "Bogoliobov transformations".

Multiplying both sides of (13.11) by $\overline{\phi}^+_\Omega(y)$, integrating over y^1 from $-c^2/a$ to ∞ and using (13.9) we have

$$\frac{b(\Omega,\lambda)}{\sqrt{2\pi}2\Omega} = \int_{-a/c^2}^\infty dy^1\left(1+\frac{ay^1}{c^2}\right)^{(-i\Omega c/a-1)} e^{i\Omega y^0/c}\,\epsilon_\lambda\cdot A(x(y)) \qquad (13.13)$$

Substituting for $\epsilon_\lambda\cdot A(x)$ using (13.10) and (13.1) to write x^μ in terms of y^μ, this becomes

$$\begin{aligned}\frac{b(\Omega,\lambda)}{\sqrt{2\pi}2\Omega} &= \int_0^\infty \frac{d\omega}{2\pi\,2\omega\sqrt{c}}\int_{-a/c^2}^\infty dy^1\left(1+\frac{ay^1}{c^2}\right)^{(-i\Omega c/a-1)} e^{i\Omega y^0/c}\\ &\qquad\times\exp\left(i\omega c/a\left[\left(1+ay^1/c^2\right)e^{-ay^0/c^2}\right]\right)a(\omega\lambda)+h.c. \qquad (13.14)\end{aligned}$$

So that comparing (13.14) with (13.12) we have

$$\begin{aligned}B(\Omega,\omega) &= \frac{1}{2\pi\,2\omega\sqrt{c}}-\int_{-a/c^2}^\infty dy^1\left(1+\frac{ay^1}{c^2}\right)^{(-i\Omega c/a-1)} e^{i\Omega y^0/c}\\ &\qquad\times\exp\left(i\omega c/a\left[\left(1+ay^1/c^2\right)e^{-ay^0/c^2}\right]\right)\\ &= \frac{c}{2\omega\,a(2\pi)^{3/2}}\left(\frac{a}{\omega c}\right)^{-i\Omega c/a} e^{\pi c\Omega/2a}\Gamma\left(-i\frac{\Omega c}{a}\right). \qquad (13.15)\end{aligned}$$

The integral is performed by making the change of variable

$$y^1 \;\rightarrow\; u \;=\; -i\frac{\omega c}{a}\left(1+\frac{ay^1}{c^2}\right)e^{-ay^0/c^2}$$

and using

$$\int_0^\infty \frac{du}{u}u^{-i\Omega c/a}e^{-u} \;=\; \Gamma\left(-i\frac{\Omega c}{a}\right)$$

Reversing the sign of ω we have

$$B(\Omega,-\omega) \;=\; -\frac{c}{2\omega a(2\pi)^{3/2}}\left(\frac{a}{\omega c}\right)^{-i\Omega c/a}e^{-\pi\Omega c/2a}\Gamma\left(-i\frac{\Omega c}{a}\right) \tag{13.16}$$

13.5 The Unruh Temperature

Consider the matrix element

$$\begin{aligned}\langle 0_A\left|\frac{b^\dagger(\Omega',\lambda)\,b(\Omega,\lambda)}{2\pi\sqrt{4\Omega\Omega'}}\right|0_A\rangle \;&=\; 2\sqrt{\Omega\Omega'}\int_0^\infty d\omega\int_0^\infty d\omega'\langle 0_A\left|\big(B^*(\Omega',\omega')a^\dagger(\omega',\lambda)\right.\\ &+\; B^*(\Omega',-\omega')a(\omega',\lambda)\big)\times\big(B(\Omega,\omega)a(\omega,\lambda)+B(\Omega,-\omega)a^\dagger(\omega,\lambda)\big)\left|\,0_A\right\rangle\end{aligned} \tag{13.17}$$

where $|0_A\rangle$ is the vacuum state in the frame of the inertial observer A, such that

$$a_\omega\,|0_A\rangle \;=\; 0$$

Using the commutation relation

$$\left[a^\dagger(\omega,\lambda),a(\omega'\lambda')\right] \;=\; 2\pi\,2\omega\,\delta_{\lambda\lambda'}\delta\left(\omega-\omega'\right)$$

and integrating over ω',

$$\begin{aligned}\langle 0_A\left|\frac{b^\dagger(\Omega',\lambda)\,b(\Omega,\lambda)}{2\pi\sqrt{4\Omega\Omega'}}\right|0_A\rangle \;&=\; \frac{4\pi c^2}{a^2}\sqrt{\Omega\Omega'}\int_0^\infty d\omega\,2\omega\,B(\Omega,-\omega)B^*(\Omega',-\omega)\\ =\;\sqrt{4\Omega\Omega'}\frac{1}{(2\pi)^2}\int_0^\infty\frac{d\omega}{2\omega}\left(\frac{a}{\omega c}\right)^{ic(\Omega-\Omega')/a}&\exp\left(-\frac{\pi c(\Omega+\Omega')}{2a}\right)\Gamma\left(\frac{-i\Omega c}{a}\right)\Gamma^*\left(\frac{-i\Omega' c}{a}\right)\\ &=\; \frac{c\,\Omega}{4\pi a}\delta\left(\Omega-\Omega'\right)\exp\left(-\frac{\pi c(\Omega+\Omega')}{2a}\right)\left|\Gamma\left(\frac{-i\Omega c}{a}\right)\right|^2\\ &=\; \frac{1}{2\pi}\delta\left(\Omega-\Omega'\right)\frac{1}{(\exp\left(2\pi\Omega c/a\right)-1)},\end{aligned} \tag{13.18}$$

where in the last step we have used the property of the Euler function

$$\left|\Gamma(iz)\right|^2 \;=\; \frac{\pi}{z\sinh(z)}$$

The density, $N(\Omega)d\Omega$ of photons in observer A's vacuum state, as seen by accelerated observer B, with frequency between Ω and $\Omega + d\Omega$ is given by

$$N(\Omega) \;=\; \sum_{\lambda=\pm}\langle 0_A \left| \frac{b^\dagger(\Omega,\lambda)\, b(\Omega,\lambda)}{2\pi\, 2\Omega} \right| 0_A\rangle \;=\; g_A 2\pi\delta(0)\frac{1}{(\exp\left(2\pi\Omega c/a\right)-1)}, \tag{13.19}$$

where $g_A = 2$ is the number of photon polarisation's. $\delta(0)$ is regularised by normalisation in a box of length L

$$\delta(0) \;\to\; \frac{2\pi L}{c}$$

so that the energy density, $\rho(E)dE$ of photon states per unit length is

$$\rho(E) \;=\; \frac{g_A}{(\exp\left(2\pi Ec/\hbar a\right)-1)} \tag{13.20}$$

This is a blackbody thermal distribution of temperature T_U where

$$T_U \;=\; \frac{\hbar a}{2\pi k_B c} \tag{13.21}$$

14 Black Holes

The Schwarzschild metric has a singularity at $r = r_S$, but the mass distribution in most bodies is sufficiently small that the metric is only valid for values of r which is much larger than the horizon, r_S.

However, there exist bodies which are sufficiently dense that the radius of the horizon, r_S, is outside the body. Such objects are called "black holes".

Any object that crosses the horizon into the black hole cannot get out again. This includes photons, so that there is no way to detect any particles which fall into the black hole.

From the relation between a time interval, dt (in the frame of a distant observer B), and the proper-time, $d\tau$, of an observer A crossing the horizon

$$dt \ = \ \frac{d\tau}{\sqrt{1 - r_S/r}},$$

we see that as the particle approaches the horizon the time interval in frame B tends to infinity. This means that the particle approaching the horizon of the black hole is seen to slow down and it never actually crosses the horizon a $r = r_S$. At the horizon, any finite proper time interval $\Delta\tau$ corresponds to an infinite time interval as measured by an external observer far away fro the horizon.

Similarly, any particle which is inside the black hole and attempts to travel back cross the horizon to the outside, would take an infinite amount of time (in frame B) – a particle inside the horizon of a black hole cannot get out.

Another way to see this, is to consider the invariant square mass, m^2, for which we have

$$m^2 c^2 \ = \ g_{\mu\nu} p^\mu p^\nu$$

Assuming that the motion is only in the radial direction, this means that near the horizon of a black hole

$$m^2 c^4 \ = \ \left(1 - \frac{r_S}{r}\right) E^2 \ - \left(1 - \frac{r_S}{r}\right)^{-1} |\mathbf{p}|^2 \,,$$

so that at the horizon the space-like momentum $|\mathbf{p}|$ vanishes and the particle has zero kinetic energy whereas the total energy – the mass energy plus the potential energy – becomes infinite. We conclude that the potential energy of the particle becomes infinite at the horizon – i.e. an infinite quantity of energy is required to free the particle from the horizon. Put another way, the escape velocity required for a body to have sufficient energy to escape the gravitational field of the black hole, becomes equal to the speed of light.

The concepts of "space" and "time" inside a black hole are reversed, in that the sign of the metric element g_{00} becomes negative, whereas the sign of the element g_{rr} becomes positive. As seen from outside a black hole extends over a finite distance ($r \leq r_S$) but lasts an infinite amount of time. Inside the horizon, the black hole is perceived to extend over an infinite distance, but lasts only a finite amount of time $\sim r_S/c$.

There are other metrics with a horizon at which g_{00} vanishes, and have a vanishing Ricci tensor (Ricci- flat) so that they are solutions to the equation of General Relativity in empty space. One such metric is the Kerr metric. In contrast to the Schwarzschild metric, which is spherically symmetric so that the horizon is the surface of a sphere, the Kerr metric horizon is an oblate spheroid with one axis of symmetry. Furthermore, the metric has a non-zero off-diagonal element $g_{0\phi}$ and is time-dependent. The asymmetry of the metric together with the off-diagonal element means that the black-hole is rotating with angular momentum which increases with the increase of the deviation of the metric from spherical symmetry.

A further possible metric is the Kerr-Newman metric, which extends the Kerr metric in such a way that the black hole possesses electric charge as well angular momentum.

A black hole is completely specifies by three quantities – mass, angular momentum, and electric charge. This is known as the "no hair theorem".

14.1 Temperature of Black Hole (Hawking Temperature)

The Schwarzschild metric, (11.1) in terms of coordinates r and t, approximates a Minkowski metric for an observer at a large distance from the centre of the black hole so that r band t are suitable coordinates in the frame of an inertial observer who is almost at rest relative to the black hole and at a large distance from the black hole.

If we want to describe and inertial observer, A who is in free fall close to the horizon, $r \approx r_S$, we should perform a coordinate transformation so that close to the horizon the metric approximates a Minkowski metric

To do this, we consider coordinates, ρ and τ (called "Kruskal-Szekeres coordinates") which are related to r and t by

$$\begin{aligned} \rho &= 2r_S \left(\frac{(r-r_S)}{r_S}\right)^{1/2} \exp\left(\frac{r-r_S}{2r_S}\right) \cosh\left(\frac{ct}{2r_S}\right) \\ c\tau &= 2r_S \left(\frac{(r-r_S)}{r_S}\right)^{1/2} \exp\left(\frac{r-r_S}{2r_S}\right) \sinh\left(\frac{ct}{2r_S}\right), \end{aligned} \tag{14.1}$$

for $r > r_S$ (outside the horizon of the black hole) and

$$\begin{aligned} \rho &= 2r_S \left(\frac{(r_s-r)}{r_S}\right)^{1/2} \exp\left(\frac{r-r_S}{2r_S}\right) \sinh\left(\frac{ct}{2r_S}\right) \\ c\tau &= 2r_S \left(\frac{(r_S-r)}{r_S}\right)^{1/2} \exp\left(\frac{r-r_S}{2r_S}\right) \cosh\left(\frac{ct}{2r_S}\right), \end{aligned} \tag{14.2}$$

for $r < r_S$ (inside the horizon of the black hole). There is no singularity or discontinuity at the horizon of the black hole and this coordinate system covers the entire space.

The metric in the coordinate system ρ, τ is

$$ds^2 = \frac{r_S}{r} \exp\left(1 - \frac{r}{r_S}\right) \left(c^2 d\tau^2 - d\rho^2\right) \tag{14.3}$$

so that near the horizon the coordinate system is approximately Minkowskian. The coordinates ρ, τ are appropriate coordinates in the fame of an observer A, who is in free fall across the horizon of the black hole . Observer B, who is also close to the horizon, but who maintains a fixed distance from the horizon (i.e. B is comoving with the distant observer) has to accelerate relative to A in order to counter the gravitational attraction of the black hole. As we can see from (14.1) this acceleration, a is given by

$$a = \frac{c^2}{2r_S}. \tag{14.4}$$

As we have seen for the Unruh effect, this means that if an observer A surrounded by a vacuum is in free-fall towards the black hole then an observer B whose distance from the centre of the black hole is fixed will see blackbody radiation with a non-zero temperature – this is called tho Hawking temperature, T_H, of the black hole.

The Hawking temperature is therefore the Unruh temperature with acceleration given by (14.4).

$$T_H = \frac{\hbar c}{4\pi k_B r_S} = \frac{\hbar c^3}{8\pi k_B G M}, \tag{14.5}$$

for a black hole of mass M.

Note that this is the temperature as observed by an observer who is very far from the horizon of the black-hole, i.e. it is the temperature corresponding to the spectrum of blackbody radiation seen by a distant observer. However, this radiation has suffered a gravitational red-shift when propagating from its source near the horizon to the distant observer. For an observer at a distance r from the centre of the black-hole the wavelength of all the radiation (see (10.1)) is longer by a factor of

$$\left(1 - \frac{2GM}{c^2 r}\right)^{-1/2},$$

so that the thermal bath of radiation measured by an observer at a distance r from the centre has temperature

$$T(r) = T_H \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}.$$

For an observer sufficiently far from the black-hole horizon, the Hawking radiation is (almost entirely) blackbody radiation at temperature T_H, whereas for observers closer to the horizon the temperature is higher– tending to infinity at the horizon.

For a black hole of mass $1\,M_\odot$ the Hawking temperature is around 6×10^{-8} K. Note that the mass of a black hole decreases with increasing temperature, meaning that a black hole has negative heat capacity!.

The radiation from an evaporating black hole is mainly blackbody radiation, but superimposed on this is a discrete spectrum which carries the information about th exact quantum state of the material which made uo the black hole and which was deposited on the surface of the black hole at the horizon.

14.2 Entropy of Black Holes

The "no hair theorem" implies that the state of a black hole is completely determined by three quantities and does not possess micro-states. The entropy, S of a system is given by

$$S = k_B \ln W,$$

where W is the number of micro-states. The no hair theorem therefore suggests that a black hole would have zero entropy.

However, Beckenstein pointed out that if that were the case, then when a body with entropy S_B falls into a black hole, its entropy disappears and the total entropy of the Universe decreases contrary to the second law of thermodynamics and could be used to construct a perpetual motion machine.

Any matter or radiation which falls into a black hole increases its mass and hence increases the radius of its horizon. Since it cannot get out again this means that classically, the area of a black hole cannot decrease.

This led Beckenstein to speculate that the non-zero entropy of a black hole was a monotonic function of the surface area, Σ, of the horizon. His argument was based in the idea that when a a bit of information its entropy increases by $\kappa_B \ln 2$ (this is Landauer's principle). If we consider a particle which fall into a black hole and ask the question "does the particle exist?" When the particle is outside the black hole is "yes". This requires one bit of information to be set. If the particle falls into a black hole then this information is lost - we cannot say if the particle doesn't exist or exists but has fallen into a black hole, from which we can retrieve no information. As a consequence of the uncertainty principle a particle with mass m can only fall into a black hole its Compton wavelength

$$\lambda_c = \frac{\hbar c}{m}$$

is smaller than the diameter of the black hole, $2r_S$. The smallest increase in mass, ΔM, of a black hole, , when a single bit of information is erased is therefore

$$\Delta M = \frac{\hbar c}{2r_S} = \frac{\hbar c^3}{2GM},$$

where M is the initial mass of the black hole. The corresponding increase in the Schwarzschild radius is

$$\Delta r_s = \frac{2\,G\,\Delta M}{c^2} = \frac{G\hbar}{c^2 r_S}$$

and the change in the horizon surface area, $\Sigma = 4\pi r_S^2$ is

$$\Delta\Sigma = 8\pi r_S \Delta r_S = \frac{8\pi G\hbar}{c^2} = 8\pi l_P^2, \tag{14.6}$$

where $l_P = \sqrt{G\hbar/xc}$ is the Planck length.

We therefore conclude that the maximum number of bits of information that are erased by a black-hole (i.e. converted into thermal entropy) of surface area Σ is

$$n_b = \frac{\Sigma}{\Delta\Sigma} = \frac{\Sigma}{8\pi l_P^2}.$$

Each of these bits of information is equivalent to an entropy of information $\kappa_B \ln 2$, so that the Beckenstein estimate of the entropy of a black hole is

$$S_{Beckenstein} = \kappa_B \ln 2 \frac{\Sigma}{8\pi G\hbar}. \tag{14.7}$$

A more rigorous derivation of the entropy of a black hole was carried out by Hawking starting with the calculation of the temperature of the blackbody radiation surrounding a black hole, due to the Unruh effect (14.5).

$$T_H = \frac{\hbar c^3}{8\pi k_B G M} = \frac{\hbar c}{4\pi k_B r_s} \tag{14.8}$$

We restrict ourselves to non-rotating black holes with zero electric charge.

The mass of the black hole is

$$M = \frac{r_S c^2}{2G},$$

The energy of the black hole E_{BH} is

$$E_{BH} = Mc^2 = \frac{r_S c^4}{2G} \tag{14.9}$$

If the radius of the black hole changes by dr_S, the energy changes by

$$dE_{BH} = \frac{dE_{BH}}{dr_S}\,dr_S = \frac{c^4}{2G}\,dr_S$$

The black hole has entropy and the change in entropy due to a change dr_S of its radius is

$$dS = \frac{dE}{T_H} = \frac{c^4}{2GT_H} dr_S = \frac{2\pi c^3 r_S k_B}{G\hbar}\,dr_S. \tag{14.10}$$

Integrating over the black hole radius from zero to r_S, we find that the entropy of the black hole with surface area Σ is

$$S \;=\; k_B \frac{c^3}{4G\hbar}\Sigma, \tag{14.11}$$

in approximate agreement with the estimate of Beckenstein.

14.3 Hawking Radiation

Whereas in classical physics a black hole is permanent, in quantum physics there exists a mechanism by which a black hole can evaporate. This mechanism is called "Hawking radiation". It arises because the horizon of a black hole has a non-zero temperature. In the coordinate system for which the distance from the centre of the black hole is fixed, the black hole acts as a cavity of blackbody radiation with temperature equal too the Hawking temperature. Note that the typical wavelength of the photons in this thermal cavity is of the order of the radius of the black-hole r_S. Photons with significantly longer wavelengths than this cannot be absorbed by the black hole.

The radiation can be can be interpreted qualitatively in the following way: Consider a state $|M;0\rangle$, consisting of a (non-rotating, uncharged) black hole of mass M surrounded by a vacuum, as seen by observer A who is in free-fall near the horizon of a black hole. As we know from the Unruh effect, this state is *not* invariant under a transformation of coordinates to a frame, B, which is acceleration relative to A in such a way as to maintain a constant distance form the centre of the black hole (a frame which is co-moving with the frame of an observer as an infinite distance from the black hole). In frame B the state is the superposition

$$|M;0\rangle + \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \int \frac{d^3\mathbf{k}'}{(2\pi)^3 2|\mathbf{k}|} B(\mathbf{k},\mathbf{k}')|(M_<;\mathbf{k},\lambda;\mathbf{k}',-\lambda\rangle + \cdots ,$$

where $M_<$ is slightly less than the mass M of the black hole so that the total energy of the photons plus the black hole is $M\,c^2$. The coefficient $B(\mathbf{k},\mathbf{k}')$ is given approximately by by the Bogoliobov coefficient, $B(\Omega,-\omega)$ with $\Omega = c|\mathbf{k}|$, $\omega = c|\mathbf{k}'|$ and acceleration given by (14.4).. The two photons are in an entangled state in which their polarizations are correlated in order to conserve the angular momentum of the state. The ellipses refer to states with more than one pairs of photons. The two-photon state is one of an infinite number of states of a radiation cavity with temperature equal to the Hawking temperature. Some of the photons are travelling towards the black hole and will be absorbed by it, whereas other photons a travelling away from the black hole and are emitted as (Hawking) radiation.

Such a cavity radiates[12] at a rate given by Stefan's law. The rate of radiation, P, from

[12] In Hawking's popular science book he presents a qualitative explanation of Hawking radiation in terms of a virtual pair production in which one of the pair is created inside the black hole horizon and the other is created outside the horizon and is radiated away. This over-simplified explanation can be somewhat misleading.

the horizon surface of the black hole is given by

$$P = \sigma T_{BH}^4 \Sigma, \tag{14.12}$$

where

$$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2},$$

is the Stefan-Boltzmann constant, the the surface area of the black hole horizon is

$$\Sigma = 4\pi r_S^2 = 8\pi \frac{G^2 M^2}{c^4}$$

so that the black hole loses mass at a rate of

$$\frac{dM}{dt} = \frac{P}{c^2} = \frac{\hbar c^4}{15360\pi G^2 M^2} \tag{14.13}$$

with solution

$$M(t) = M(0)\left(1 - \frac{t}{t_L}\right)^{1/3}, \tag{14.14}$$

with

$$t_L = \frac{5120\pi G^2 M_)^2}{\hbar c^4}$$

For a black hole with mass $\sim 1M_\odot$ t_L is of order 10^{67} years.

Because the radiation from any hitherto observed black hole is extremely weak, Hawking radiation has not been observed. However an analogous radiation has been observed by Jeff Steinhauer. The analogue of the black hole is a Bose-Einstein condensate of Rubidium in which sound travels very slowly. Steinhauer used lasers to create a region in which the cold atoms at a supersonic speed for which classically no phonons could escape[13]. The boundary between the atoms moving with subsonic speed and those moving with supersonic speed is the analogue of the horizon of the black hole. By observing density-density correlations between two points either side of the horizon, he identified entangled pairs of phonons either side of the horizon. Such an entangled pair arises from pair creation either side of the horizon as described in Hawking's qualitative explanation of Hawking radiation.

[13]In this simulation of a black hole the analogue of electromagnetic radiation is the emission of sound.

15 The Cosmological Constant

The equation of General Relativity (9.5) is not unique. It is possible to add the the LHS a "cosmological" term

$$\Lambda g_{\mu\nu}$$

where Λ is a constant (the "cosmological constant") so that (9.5) becomes

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^2} T_{\mu\nu}. \tag{15.1}$$

Both sides of (15.1) are divergence-free because the covariant derivative of the metric is always zero.

A metric, which satisfies (15.1) in free space ($T_{\mu\nu} = 0$), is the metric for "de Sitter space"

$$d\tau^2 = \left(1 - \Lambda r^2\right) dt^2 - \frac{1}{c^2}\left(\left(1 - \Lambda r^2\right)^{-1} dr^2 + r^2 \Omega^2\right) \tag{15.2}$$

The Riemann tensor form this metric is

$$R_{\mu\nu\rho\sigma} = \frac{\Lambda}{3}\left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right)$$

So that the Einstein tensor is

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}$$

confirming that (15.2) is a solution to the equation for General Relativity in free space.

The curvature $R = 4\Lambda$. If Λ is negative this curvature is negative and the space is called "anti-de Sitter space."

The geodesic equation for time is

$$\frac{dt}{d\tau} = \frac{1}{(1 - \Lambda r^2)} \tag{15.3}$$

and for radial distance we have

$$\frac{d^2 r}{d\tau^2} - \frac{\Lambda r}{(1 - \Lambda r^2)}\left(\frac{dr}{d\tau}\right)^2 + \left(1 - \Lambda r^2\right)\left(\Lambda r \left(\frac{dt}{d\tau}\right)^2 c^2 - r\left(\frac{d\theta}{d\tau}\right)^2 - r \sin^2\theta \left(\frac{d\phi}{d\tau}\right)^2\right) \tag{15.4}$$

If we restrict the motion to the space-like plane $\sin\theta = 1$, $d\theta = 0$ and use (15.3)

$$\frac{d^2 r}{d\tau^2} - \frac{\Lambda r}{(1 - \Lambda r^2)}\left(\left(\frac{dr}{d\tau}\right)^2 - c^2\right) + \left(1 - \Lambda r^2\right)\frac{L^2}{r^3} \tag{15.5}$$

In the presence of a non-rotating black hole or outside some other spherically symmetric mass distribution we can combine the de Sitter metric with the Schwarzschild metric to obtain

$$d\tau^2 = \left(1 - \frac{r_S}{r} - \Lambda r^2\right) dt^2 - \frac{1}{c^2}\left(\left(1 - \frac{r_S}{r} - \Lambda r^2\right)^{-1} dr^2 + r^2 d\Omega^2\right) \tag{15.6}$$

16 Expanding Universe

At distance scales which are larger than the size of clusters of galaxies (hundreds of Mpc) the Universe can be approximated as a homogenous isotropic energy distribution with energy density, $T_{00} = \rho$ and pressure $T_{ii} = P$ From the equation of General Relativity (15.1) this means that the Einstein tensor in *not* zero and so we need a metric with spherical symmetry whose Ricci tensor does not vanish but which generates an Einstein tensor that satisfies (9.5).

Furthermore, we know from Hubble's law that the Universe is expanding - which means that the spatial components of the metric have a time-dependent scale factor $a(t)$, which encodes the expansion of the Universe.

Such a metric was constructed by Friedman, LeMaître, Robertson and Walker - usually known as the "Robertson-Walker" metric.

$$d\tau^2 \; = \; dt^2 - \frac{a(t)^2}{c^2}\left(\frac{dr^2}{(1-\kappa r^2)} + r^2 d\Omega^2\right) \tag{16.1}$$

The parameter κ encodes the curvature of the Universe. For positive values it describes a closed Universe with a maximum radius $1/\sqrt{\kappa}$, whereas for negative κ the universe is open and there is no maximum radius. The value $\kappa = 0$ describes a Universe which is exactly flat.

The non-zero components of the Ricci tensor from this metric are

$$\begin{aligned}
R_{00} &= \frac{3}{c^2}\frac{\ddot{a}}{a} \\
R_{rr} &= -\frac{1}{c^2}\frac{(a\ddot{a} + 2\dot{a}^2 + 2\kappa c^2)}{(1-\kappa r^2)} \\
R_{\theta\theta} &= -\frac{r^2}{c^2}\left(a\ddot{a} + 2\dot{a}^2 + 2\kappa c^2\right) \\
R_{\phi\phi} &= -\frac{r^2\sin^2\theta}{c^2}\left(a\ddot{a} + 2\dot{a}^2 + 2\kappa c^2\right),
\end{aligned} \tag{16.2}$$

where $\cdot$ indicates differentiation w.r.t. time.

The curvature scalar is

$$R \; = \; \frac{6}{a^2c^2}\left(a\ddot{a} + \dot{a}^2 + \kappa c^2\right) \tag{16.3}$$

Constructing the Einstein tensor, $G_{\mu\nu}$ and inserting into (15.1) we find from the equation for the component G_{00}

$$-\frac{3}{c^2a^2}\left(\dot{a}^2 + \kappa c^2\right) + \Lambda \; = \; -\frac{8\pi G}{c^2}\rho \tag{16.4}$$

and from the equation for the diagonal space-like components G_{ii}

$$\frac{1}{c^2a^2}\left(2a\ddot{a} + \dot{a}^2 + \kappa c^2\right) - \Lambda \; = \; -\frac{8\pi G}{c^2}P \tag{16.5}$$

Inserting (16.4) in to (16.5), we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3P) + \frac{2}{3}\Lambda.$$

We note that in the absence of the cosmological constant Λ the acceleration $\ddot{a}$ is negative indicating that the rate of expansion of the Universe is slowing down, but for a sufficiently large Λ it can be positive. The cosmological constant acts as "negative pressure".

In 1998, from observations of distance supernovae, Perlmutter, Schmidt, and Riess discovered that the Universe was indeed accelerating, thereby providing strong evidence for the existence of a cosmological constant. This constant term is also known as "dark energy".

Differentiating (16.4) w.r.t time and using (16.5) we find

$$\dot{\rho} = -3H(\rho + P) = -3H\rho(1 + w), \tag{16.6}$$

where

$$H \equiv \frac{\dot{a}}{a},$$

is the "Hubble constant" (N.B. *it isn't constant*), and we have used the fact that for any system the pressure, P is always proportional to the energy density ρ so we can write $P = w\rho$. (16.6) gives a relation between the time dependence of the energy density and the time dependence of the scale factor

$$\frac{d\rho}{\rho} = -3(1 + w)\frac{\dot{a}}{a} \tag{16.7}$$

For a flat Universe ($\kappa = 0$) there is a particularly simple solution for the time dependence of the scale factor.

Adding (16.4) and (16.5) in the $\kappa = 0$ case, the cosmological constant cancels and we get

$$\frac{1}{a^2}\left(a\ddot{a} - \dot{a}^2\right) = -8\pi G\rho(1 + w) \tag{16.8}$$

If we assume that $a(t)$ varies as a power of t, we find that (16.7) and (16.8) are satisfied by

$$a(t) \propto \left(t^{2/3}\right)^{(1+w)} \tag{16.9}$$

For a matter-dominated Universe, in which the matter is moving non-relativistically in the frame of the 'Hubble flow', which is the case today, the pressure can be neglected in comparison with the energy density. In this case we set $w = 0$ and for a flat Universe the scale parameter has a time dependence

$$a(t) \propto t^{2/3}.$$

If the total energy of the Universe remains unchanged as the Universe expands then we have

$$\frac{d}{dt}\left(\rho a^3\right) \;=\; 0,$$

which is consistent with (16.6) (with $w = 0$) and the energy density has a time dependence

$$\rho \propto t^{-2}.$$

In the early Universe (first 47000 years after, the the big bang, the Universe was dominated by (electromagnetic) radiation. For electromagnetic field for which the trace of the stress-energy tensor vanishes, we have

$$P = \frac{1}{3}\rho, \quad \left(w = \frac{1}{3}\right)$$

In this case we have

$$\begin{aligned} \rho &\propto \frac{1}{a^4}, \\ a(t) &\propto \sqrt{t}. \end{aligned}$$

17 Gravitational Waves

17.1 Wave Components

In a gravitational field which is not too strong we can write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $\eta_{\mu\nu}$ is the Minkowski space metric, and work only to linear order in $h_{\mu\nu}$

To this order, the Riemann tensor is

$$R^{\tau}_{\mu\nu\rho} = \frac{1}{2}\eta^{\sigma\tau}\left(\partial_\rho\partial_\sigma h_{\mu\nu} + \partial_\mu\partial_\nu h_{\rho\sigma} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho}\right), \tag{17.1}$$

The Ricci tensor is given by

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\nu\partial^\rho h_{\rho\mu} + \partial_\mu\partial^\rho h_{\rho\nu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h^\rho_\rho\right), \tag{17.2}$$

where $\Box \equiv \partial^\rho\partial_\rho$

The curvature scalar is

$$R = \partial^\rho\partial^\sigma h_{\rho\sigma} - \Box h^\rho_\rho \tag{17.3}$$

These tensors are invariant under the transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \left(\partial_\mu v_\nu + \partial_\nu v_\mu\right), \tag{17.4}$$

where v_μ is any (covariant) vector. This transformation is equivalent to the coordinate transformation

$$x^\mu \to x^\mu + v^\mu,$$

which always leaves the Riemann tensor invariant.

This invariance reduces the number of physical components (components which can affect the curvature) of the symmetric tensor $h_{\mu\nu}$ from 10 to 6 components.

Define the reduced tensor

$$\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\rho_\rho \tag{17.5}$$

The Einstein tensor can be written in terms of this reduced tensor

$$G_{\mu\nu} = -\frac{1}{2}\Box\overline{h}_{\mu\nu} + \frac{1}{2}\left(\partial_\mu\partial^\rho\overline{h}_{\rho\nu} + \partial_\nu\partial^\rho\overline{h}_{\rho\mu} - \eta_{\mu\nu}\partial^\rho\partial_\sigma\overline{h}_{\rho\sigma}\right) \tag{17.6}$$

The invariance under the (gauge) transformation (17.4) means that we can choose convenient gauge condition which eliminates four components. A very convenient gauge is the equivalent of the Lorentz gauge in electromagnetism, namely

$$\partial^\rho\overline{h}_{\rho\mu} = 0 \tag{17.7}$$

In this gauge the Einstein tensor simplifies enormously to

$$G_{\mu\nu} \;=\; -\frac{1}{2}\Box\overline{h}_{\mu\nu}$$

and the (linearized) equation of General Relativity is

$$\Box\overline{h}_{\mu\nu} \;=\; \frac{16\pi G}{c^2}T_{\mu\nu} \tag{17.8}$$

This is the inhomogeneous wave equation for fluctuations of the metric. In analogy with electromagnetic waves, whose source is a localised time-dependent charge and current density distribution, a time-dependent energy density and momentum density acts as a source of gravitational waves, which propagate through space-time at the speed of light.

The gauge condition (17.7) does not totally eliminate the ambiguity of the metric. There remains an invariance under the coordinate transformation

$$x^\mu \;\rightarrow\; x^\mu + \chi^\mu,$$

which leaves $\partial^\rho\overline{h}_{\rho\mu}$ unchanged provided χ^μ obeys the wave equation

$$\Box\chi^\mu = 0.$$

This provides for more constraints, further reducing the number of independent components of $\overline{h}_{\mu\nu}$. It is convenient to choose the "traceless transverse gauge". In this gauge the trace of $\overline{h}_{\mu\nu}$ vanishes, i.e.

$$\overline{h}^\rho_\rho \;=\; 0,$$

and in the rest frame of the observer the components h_{0i} are zero. This last condition can be expressed in a covariant form by

$$u^\rho\overline{h}_{\rho\mu} = 0,$$

where u^μ is the four-velocity of the observer,

In the rest frame of the observer $u = (1, 0, 0, 0)$. these constraints become

$$\begin{array}{rcl} \partial^\rho\overline{h}_{\rho\mu} \;=\; 0 & \text{becomes} & \dfrac{d}{dt}\overline{h}_{0\mu} - \nabla^i\overline{h}_{i\mu} \;=\; 0 \\ u^\rho\overline{h}_{\rho\mu} & \text{becomes} & \overline{h}_{0\mu} \;=\; 0 \\ \overline{h}^\rho_\rho = 0 & \text{becomes} & \overline{h}_{00} \;=\; \overline{h}_{xx} + \overline{h}_{yy} + \overline{h}_{zz} \end{array} \tag{17.9}$$

For a wave moving in the z-direction, the first two of these constraints leaad to $\overline{h}_{z\mu} = 0$. so that the only non-zero components of $\overline{h}_{\mu\nu}$ which satisfy these constraints are

$$\overline{h}_{xx} = -\overline{h}_{yy} = h_+, \;\; \overline{h}_{xy} = \overline{h}_{yx} = \overline{h}_\times$$

The two independent components – the "plus" polarisation, h_+ and the "cross" polarisation, $h_\times$ – represent the two independent polarisations of the gravitational wave. For the plus

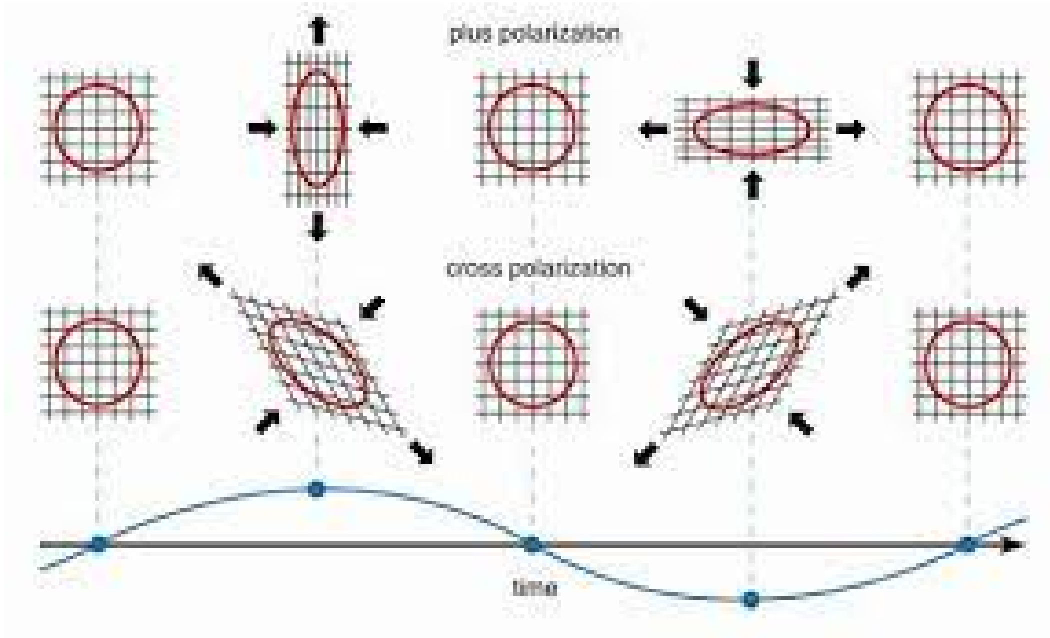

Figure 16: Stretching and contraction of space transverse to the direction of motion of the gravitational wave for plus polarisation (upper) and cross polarisation (lower)

popularisation space is stretched in the x-direction and contracted in the y-direction of vice versa. If we perform a rotation by 90° we get the same pattern but phase shifted. But if we perform a rotation by 45° we obtain the cross polarisation in which the stretching and compression is along the diagonal direction. These two polarisation are shown diagrammatically in Figure 16). This differs from electromagnetic waves in which the two independent popularisation are at right-angles. The amplitudes of the oscillations h_+ or $h_\times$ are known as the "strain" of the gravitational wave and are showed diagrammatically in Figure 16). . This is a dimensional quantity equal to the maximum stretching of the metric in any one direction.

Unlike electromagnetic waves, the gravitational wave equation (17.8) is an approximation which is valid if the amplitude of the wave is small. The full theory gives rise to a wave equation which contains (an infinite number of) terms that are quadratic or higher power. in $\overline{h}_{\mu\nu}$. It is because of this that a consistent quantum theory of gravity has proved so recalcitrant - although modern string theory does include gravity

Fortunately, because of the very small size of the prefactor G/c^2 on the RHS of (17.8) this linear approximation is nearly always sufficient.

17.2 LIGO

The weakness of gravitational waves (smallness of the stress amplitude) meant that a century elapsed between the prediction of gravitational waves from the theory of General Relativity and their discovery by the Laser Interferometer Gravitational-Wave Observatory (LIGO) in 2015.

The source of the detected gravitational wave was the merging of two back holes, each of mass around $30\,M_{\odot}$, at a distance of 400 MPc. This produced waves with a spectrum of many different frequencies. The frequencies observed by LIGO were between 35 Hz and 250 HZ (corresponding to wavelengths between 1000 and and 7000 kilometres. The strain (amplitude) of the fluctuations was only 10^{21} .

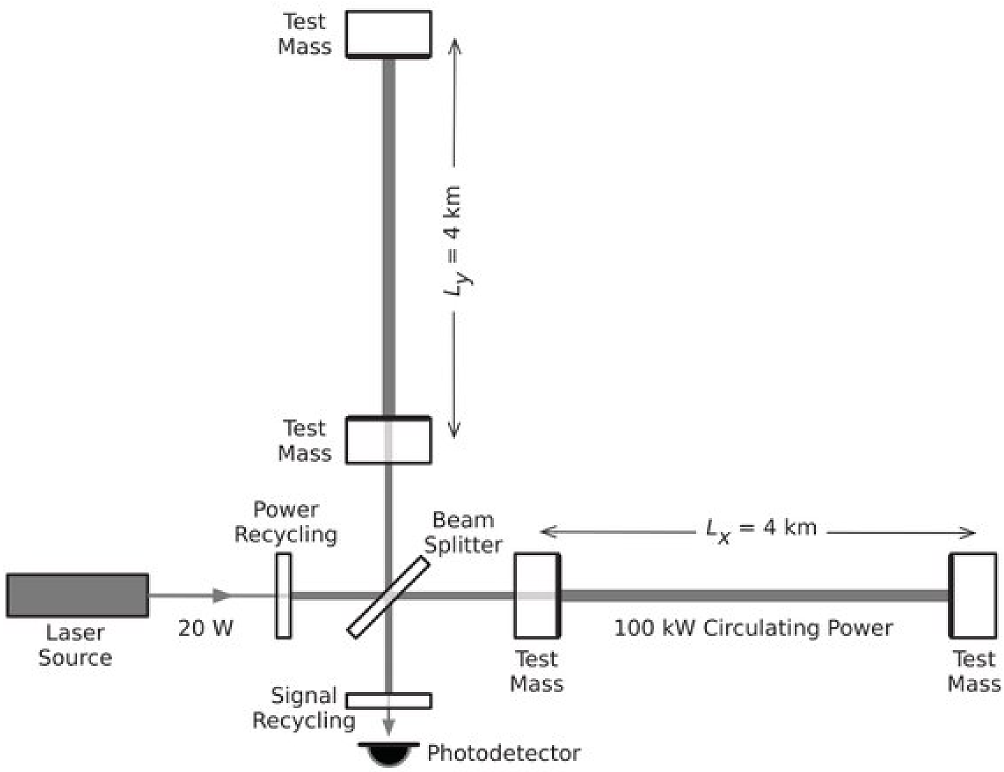

Figure 17: The LIGO interferometer, with 4 km long arms, using 1024 nm laser light.

This tiny strain was detected using a scaled up version of the Michelson interferometer, shown schematically in Figure 17. It uses a laser beam and observes a shift in the interference pattern between laser light which travels along the two arms. The gravitational wave from the merger of the two black holes stretches one of the arms, thereby altering the distance between two test masses. Even with arms of length 4 km the change in separation of the test masses is 10^{-18} m. - one thousandth of the diameter of a proton. The effective length of the arms is enhanced by means of a Fabry-Perot cavity which causes the laser beam to bounce to and fro along the arms 300 times, thereby increasing the effective length to 1200 km. The cavity will only transmit light if the distance between two mirrors at the ends of the cavity is precisely an integer number of half-wavelengths of the laser light. The power of the laser is enhanced from 40 W to 750 W by placing recycling mirrors between the laser source and the beam-splitter. This enhancement of the laser power is required in order to be able to detect the very small changes in effective length of the arms.

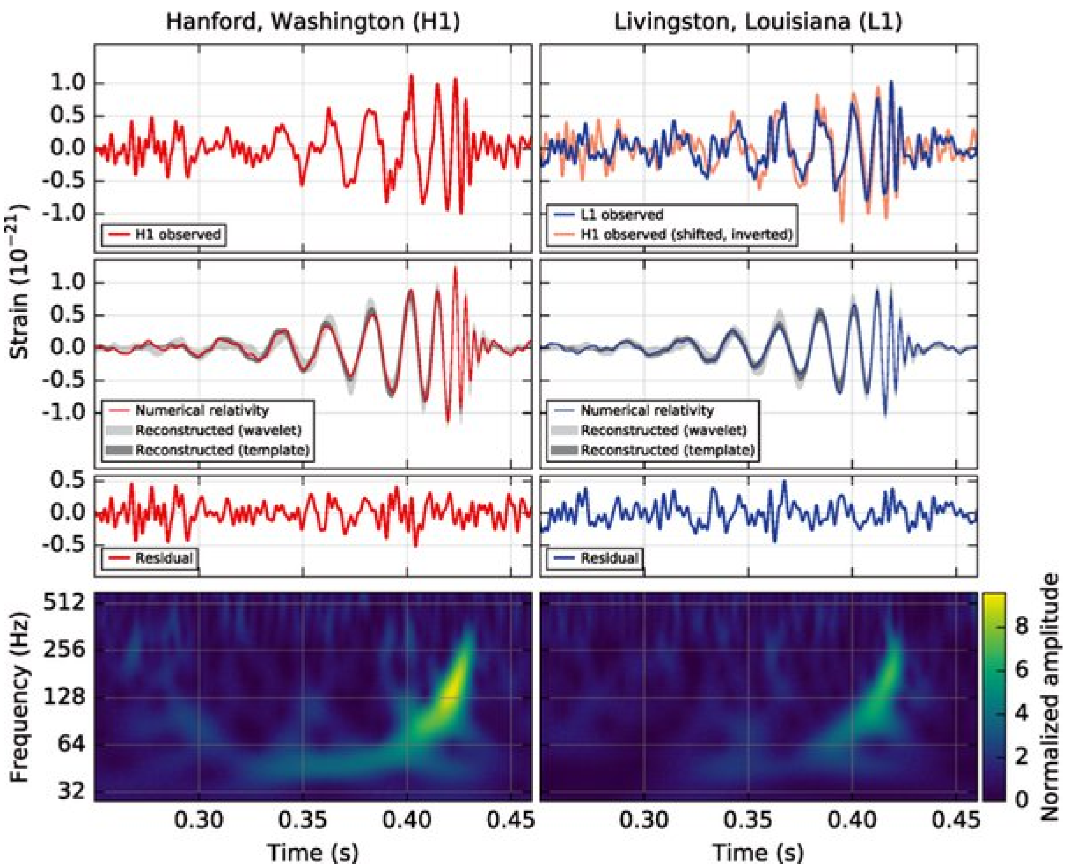

Figure 18: The observed signal of gravitational waves from the two LIGO observatories. The upper graphs show the observed signal and the middle graphs show the signal after the noise has been removed.

The experiment was set up at two sites – Hanford, Washington and Livingstone, Louisiana which are about 3000 km apart. Both laboratories observed the same signal, as can be seen in Figure 18 which displays distinct oscillations of the strain as a function of time with frequencies between $\sim$ 40 Hz. and 100 Hz.