who report a ballot in the profile A and we say that two profiles A, A' are disjoint if $N_A \cap N_{A'} = \emptyset$. Moreover, for two disjoint profiles A and A', we define $A + A'$ as the profile with $N_{A+A'} = N_A \cup N_{A'}$, $(A + A')_i = A_i$ for all $i \in N_A$, $(A + A')_i = A'_i$ for all $i \in N_{A'}$.

Given an approval profile, the goal is to choose a committee. Formally, a *committee* is a subset of the candidates with a specific size. We denote by $\mathcal{W}_k$ the set of all committees of size k and by $\mathcal{W} = \bigcup_{k=0}^{m} \mathcal{W}_k$ the set of all committees. For selecting the winning committees for an approval profile A, we use *approval-based committee (ABC) voting rules.* These rules are functions which take an arbitrary approval profile $A \in \mathcal{A}^*$ and target committee size $k \in \{0, \dots, m\}$ as input and return a non-empty subset of $\mathcal{W}_k$. Intuitively, the chosen set contains the winning committees and we allow for sets of committees as output to indicate that multiple committees are tied for the win. Furthermore, note that ABC voting rules are also defined for committees of size 0: $f(A, 0) = \{\emptyset\}$ for all profiles A since the empty set is the only committee of size 0. This definition is only used for notational convenience.

In this paper, we will restrict our attention to *proper* ABC voting rules which satisfy the following four conditions. Note that almost all commonly studied ABC voting rules are proper voting rules as the subsequent axioms are extremely mild.[1]

- Anonymity: An ABC voting rule f is *anonymous* if $f(A, k) = f(\pi(A), k)$ for all $A \in \mathcal{A}^*$, $k \in \{0, \dots, m\}$, and permutations $\pi : \mathbb{N} \to \mathbb{N}$. Here, $A' = \pi(A)$ denotes the profile such that $N_{A'} = \pi(N_A)$ and $A'_{\pi(i)} = A_i$ for all $i \in N_A$.
- Neutrality: An ABC voting rule f is *neutral* if $f(\tau(A), k) = \{\tau(W) : W \in f(A, k)\}$ for all $A \in \mathcal{A}^*$, $k \in \{0, \dots, m\}$, and permutations $\tau : C \to C$. $A' = \tau(A)$ denotes here the profile such that $N_{A'} = N_A$ and $A'_i = \tau(A_i)$ for all $i \in N_A$.
- Continuity: An ABC voting rule f is *continuous* if for all disjoint profiles $A, A' \in \mathcal{A}^*$ and committee sizes $k \in \{0, \dots, m\}$ such that $|f(A, k)| = 1$, there is an integer $j \in \mathbb{N}$ such that $f(jA + A', k) = f(A, k)$. Here, jA denotes a profile consisting of j disjoint copies of A; the identities of the voters are irrelevant for proper rules due to anonymity.
- Non-imposition: An ABC voting rule f is *non-imposing* if for every committee $W \in \mathcal{W}$, there is a profile $A \in \mathcal{A}^*$ such that $f(A, |W|) = \{W\}$.

Anonymity and neutrality are common fairness conditions which require that voters and candidates, respectively, are treated equally. Continuity, also known as overwhelming majority axiom [22], requires that a sufficiently large group can force the voting rule to choose their desired committee. Finally, non-imposition states that each committee has a chance to be uniquely chosen.

Aside of these standard conditions, we will use two new axioms in our analysis: independence of losers and committee separability. The idea of independence of losers is that a chosen committee $W \in f(A, k)$ should still be chosen if some voters change their preferences by disapproving candidates $c \notin W$ because, intuitively, this does not affect the quality of W. Formally, we say an ABC voting rule f is *independent of losers* if $W \in f(A, |W|)$ implies that $W \in f(A', |W|)$ for all profiles $A, A' \in \mathcal{A}^*$ and committees $W \in \mathcal{W}_k$ with $N_A = N_{A'}$, $W \cap A_i = W \cap A'_i$, and $A'_i \subseteq A_i$ for all $i \in N_A$. Note that this axiom is well-known in single winner voting and choice theory [e.g., 5, 6]. While this axiom has not been considered for ABC elections before, we find it intuitive and it is satisfied by all commonly considered ABC voting rules which do not depend on the ballot size (e.g., Thiele rules, sequential Thiele rules, Phragmen's rule). On the other hand, satisfaction approval voting fails independence of losers as it depends on the sizes of the voters' approval ballots (see [19] for definitions of these rules).

Our second non-standard axiom is committee separability. The rough intuition of this axiom is that if there are two disjoint profiles A and B such that no voters $i \in N_A$, $j \in N_B$ approve a common candidate, we can decompose every chosen committee W into two subcommittees which are chosen for A and B separately. For formally defining this axiom, let $C_A = \bigcup_{i \in N_A} A_i$ denote the set of candidates that are approved by the voters in a profile A. Then, an ABC voting rule f is *committee separable* if $W \in f(A + B, |W|)$ implies that $W \cap C_A \in f(A, |W \cap C_A|)$ and $W \cap C_B \in f(B, |W \cap C_B|)$ for all disjoint profiles A, B with $C_B = C \setminus C_A$ and committees $W \in \mathcal{W}$. Indeed, since $C_A \cap C_B = \emptyset$, it seems reasonable that the choice of candidates from C_A (resp. C_B) only depends on A (resp. B). All proper rules named in this paper satisfy committee separability.

2.1 Consistent Committee Monotonicity

The key axiom for our results is consistent committee monotonicity, which is a strengthening of the well-known axiom of committee monotonicity. The idea of the latter property is that the winning committees of size k are derived by adding candidates to those of size $k - 1$. While this is straightforward to define for ABC voting rules that always choose a single winning committee, it becomes less clear how to formalize committee monotonicity when allowing for multiple tied winning committees. We use the definition of Elkind et al. [11] in this paper which requires that every winning committee of size k is derived from a winning committee of size $k - 1$ and every winning committee of size $k - 1$ is extended to a winning committee of size k.

Definition 1. An ABC voting rule f is *committee monotone* if for every profile $A \in \mathcal{A}^*$ and $k \in \{1, \dots, m\}$, it holds that:

(1) $W \in f(A, k)$ implies that there is $W' \in f(A, k-1)$ with $W' \subseteq W$.
(2) $W \in f(A, k-1)$ implies that there is $W' \in f(A, k)$ with $W \subseteq W'$.

Committee monotone ABC voting rules are closely connected to *generator functions* g, which take a profile A and a committee $W \neq C$ as input and output a possibly empty subset $g(A, W)$ of $C \setminus W$. In particular, generator functions induce committee monotone ABC voting rules in a natural way: a generator function g *generates* an ABC voting rule f if $W \in f(A, k - 1)$ implies $g(A, W) \neq \emptyset$ and $f(A, k) = \{W \cup \{x\} : W \in f(A, k - 1), x \in g(A, W)\}$ for all $k \in \{1, \dots, m\}$ and $A \in \mathcal{A}^*$. Since $f(A, 0) = \{\emptyset\}$, this recursion is well-defined. As we show next, committee monotonicity is equivalent to the existence of a generator function.

Proposition 1. *An ABC voting rule f is committee monotone if and only if it is generated by a generator function g.*

Proof. Consider an arbitrary ABC voting rule f and first assume that f is generated by a generator function g, i.e., $f(A, k) =$

[1] Indeed, we are only aware of a single studied voting rule that fails to be proper: the minimax rule [3], which chooses the committees that minimize the maximal Hamming-distance to a ballot. This rule fails continuity as it completely ignores how many voters report a specific ballot. We view this rule as unreasonable in light of our axioms.

$\{W \cup \{x\} \colon W \in f(A, k-1), x \in g(A, W)\}$ for all profiles A and committee sizes k. Now, fix a profile $A \in \mathcal{A}^*$ and a committee size $k \in \{1, \dots, m\}$. If $W \in f(A, k)$, then there is $W' \in f(A, k-1)$ and $x \in g(A, W')$ such that $W = W' \cup \{x\}$ because g generates f. Conversely, if $W' \in f(A, k-1)$, then $g(A, W')$ cannot be empty and there is a candidate $x \in C \setminus W'$ such that $W \cup \{x\} \in f(A, k)$. This shows that f is committee monotone.

Next, suppose that f is committee monotone. We define the generator function of g as follows: if $W \notin f(A, |W|)$, then $g(A, W) = \emptyset$. On the other hand, if $W \in f(A, |W|)$ and $W \neq C$, there is a committee $W' \in f(A, |W|+1)$ with $W \subsetneq W'$ due to the committee monotonicity of f. We thus define $g(A, W) = \{x \in C \setminus W \colon W \cup \{x\} \in f(A, |W|+1)\}$ if $W \in f(A, |W|)$ and let f_g denote the ABC voting rule defined by $f_g(A, 0) = \{\emptyset\}$ and $f_g(A, k) = \{W \cup \{x\} \colon W \in f_g(A, k-1), x \in g(A, W)\}$ for all $k > 0$. We prove inductively that $f_g(A, k) = f(A, k)$ for all profiles A and $k \in \{0, \dots, m\}$, which implies that f_g is well-defined and that g generates f. The induction basis $k = 0$ is true since $f_g(A, 0) = \{\emptyset\} = f(A, 0)$ for all profiles A. Hence, consider a fixed $k \in \{0, \dots, m-1\}$ and $A \in \mathcal{A}^*$ and suppose that $f_g(A, k) = f(A, k)$. First, let $W \in f(A, k+1)$. Due to committee monotonicity, there is $W' \in \mathcal{W}_k$ and $x \in W \setminus W'$ such that $W' \in f(A, k) = f_g(A, k)$ and $W' \cup \{x\} = W$. This implies that $x \in g(A, W')$ and hence $W \in f_g(A, k+1)$. For the other direction, let $W \in f_g(A, k+1)$, which means that there are $W' \in f_g(A, k) = f(A, k)$ and $x \in g(A, W')$ such that $W = W' \cup \{x\}$. Hence, $f(A, k+1) = f_g(A, k+1)$ and we infer inductively that g generates f. □

Since a generator function completely describes its generated ABC voting rule, we can expect that a well-behaved generator function yields an attractive committee monotone ABC voting rule. Consequently, we now introduce axioms for generator functions. Our main condition on these functions is *consistency*, which is concerned with the behavior of the generator function when combining two disjoint profiles. In more detail, suppose that the choice of the generator g intersects for two disjoint profiles A and A' and a committee W. Intuitively, the best candidates in the combined profile $A + A'$ should be exactly those in the intersection as they are winning for the individual electorates. Hence, consistency requires for such situations that, if $g(A+A', W) \neq \emptyset$, it contains precisely the elements in the intersection of $g(A, W)$ and $g(A', W)$. Note that such consistency axioms have already led to several prominent results [e.g., 4, 13, 18, 27]. Subsequently, we formally define consistency and introduce the notion of consistent committee monotonicity. The latter axiom strengthens committee monotonicity by requiring that the voting rule is generated by a consistent generator function.

Definition 2. A generator function g is *consistent* if $g(A, W) \cap g(A', W) \neq \emptyset$ and $g(A + A', W) \neq \emptyset$ imply that $g(A + A', W) = g(A, W) \cap g(A', W)$ for all disjoint profiles $A, A' \in \mathcal{A}^*$ and committees $W \in \mathcal{W} \setminus \{C\}$. An ABC voting rule f is *consistently committee monotone* if it is generated by a consistent generator function.

Furthermore, analogous to ABC voting rules, we call a generator function g *proper* if it satisfies the following conditions:

- *anonymous*: $g(A, W) = g(\pi(A), W)$ for all $A \in \mathcal{A}^*$, $W \in \mathcal{W} \setminus \{C\}$, and permutations $\pi : \mathbb{N} \to \mathbb{N}$,
- *neutral*: $g(\tau(A), \tau(W)) = \tau(g(A, W))$ for all $A \in \mathcal{A}^*$, $W \in \mathcal{W} \setminus \{C\}$, and permutations $\tau : C \to C$,
- *continuous*: for all $A, A' \in \mathcal{A}^*$ and $W \in \mathcal{W} \setminus \{C\}$ with $|g(A, W)| = 1$ and $g(A', W) \neq \emptyset$, there is $j \in \mathbb{N}$ such that $g(jA + A', W) = g(A, W)$, and
- *non-imposing*: for every $W \in \mathcal{W} \setminus \{C\}$ and $x \in C \setminus W$, there is $A \in \mathcal{A}^*$ such that $g(A, W) = \{x\}$.

Just as for ABC voting rules, all these axioms are very mild. Finally, we say that a generator function g is *complete* if $g(A, W) \neq \emptyset$ for all profiles $A \in \mathcal{A}^*$ and committees $W \in \mathcal{W}$.

2.2 Sequential Valuation Rules

The main goal of this paper is to characterize the class of sequential valuation rules. These rules rely on *valuation functions* v, which are mappings of the type $v : \mathcal{A} \times \mathcal{W} \to \mathbb{R}$, to compute the outcome. Less formally, a valuation function specifies for every ballot A_i and committee W the number of points that a voter with ballot A_i assigns to the committee W. The score of a committee W in a profile A is defined as $s_v(A, W) = \sum_{i \in N_A} v(A_i, W)$. Now, a *sequential valuation function* f works as follows: $f(A, 0) = \{\emptyset\}$ and for $k \geq 1$, $f(A, k) = \{W \cup \{x\} \colon W \in f(A, k-1) \wedge \forall y \in C \setminus W \colon s_v(A, W \cup \{x\}) \geq s_v(A, W \cup \{y\})\}$, i.e., f extends in each step the currently chosen committees with the candidates that increase the score by the most.[2]

Note that our definition of sequential valuation functions is so general that it includes even non-proper ABC voting rules. For instance, if v is constant, the corresponding sequential valuation rule always chooses all committees of the given size and thus fails non-imposition. Nevertheless, we will focus only on proper sequential valuation rules and in particular on the following three subclasses.

- *Sequential Thiele rules* rely on a Thiele counting function to compute the outcome. A Thiele counting function is a mapping $h(x) : \{0, \dots, m\} \to \mathbb{R}$ which is non-negative, non-decreasing, and satisfies $h(1) > h(0)$. Then, the valuation function of a sequential Thiele rule is $v(A_i, W) = h(|A_i \cap W_i|)$. In other words, every voter values a committee only based on how many of its members she approves.[3]
- *Step-dependent sequential Thiele rules* use a step-dependent Thiele counting function as valuation function. A step-dependent Thiele counting function is a mapping $h(x, y) : \{0, \dots, m\} \times \{1, \dots, m\} \to \mathbb{R}$ which is non-negative, non-decreasing in x, and satisfies for each $y \in \{1, \dots, m-1\}$ that there is $x \in \{1, \dots, y\}$ with $h(x, y) > h(x-1, y)$. The valuation function of a step-dependent sequential Thiele rule is then $v(A_i, W) = h(|A_i \cap W|, |W|)$. Intuitively, these rules can use in every step a different Thiele counting function.
- *Step-dependent sequential scoring rules* compute the winner based on a step-dependent counting function. A step-dependent counting function is a mapping $h(x, y, z) : \{0, \dots, m\} \times \{1, \dots, m\} \times \{1, \dots, m\} \to \mathbb{R}$ such that for every $y \in \{1, \dots, m-1\}$, there is $x \in \{1, \dots, y\}$ and $z \in \{x, \dots, m-1-(y-x)\}$ with $h(x, y, z) \neq h(x-1, y, z)$. Then, the valuation function of a step-dependent sequential scoring rule is $v(A_i, W) = h(|A_i \cap W|, |W|, |A_i|)$.

[2] It is also possible to choose the committees that maximize the score for a given valuation function. These rules are proper and satisfy a consistency property for chosen committees (see [18]). However, they fail consistent committee monotonicity and it is not clear why they should be more desirable than their sequential variants.

[3] There are multiple different definitions of Thiele counting functions in the literature (e.g., [9, 19]). Our definition agrees with the one of Aziz et al. [1].

The class of sequential Thiele rules contains many prominent ABC voting rules, such as *sequential approval voting*[4] (seqAV) defined by $h(x) = x$, *sequential proportional approval voting* (seqPAV) defined by $h(0) = 0$ and $h(x) = \sum_{i=1}^{x} \frac{1}{i}$ for $x > 0$, and *sequential Chamberlin-Courant approval voting* (seqCCAV) defined by $h(0) = 0$ and $h(x) = 1$ for $x > 0$. An example of a step-dependent sequential Thiele rule can be constructed by switching between seqAV and seqCCAV in the different steps. Finally, sequential satisfaction approval voting (seqSAV), defined by $h(x, y, z) = \frac{x}{z}$, is an example of a step-dependent sequential scoring rule.

It is easy to see that every sequential valuation function f is consistently committee monotone as it can be verified that its generator function $g(A, W) = \{x \in C \setminus W : \forall y \in C \setminus W : s_v(A, W \cup \{x\}) \geq s_v(A, W \cup \{y\}\}$ is consistent (here, v denotes the valuation function of f). Furthermore, all step-dependent sequential scoring rules are proper ABC voting rules. In particular, the technical condition on h is necessary to ensure that step-dependent sequential scoring rules are non-imposing. Finally, note that every sequential Thiele rule is a step-dependent sequential Thiele rule, which are in turn step-dependent sequential scoring rules. Consequently, all three classes of sequential valuation rules only contain proper ABC voting rules. We can even make the relation between these different types of rules precise as shown in the next proposition.

Proposition 2. *The following equivalences hold:*

1. *A sequential valuation rule is a step-dependent sequential scoring rule if and only if it is proper.*
2. *A step-dependent sequential scoring rule is a step-dependent sequential Thiele rule if and only if it is independent of losers.*
3. *A step-dependent sequential Thiele rule is a sequential Thiele rule if and only if it is committee separable.*

Proof Sketch. The "only if" part of the claims is always easy to prove as it is, e.g., straightforward to see that every step-dependent sequential scoring rule is a proper sequential valuation rule. Hence, we focus on the "if" part. The key insight for (1) is that the valuation function v of a proper sequential valuation rule is neutral, i.e., $v(A_i, W) = v(\tau(A_i), \tau(W))$ for all ballots A_i, committees W, and permutations $\tau : C \to C$. Since $|A_i| = |\tau(A_i)|$, $|W| = |\tau(W)|$, and $|A_i \cap W| = |\tau(A_i \cap W)|$, for all ballots A_i, committees W, and permutations τ, the corresponding sequential valuation rule is a step-dependent sequential scoring rule. For (2), the "if" part intuitively holds because independence of losers excludes the possibility that the step-dependent Thiele counting function h depends on the size of the ballot. By formalizing this insight, we can construct a step-dependent Thiele counting function that induces f, which proves (2). Finally, the "if" part of (3) follows since committee separability relates the different steps of the rule. In more detail, we can construct two disjoint profiles A, B such that $f(A+B, |C_A|) = \{C_A\}$ and then, committee separability shows that all following steps must be equal to the choice for B. Formalizing this argument rules out that h depends on $|W|$ and we thus end up with a sequential Thiele rule. □

[4]Sequential approval voting is often called approval voting since the sequential and the optimizing variant coincide. For consistency in the names of our rules, we prefer to call it sequential approval voting.

3 CHARACTERIZATIONS OF SEQUENTIAL VALUATION RULES

We are now ready to discuss our main result, a characterization of step-dependent sequential scoring rules: an ABC voting rule is a step-dependent sequential scoring rule if and only if it is proper and consistently committee monotone. Combined with Proposition 2, we infer as corollary also characterizations of step-dependent sequential Thiele rules and sequential Thiele rules. Moreover, this proposition also emphasizes the generality of our result since characterizing step-dependent sequential scoring rules is equivalent to characterizing all proper sequential valuation rules. Due to space constraints, we defer the proofs of all auxiliary propositions to the full version [10] and discuss here proof sketches instead.

While it is quite easy to show that every step-dependent sequential scoring rule is proper and consistently committee monotone, the converse claim is much more involved. Our main idea for proving this direction is to investigate the generator function of consistently committee monotone and proper ABC voting rules. Hence, we first verify the conjecture that attractive committee monotone ABC voting rules are generated by well-behaved generator functions.

Proposition 3. *An ABC voting rule is proper and consistently committee monotone if and only if it is generated by a proper, consistent, and complete generator function.*

Proof Sketch. If f is generated by a proper, consistent, and complete generator function, it is fairly straightforward that it is consistently committee monotone and proper. We thus focus on the inverse direction and suppose that f is a proper and consistently committee monotone ABC voting rule. The key insight for this direction is that non-imposition and continuity can be generalized to sequences of committees $W_1, \dots, W_\ell$ with $|W_k| = k$ and $W_{k-1} \subseteq W_k$ for all $k \in \{1, \dots, \ell\}$ (we assume subsequently that $W_0 = \emptyset$):

1. If $\ell < m$, there is a profile A such that $f(A, k) = \{W_k\}$ for all $k \in \{1, \dots, \ell\}$ and $f(A, \ell + 1) = \{W_\ell \cup \{x\} : x \in C \setminus W_\ell\}$.
2. For any two profiles A, A' such that $f(A, k) = \{W_k\}$ for all $k \in \{1, \dots, \ell\}$, there is an integer j such that $f(jA + A', k) = \{W_k\}$ for all $k \in \{1, \dots, \ell\}$.

For instance, we prove (1) by an induction on the length of the sequence: by non-imposition, there is a profile A^1 for every committee $W_{\ell+1} \in \mathcal{W}_{\ell+1}$ such that $f(A^1, \ell + 1) = \{W_{\ell+1}\}$. Committee monotonicity implies then that there is a sequence of committees $W_1, \dots, W_\ell$ such that $W_k \in f(A^1, k)$ and $W_{k+1} \setminus W_k \subseteq g(A, W_k)$ for all $k \in \{1, \dots, \ell\}$, where g is a consistent generator function of f. By the induction hypothesis, there is a profile A^2 such that $f(A^2, k) = \{W_k\}$ for all $k \in \{1, \dots, \ell\}$ and $f(A^2, \ell + 1) = \{W_\ell \cup \{x\} : x \in C \setminus W_\ell\}$. We can now use the consistency of g to infer that $f(A^1 + A^2, k) = \{W_k\}$ for all $k \in \{1, \dots, \ell + 1\}$. Finally, we can further modify the profile to ensure that $W_{\ell+1}$ is extended by all remaining candidates due to anonymity and neutrality.

Now, we will extend the consistent generator function g of f to make it complete. Consider for this a sequence of committees $W_1, \dots, W_\ell$ with $|W_k| = k$ and $W_{k-1} \subseteq W_k$ for all $k \in \{1, \dots, \ell\}$. Due to (1), there is a profile A^{W_ℓ} with $f(A^{W_\ell}, k) = \{W_k\}$ for all $k \in \{1, \dots, \ell\}$ and $f(A^{W_\ell}, \ell+1) = \{W_\ell \cup \{x\} : x \in C \setminus W_\ell\}$. We define the function $\hat{g}(A, W_\ell) = g(A + jA^{W_\ell}, W_\ell)$, where j is the smallest integer such that $f(A+jA^{W_\ell},k)=\{W_k\}$ for all $k\in\{1,\dots,\ell\}$; such an integer exists because of (2). First, note that $\hat{g}$ generates f since $\hat{g}(A,W)=g(A,W)$ for all $A\in\mathcal{A}^*$ and $W\in f(A,|W|)$. This follows from consistent committee monotonicity as $g(jA^W,W)=C\setminus W$, $g(A,W)\neq\emptyset$, and $g(A+jA^W,W)\neq\emptyset$. Finally, $\hat{g}$ satisfies anonymity, neutrality, non-imposition, and continuity as it generates f and f would fail these properties otherwise. □

As second step, we characterize the class of proper, consistent, and complete generator functions. In particular, we show that for every committee $W\neq C$, $g(A,W)$ can be described by a weighted variant of single winner approval voting. For making this formal, let $v(x,y):\{0,\dots,m\}\times\{1,\dots,m\}\to\mathbb{R}$ be a weight function. Then, v-weighted approval voting is defined as the generator function $AV_v(A,W)=\{c\in C\setminus W\colon \forall d\in C\setminus W\colon \sum_{i\in N_A\colon c\in A_i} v(|W\cap A_i|,|A_i|)\geq\sum_{i\in N_A\colon d\in A_i} v(|W\cap A_i|,|A_i|)\}$.

Proposition 4. *Let g denote a proper, consistent, and complete generator function. For every committee $W\neq C$, there is a weight function v^W such that $g(A,W)=AV_{v^W}(A,W)$ for all profiles $A\in\mathcal{A}^*$.*

Proof Sketch. Let g denote a proper, consistent, and complete generator function and fix a committee $W\neq C$. We show the proposition by applying a separating hyperplane argument analogous to how Young [27] derives his characterization of scoring rules.

For doing so, we first transform the domain of $g(\cdot,W)$ from preference profiles to a numerical space and we show thus that $g(\cdot,W)$ can be computed only based on the values $n(c,A,W,k,\ell)=|\{i\in N\colon c\in A_i\wedge|A_i\cap W|=k\wedge|A_i|=\ell\}|$ for $c\in C\setminus W$, $k\in\{0,\dots,|W|\}$, and $\ell\in\{k+1,\dots,m-1-|W|+k\}$. For proving this, we first show that if $A_i\cap W=A_j\cap W$ and $|A_i|=|A_j|$ for all $i,j\in N_A$ and all candidates $x\in C\setminus W$ are approved by the same number of voters, then $g(A,W)=C\setminus W$. Once this restricted claim is proven, we can use our axioms to generalize it; e.g., consistency, neutrality, and anonymity then entail that $g(A^{k,\ell},W)=C\setminus W$ for all k,ℓ and profiles $A^{k,\ell}$ in which $|A^{k,\ell}_i\cap W|=k$ and $|A^{k,\ell}_i|=\ell$ for all $i\in N_A$. Finally, this means that if there are constants $c_{k,\ell}$ such that $n(x,A,W,k,\ell)=c_{k,\ell}$ for all candidates $c\in C\setminus W$ and indices k and ℓ, then $g(A,W)=C\setminus W$ as we can decompose A with respect to k and ℓ into these profiles $A^{k,\ell}$. Together with consistency, we infer from this observation that $g(\cdot,W)$ can indeed be computed based on on the matrix $N(A,W)$ that contains all the values $n(c,A,W,k,\ell)$.

As next step, we use standard constructions to extend the domain of g further from integer matrices $N(A,W)$ to rational matrices. To this end, let Q_2 be the matrix that corresponds to the profile in which each ballot is reported once and note that $g(Q_2,W)=C\setminus W$ due to anonymity and neutrality. Based on this matrix, we extend g to negative numbers by defining $g(Q_1,W)=g(Q_1+jQ_2,W)$ (where $j\in\mathbb{N}$ is a scalar such that Q_1+jQ_2 contains only positive integers) and as second step to g to rational numbers by defining $g(Q_1,W)=g(jQ_1,W)$ (where j is the smallest integer such that jQ_1 only contains integers). For both steps, consistency ensures that g remains well-defined. Moreover, the extension of $g(\cdot,W)$ to rational numbers preserves all desirable properties of g.

Finally, we partition the feasible input matrices Q into sets $R_c=\{Q\colon c\in g(Q,W)\}$ for $c\in C\setminus W$. These sets are convex (with respect to $\mathbb{Q}$) and symmetric since g is consistent, anonymous, and neutral. Moreover, the interior of R_c and R_d is disjoint for $c,d\in C\setminus W$ with $c\neq d$ and we can thus derive a separating hyperplane between these sets (see, e.g., [21]). As last step, we infer from these hyperplanes the weight function v^W. □

Based on Proposition 4, we finally prove our main result.

Theorem 1. *An ABC voting rule is a step-dependent sequential scoring if and only if it is proper and consistently committee monotone.*

Proof. We show in Proposition 2 that every step-dependent sequential scoring rule f is proper. For proving that f is consistently committee monotone, let h denote its step-dependent counting function. Moreover, let $W^x=W\cup\{x\}$ for every committee W and candidate $x\in C\setminus W$. By definition, $f(A,0)=\emptyset$ and $f(A,k)=\{W^c\colon W\in f(A,k-1),c\in C\setminus W\colon\forall d\in C\setminus W\colon s_h(A,W^c)\geq s_h(A,W^d)\}$. Thus, $g(A,W)=\{c\in C\setminus W\colon\forall d\in C\setminus W\colon s_h(A,W^c)\geq s_h(A,W^d)\}$ is complete and generates f. Moreover, g is consistent since the scores are additive, i.e., $s_h(A+A',W)=s_h(A,W)+s_h(A',W)$ for all profiles A,A' and committees W. Hence, if $s_h(A,W^c)\geq s_h(A,W^d)$ and $s_h(A',W^c)\geq s_h(A',W^d)$, then $s_h(A+A',W^c)\geq s_h(A+A',W^d)$. Moreover, if one of the inequalities is strict for A or A', so it is for $A+A'$. Thus, $g(A+A',W)=g(A,W)\cap g(A',W)$ if $g(A,W)\cap g(A',W)\neq\emptyset$, which proves that g is consistent.

For the other direction, consider a proper and consistently committee monotone ABC voting rule f. By Proposition 3, f is generated by a proper, consistent, and complete generator function g. Furthermore, by Proposition 4, there is for every committee $W\neq C$ a weight function v^W such that $g(A,W)=AV_{v^W}(A,W)$ for all $A\in\mathcal{A}^*$. Now, consider two committees W and W' with $|W|=|W'|<m$ and let v^W and $v^{W'}$ denote the corresponding weight functions. We first show that $AV_{v^W}(A',W')=AV_{v^{W'}}(A',W')$ for every profile A'. For this, let $c'\in AV_{v^{W'}}(A',W')$ which is the case if and only if $\sum_{i\in N_{A'}\colon c'\in A'_i} v^{W'}(|W'\cap A'_i|,|A'_i|)\geq\sum_{i\in N_{A'}\colon d'\in A'_i} v^{W'}(|W'\cap A'_i|,|A'_i|)$ for all $d'\in C\setminus W'$. Next, let $\tau: C\to C$ denote a permutation such that $\tau(W)=W'$, and let $A\in\mathcal{A}^*$ and $c\in C$ such that $\tau(A)=A'$ and $\tau(c)=c'$. Because of $g(A,W)=AV_{v^W}(A,W)$, $g(A',W')=AV_{v^{W'}}(A',W')$, and the neutrality of g, it holds that $c'\in AV_{v^{W'}}(A',W')$ if and only if $c\in AV_{v^W}(A,W)$. By the definition of AV_{v^W}, the last claim is true if and only if $\sum_{i\in N_A\colon c\in A_i} v^W(|W\cap A_i|,|A_i|)\geq\sum_{i\in N_A\colon d\in A_i} v^W(|W\cap A_i|,|A_i|)$ for all $d\in C\setminus W$. Finally, observe that $x\in A_i$ if and only if $\tau(x)\in A'_i$, $|A_i|=|A'_i|$, and $|W\cap A_i|=|W'\cap A'_i|$ for all candidates $x\in C\setminus W$ and voters $i\in N_A$. Hence, we conclude that $\sum_{i\in N_A\colon c\in A_i} v^W(|W\cap A_i|,|A_i|)\geq\sum_{i\in N_A\colon d\in A_i} v^W(|W\cap A_i|,|A_i|)$ if and only if $\sum_{i\in N_A\colon c'\in A'_i} v^W(|W'\cap A'_i|,|A'_i|)\geq\sum_{i\in N_A\colon \tau(d)\in A'_i} v^W(|W'\cap A'_i|,|A'_i|)$ for all $d\in C\setminus W$. So, c' obtains the maximal score in A' with respect to $v^{W'}$ if and only if the same holds with respect to v^W. This proves that $AV_{v^W}(A',W')=AV_{v^{W'}}(A',W')$ for all profiles A' and committees W,W' with $|W|=|W'|<m$.

Next, let $W_0,\dots,W_{m-1}$ denote committees such that $|W_i|=i$ and let $v^i=v^{W_i}$. We define the function $v(x,y,z):\{0,\dots,m\}\times\{0,\dots,m-1\}\times\{1,\dots,m\}\to\mathbb{R}$ by $v(x,y,z)=v^y(x,z)$. By our previous reasoning, it holds that $g(A,W)=AV_{v^{|W|}}(A,W)=\{c\in C\setminus W\colon\forall d\in C\setminus W\colon\sum_{i\in N_A\colon c\in A_i} v(|A_i\cap W|,|W|,|A_i|)\geq\sum_{i\in N_A\colon d\in A_i} v(|A_i\cap W|,|W|,|A_i|)\}$. Our next goal is to derive a valuation function from v. For doing so, define the function

$h(x, y, z) : \{0, \dots, m\} \times \{1, \dots, m\} \times \{1, \dots, m\} \to \mathbb{R}$ as follows: $h(0, y, z) = 0$ for all $y, z \in \{1, \dots, m\}$ and $h(x, y, z) = h(x-1, y, z) + v(x-1, y-1, z)$ for all $x, y, z \in \{1, \dots, m\}$. We claim that f is the sequential valuation rule induced by the valuation function $w(A_i, W) = h(|A_i \cap W|, |W|, |A_i|)$. For this, let $g_w(A, W) = \{c \in C \setminus W : \forall d \in C \setminus W : \sum_{i \in N_A} w(A_i, W \cup \{c\}) \geq \sum_{i \in N_A} w(A_i, W \cup \{d\})\}$. We will show that $g_w(A, W) = g(A, W)$ for all profiles $A \in \mathcal{A}^*$ and committees $W \neq C$. Note for this that for all profiles A, committees W, and candidates $c \in C \setminus W$, the following equation holds:

$$\begin{aligned}
&\sum_{i \in N_A} h(|W^c \cap A_i|, |W^c|, |A_i|) - h(|W \cap A_i|, |W^c|, |A_i|) \\
&= \sum_{i \in N_A : c \in A_i} h(|W \cap A_i| + 1, |W^c|, |A_i|) - h(|W \cap A_i|, |W^c|, |A_i|) \\
&\quad + \sum_{i \in N_A : c \notin A_i} h(|W \cap A_i|, |W^c|, |A_i|) - h(|W \cap A_i|, |W^c|, |A_i|) \\
&= \sum_{i \in N_A : c \in A_i} v(|W \cap A_i|, |W|, |A_i|).
\end{aligned}$$

Now, define $C(A, W) = \sum_{i \in N_A} h(|W \cap A_i|, |W| + 1, |A_i|)$. Then, the above equation shows that $s_w(A, W^c) \geq s_w(A, W^d)$ if and only if $s_w(A, W^c) - C(A, W) \geq s_w(A, W^d) - C(A, W)$ if and only if $\sum_{i \in N_A : c \in A_i} v(|W \cap A_i|, |W|, |A_i|) \geq \sum_{i \in N_A : d \in A_i} v(|W \cap A_i|, |W|, |A_i|)$. Hence, $g_w(A, W) = g(A, W)$ for all profiles A and committees W and f is the sequential valuation rule generated by g. Finally, since f is proper, Proposition 2 shows that it is a step-dependent sequential valuation rule. □

Due to Proposition 2, Theorem 1 entails also characterizations of step-dependent sequential Thiele rules and sequential Thiele rules.

Corollary 1. *The following statements hold:*

(1) An ABC voting rule is a step-dependent sequential Thiele rule if and only if it is consistently committee monotone, independent of losers, and proper.
(2) An ABC voting rule is a sequential Thiele rule if and only if it is consistently committee monotone, independent of losers, committee separable, and proper.

Remark 1. All axioms are required for Theorem 1 as there are ABC voting rules other than step-dependent sequential scoring rules that satisfy all but one condition. If we omit anonymity, we can use seqAV but count the vote of voter 1 twice. When omitting neutrality, we can use seqAV but count the votes for candidate a twice. When omitting non-imposition, the rule that always returns all committees of the given size satisfies all remaining conditions. The rule that refines the generator of seqAV by breaking ties based on the Chamberlin-Courant score only fails continuity. Finally, when omitting consistent committee monotonicity, Thiele rules satisfy all remaining conditions. We can also not weaken consistent committee monotonicity to committee monotonicity as reverse sequential Thiele rules then satisfy all given conditions.

Remark 2. Our hierarchy of sequential valuation rules misses the class of sequential scoring rules, which are defined by a valuation function of the form $v(A_i, W) = h(|A_i \cap W|, |A_i|)$. These rules form a subclass of step-dependent sequential scoring rules, but committee separability does not characterize them within the class of step-dependent sequential scoring rules.

Remark 3. A natural follow-up question to Theorem 1 is whether sequential valuation rules can be characterized by consistent committee monotonicity, anonymity, and continuity since they satisfy these three axioms. Unfortunately, this is not the case as we can still treat candidates differently (see Remark 1). On the other hand, it might be possible to characterize the rules that satisfy anonymity, neutrality, continuity, and consistent committee monotonicity.

4 CHARACTERIZATIONS OF SPECIFIC ABC VOTING RULES

Finally, we leverage our results to derive characterizations of specific voting rules. First, we note that our characterizations can be combined with known results that single out rules within the class of, e.g., sequential Thiele rules, to derive full characterizations [e.g., 16, 18]. Nevertheless, we prefer to present our own characterizations for seqCCAV, seqAV, and seqPAV to highlight new aspects of these rules. We state our results restricted to the class of sequential Thiele rules; Corollary 1 turns them into full characterizations by adding the necessary axioms. Moreover, we focus on the case $m \geq 3$ since every sequential Thiele rule coincides with seqAV if $m = 2$.

The main idea for our characterizations is to study how ABC voting rules treat clones. To this end, we say that two candidates c, d are *clones* in a profile A if $c \in A_i$ if and only if $d \in A_i$ for all voters $i \in N_A$. Depending on the goal of the election, clones should be treated differently. For instance, if our goal is to choose a committee that is as diverse as possible, there is no point in choosing both clones. We formalize this new condition as follows: an ABC voting rule f is *clone-rejecting* if $f(A, |W|) = \{W\}$ implies that $\{c, d\} \not\subseteq W$ for all profiles A with clones c, d and committees $W \neq C$. The requirement that a single committee is chosen is necessary since, for instance, in the profile where all voters approve all candidates, we need to choose clones but we will also choose multiple committees. As our next result shows, this axiom characterizes seqCCAV.

Theorem 2. seqCCAV *is the only sequential Thiele rule that satisfies clone-rejection if $m \geq 3$.*

Proof. Since seqCCAV clearly satisfies clone-rejection, we focus on the inverse direction. Hence, consider a sequential Thiele rule f other than seqCCAV and let h denote its Thiele counting function. Since sequential Thiele functions are invariant under scaling and shifting h, we can suppose that $h(0) = 0$ and $h(1) = 1$. Moreover, because f is not seqCCAV, there is an integer $x \in \{2, \dots, m-1\}$ such that $h(x) > 1$ and $h(x') = 1$ for all $x' \in \{1, \dots, x-1\}$. Now, let $\Delta = h(x) - 1$ and $\ell \in \mathbb{N}$ such that $\ell\Delta > 1$. We consider the following profile A to show that f fails clone-rejection: there are ℓ voters who approve the candidates $c_1, \dots, c_x$, x voters who approve c_1 and c_2, and for each $i \in \{3, \dots, x+1\}$ there are $x + 2 - i$ voters who approve only c_i. Now, due to the minimality of x, f agrees in the first $x - 1$ rounds with seqCCAV and we thus have that $f(A, x-1) = \{\{c_1, c_3, \dots, c_x\}, \{c_2, c_3, \dots, c_x\}\}$. On the other hand, it holds that $s_h(A, \{c_1, \dots, c_x\}) \geq s_h(A, \{c_1, c_3, \dots, c_x\}) + \ell\Delta > s_h(A, \{c_1, c_3, \dots, c_x\}) + 1$ and $s_h(A, \{c_1, c_3, \dots, c_x, c_{x+1}\}) = s_h(A, \{c_2, c_3, \dots, c_x, c_{x+1}\}) = s_h(A, \{c_1, c_3, \dots, c_x\}) + 1$. Thus,

$f(A,x) = \{\{c_1,\dots,c_x\}\}$. However, this committee contains the clones c_1 and c_2 which proves that f fails clone-rejection. ◻

The polar opposite to diverse committees are quality-based ones, where the goal is to find the k best candidates regardless of how well they represent the voters. In such a setting, clones should be treated as equal as possible and we thus propose the following notion: an ABC voting rule f is *clone-accepting* if for all profiles A with clones c, d and committees $W \subseteq C\setminus\{c,d\}$, it holds that $W\cup\{c\} \in f(A, |W\cup\{c\}|)$ implies that $W \cup \{c,d\} \in f(A, |W \cup \{c,d\}|)$. Or, in words, the only reason that a winning committee does not contain both clones is if this conflicts with the committee size. Perhaps surprisingly, clone-acceptance does not characterize seqAV as, e.g., the sequential Thiele rule defined by $h(0) = 0$, $h(1) = 1$, and $h(x) = 2x + 1$ for $x \geq 2$ satisfies this axiom, too. However, this rule prefers to choose candidates that are approved by voters who already approve a chosen candidate. This behavior can be interpreted as trust in a voter's recommendation and can be reasonable for quality-based elections. Nevertheless, to single out seqAV, we use a mild condition prohibiting this behavior: an ABC voting rule f is *distrusting* if for all profiles A, committees $W \neq C$ with $f(A,|W|) = \{W\}$, and candidates b, c, it holds that $b \in W$ implies $c \in W$ if more voters in A report the ballot $\{c\}$ than there are voters who approve b. Based on these two axioms, we derive the following theorem.

Theorem 3. seqAV *is the only sequential Thiele rule that is clone-accepting and distrusting if $m \geq 3$.*

Proof Sketch. We focus on the direction from right to left and thus consider a sequential Thiele rule f other than seqAV. Moreover, let h denote the corresponding Thiele counting function and suppose again that $h(0) = 0$ and $h(1) = 1$. Since f is not seqAV, there is a integer $x \in \{2,\dots,m-1\}$ such that $h(x) \neq x$ but $h(x') = x'$ for $x' \in \{1,\dots,x-1\}$. Now, let $\Delta = |h(x) - x|$ and $\ell \in \mathbb{N}$ such that $\ell\Delta > 1$. If $h(x) > x$, f fails distrust in the following profile A, where W is a committee of size $x - 1 \leq m - 2$ and $c, d \in C \setminus W$: ℓ voters approve $W \cup \{c\}$, $\ell + 1$ voters approve d, and two voters approve W. Indeed, it can be checked that $f(A,x) = \{W \cup \{c\}\}$ but distrust requires that d is not chosen after c. On the other hand, if $h(x) < x$, f fails clone-acceptance in the following profile A, where W is a committee W with $|W| = x - 2 \leq m - 3$ and $b, c, d \in C \setminus W$: ℓ voters report $W \cup \{c,d\}$ and $\ell - 1$ voters report b. Indeed, $f(A,x-1) = \{W \cup \{c\}, W \cup \{d\}\}$ but $f(A,x) = \{W \cup \{b,c\}, W \cup \{b,d\}\}$. Thus, seqAV is the only distrusting and clone-accepting sequential Thiele rule. ◻

Finally, a large stream of research on ABC voting rules tries to find proportional committees, i.e., the chosen committee should proportionally reflect the voters' preferences. For defining this concept, we rely on heavily restricted profiles A in which n_1 voters report the same ballot A_1 and n_2 voters approve a single candidate $c \notin A_1$. In such a profile, each clone $d \in A_1$ that is in the elected committee W represents on average $\frac{n_1}{|A_1\cap W|}$ voters, whereas the candidate c represents n_2 voters. Following the idea of proportionality, we should choose a subset of A_1 for a committee size k if $\frac{n_1}{k} > n_2$ as every candidate $d \in A_1$ represents on average more voters than c. Conversely, if $\frac{n_1}{k} < n_2$, the chosen committee should contain c. Thus, we say an ABC voting rule is *clone-proportional* if for all such profiles A, committee sizes $k \leq |A_1|$, and committees $W \in f(A,k)$, it holds that $c \notin W$ if $\frac{n_1}{k} > n_2$ and $c \in W$ if $\frac{n_1}{k} < n_2$. Note that clone-proportionality is closely related to D'Hondt proportionality [8, 18]. Next, we show that this axiom characterizes seqPAV.

Theorem 4. seqPAV *is the only sequential Thiele rule that satisfies clone-proportionality if $m \geq 3$.*

Proof Sketch. We only show that no other sequential Thiele rule f but seqPAV satisfies clone-proportionality. For this, let h denote the Thiele counting function of f and normalize h such that $h(0) = 0$ and $h(1) = 1$. Since f is not seqPAV, there is a minimal integer $x \in \{2,\dots,m-1\}$ such that $h(x) \neq \sum_{i=1}^{x} \frac{1}{i}$. As in the proofs of Theorems 2 and 3, we can now construct a profile in which f fails clone-proportionality. For instance, if $h(x) > \sum_{i=1}^{x} \frac{1}{i}$, let $\Delta = h(x) - \sum_{i=1}^{x} \frac{1}{i}$ and $\ell \in \mathbb{N}$ such that $\ell x \cdot \Delta > 1$ and consider the following profile A: ℓx voters report $\{c_1,\dots,c_x\}$ and $\ell + 1$ voters approve a single candidate $c \notin \{c_1,\dots,c_x\}$. It can be checked that $f(A,x) = \{\{c_1,\dots,c_x\}\}$ but clone-proportionality requires that $c \in W$ for $W \in f(A,x)$ as $\ell + 1 > \frac{\ell x}{x}$. A similar counter example can be constructed if $h(x) < \sum_{i=1}^{x} \frac{1}{i}$ and thus, seqPAV is the only sequential Thiele rule that satisfies this axiom. ◻

Remark 4. Notably, clone-acceptance characterizes seqAV within the class of sequential Thiele rules with non-increasing partial sums $h(j) - h(j-1)$. In the literature, the definition of sequential Thiele rules often includes this condition.

5 CONCLUSION

In this paper, we provide axiomatic characterizations for the new class of sequential valuation rules. These rules are based on valuation functions, which assign each pair of ballot and committee a score and compute the winning committees greedily by extending the current winning committees with the candidates that increase the score by the most. Clearly, sequential valuation rules generalize the prominent class of sequential Thiele rules whose valuation function only depends on the size of the intersection between the given ballot and committee. Our main result characterizes the class of proper (=anonymous, neutral, continuous, and non-imposing) sequential valuation rules based on a new axiom called consistent committee monotonicity. This axiom combines the well-known notions of committee monotonicity and consistency by requiring that the winning committees of size k are derived from those of size $k-1$ by only adding new candidates, and that these newly added candidates are chosen in a consistent way across the profiles. By adding additional conditions, we also derive characterizations of important subclasses such as sequential Thiele rules and of prominent ABC voting rules such as sequential proportional approval voting. For a full overview of our results, we refer to Figure 1.

Our theorems address one of the major open problems in the field of ABC voting: while there is an enormous number of different voting rules, there are almost no characterizations. Such characterizations are crucial for reasoning about which rule to use because without a characterization, there is always the possibility that a more attractive rule exists. Moreover, many ideas of our results seem rather universal and it might be possible to reuse them to characterize other rules such as Phragmen's rule or Thiele rules.

ACKNOWLEDGEMENTS

We thank Felix Brandt and the anonymous reviewers for helpful feedback. This work was supported by the Deutsche Forschungsgemeinschaft under grants BR 2312/11-2 and BR 2312/12-1.

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