Margin of Victory for Weighted Tournament Solutions

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ABSTRACT

Determining how close a winner of an election is to becoming a loser, or distinguishing between different possible winners of an election, are major problems in computational social choice. We tackle these problems for so-called weighted tournament solutions by generalizing the notion of margin of victory (MoV) for tournament solutions by Brill et al. [8][Artificial Intelligence] to weighted tournament solutions. For these, the MoV of a winner (resp. loser) is the total weight that needs to be changed in the tournament to make them a loser (resp. winner). We study three weighted tournament solutions: Borda's rule, the weighted Uncovered Set, and Split Cycle. For all three rules, we determine whether the MoV for winners and non-winners is tractable and give upper and lower bounds on the possible values of the MoV. Further, we axiomatically study and generalize properties from the unweighted tournament setting to weighted tournaments.

KEYWORDS

Computational Social Choice; Voting; Tournaments

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1 INTRODUCTION

Social choice theory is primarily concerned with choosing a socially desirable outcome from a given set of alternatives based on preference information. Such alternatives can be human beings, like politicians or athletes, or more abstract choices, like projects or desired goods. The preference information can be acquired from the opinion of voters or through other methods of assessment, e. g., sports matches. In many practical scenarios, the information comes in the form of pairwise comparisons between the alternatives, resulting in so called *tournaments*. Problems pertaining to tournaments, and more generally to collective decision-making, often involve the use of algorithms and graph-theoretic approaches, and have therefore attracted significant attention from computational social choice researchers over the past few decades [22].

Tournaments are omnipresent in sports competitions, where players or teams compete against each other in head-to-head matches in order to determine one single winner. They are also applied in voting scenarios, where each voter contributes a preference list over all alternatives from which the pairwise preferences are read off. In order to determine the set of 'best' choices among the alternatives, most preferably one winner, several *tournament solutions* have been proposed [4, 19].

Consider a tennis tournament in which everyone plays against everyone else in exactly one match. The result of each match is binary; either x wins against y or x loses against y. This can be illustrated in a graph: Every player is represented by a vertex, and we draw an edge from each match-winner towards the corresponding match-loser. The resulting directed graph is called a *tournament graph*. Using this information, the pre-chosen tournament solution determines the winner, let's call it a. Now, assume there is an edge in the graph that, when reversed, leads to a different tournament in which a is not a winner anymore. In that case, the win of a would be much less perspicuous compared to winning a tournament where we would need to reverse twenty edges before a drops out of the winning set.

That is what we call the *margin of victory* (MoV) – the minimum number of edges needed to be reversed, such that a winner drops out of the winning set, or a loser gets into the winning set. This notion was introduced and analyzed by Brill et al. [8] for such so-called *unweighted* tournaments, that is, tournaments in which one alternative either wins or loses against another alternative. However, in many applications such as voting, and many tournaments in sports, alternatives are not compared only once, but multiple times, leading to *weighted tournaments*.

In our work, we extend the notion of MoV to n-weighted tournaments, where everyone is compared to everyone else exactly n times. The resulting tournament graph contains two edges between any two alternatives a and b. One edge of weight k from a to b if a won k out of the n comparisons against b, and another edge of weight $n - k$ from b to a. Now, the MoV of a winning alternative in regards to the tournament solution is equal to the minimal sum of weight necessary to be changed on the edges for this alternative to fall out of the winning set, and for a non-winning alternative the negative of the minimimal sum of weight needed to get into the winning set.

Using this notion we can study how close a winning alternative is to dropping out of the winning set and, more generally, asses the robustness of a given outcome. If the MoV values of the winning alternatives and runner ups are close to one, the risk of a wrongly chosen winner due to errors in the aggregation process or small manipulations is elevated. Thus, low absolute MoV values might indicate the need for a recount or reevaluation of the given tournament. Furthermore, the MoV allows us to better distinguish between all alternatives while adhering to the principle ideas of the chosen tournament solution. It can therefore be used as a refinement of any tournament solution, generating a full ranking of the alternatives. This solves the problem of some prevalent tournament solutions which tend to choose a large winning set, which so far is reduced to one winner by some tie-breaker.

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1.1 Our Results

We investigate the MoV of three weighted tournament solutions: First, we study Borda's rule (BO), one of the most ubiquitous weighted tournament solutions. Variants of it are used in the Eurovision Song Contest, in various sports awards, such as the award for most valuable player in Major League Baseball, and essentially in every scoring-based tournament. Second, we study the weighted Uncovered Set (wUC) due to its interesting properties in the unweighted setting, for instance that determining the MoV for non-winners in the unweighted setting is one of the rare problems solvable in quasi-polynomial time [8]. Finally, we study the recently introduced Split Cycle (SC), which was shown to admit quite promising axiomatic properties, such as Condorcet consistency and spoiler immunity [17].

In Section 3 we determine the complexity of computing the MoV for each tournament solution, first for winners (*destructive* MoV) and then non-winners (*constructive* MoV). Destructive MoV can be solved in polynomial time for all three tournament solutions. Constructive MoV is only polynomial time solvable for BO and NP-complete for SC and wUC. Whenever we prove a problem to be polynomial time solvable, the provided algorithm not only computes the MoV value for the given alternative, but also a set of edges with corresponding weight witnessing that value.

In Section 4 we provide insight into some structural properties of the MoV. First, in Section 4.1 we prove that all three tournament solutions and their MoV functions satisfy *monotonicity*. This is a desired property as it ensures a basic principle of social choice theory, where an alternative should not become unfavoured if reinforced. The second notion of monotonicity, *transfer-monotonicity*, holds only for BO and wUC, while SC fails it. In Section 4.2 we prove the consistency of all three tournament solutions with the weighted extension of the covering relation. In Section 4.3 we show that none of the considered tournament solutions are in any way *degree-consistent*.

Finally, in Section 5, we derive bounds for the MoV of each tournament solution, i. e., how much weight needs definitely to be changed to get an alternative out of, resp. into, the winning set.

For a summary of these results, we refer to Table 1. Note that due to space constraints, many of our proofs had to be moved to the appendix.

1.2 Related Work

Our work generally fits into the field of computational social choice [5], in which both weighted [14] and unweighted tournaments [4] have found myriads of applications. For a general overview on recent work on tournaments we refer the reader to [22]. The closest related work to ours is the aforementioned work by Brill et al. [8] (and their two preceeding conference papers Brill et al. [6, 7]) on the MoV for unweighted tournaments. We use their framework and structural axioms as grounds for generalization to weighted tournaments. Further, our notion of MoV is very similar to the *microbribery* setting of Faliszewski et al. [12]. In their setting, voters (with rankings) can be bribed to change individual pairwise comparisons between alternatives, even if this results in intransitive preferences of the voter. This is very close to our reversal set notion, as the weight that needs to be reversed corresponds to the pairwise comparisons that need to be manipulated. Faliszewski et al. studied the complexity of that microbribery problem for a parameterized version of Copeland's rule. Further, Erdélyi and Yang [11] considered microbribery under the model of group identification. In that setting, one does not have distinct voter and alternative groups, but rather one set of individuals which approve or disapprove of each other (including themselves).

The problem of determining the MoV a important when studying of robustness of election outcomes. A winner with a low MoV value is in some sense less robust and more prone to changes than a winner with a high MoV value. For recent papers on robustness in elections, we refer the reader to the works of Boehmer et al. [2, 3], Shiryaev et al. [21], Xia [23], or Baumeister and Hogrebe [1], who computationally and experimentally studied the robustness of election winners.

Finally, our problem is closely related to the classical study of bribery and manipulation in social choice, since the MoV can be considered as a measure of how many games in a sports tournament need to be rigged, in order for a competitor to become the winner of the tournament. For an overview on this topic in social choice, we refer the reader to the chapter by Faliszewski and Rothe [13].

2 PRELIMINARIES

A *tournament* is a pair $T = (V, E)$ where V is a nonempty finite set of $|V| = m$ alternatives and $E \subseteq V \times V$ is an irreflexive asymmetric complete relation on V, i. e., either $(x, y) \in E$ or $(y, x) \in E$ for all distinct $x, y \in V$. Let n be a positive integer. An *n-weighted tournament* is a pair $T = (V, w)$ consisting of a finite set V of alternatives and a weight function $w\colon V \times V \to \{0, \dots, n\}$ such that for each pair of distinct alternatives $(x, y) \in V \times V$ we have $w(x, y)+w(y, x) = n$. Observe, that a 1-weighted tournament (V, w) can be associated with an **un**weighted tournament (V, E) by setting $E = \{(x, y) \in V \times V \colon w(x, y) = 1\}$.

Given two distinct alternatives $x, y \in V$, we define the *(majority) margin* of x over y as the difference between the number of wins by x over y and the number of wins by y over x, that is $m(x, y) = w(x, y) - w(y, x)$. Note that $m(x, y) = -m(y, x)$ holds and that the margins are either all even or all odd. Given those margins we denote by $\mathcal{M} = (V, E)$, $E = \{(x, y) \in V \times V \colon x \neq y, m(x, y) > 0\}$, the *margin graph* corresponding to the tournament.

The edges of the margin graph define an asymmetric *weighted dominance relation* between the alternatives. If $(x, y) \in E$, we say that x *dominates* y. An alternative who dominates every other alternative is called a *Condorcet winner*, and a *Condorcet loser*, if it is dominated by every other alternative.

In an **un**weighted tournament $T = (V, E)$ an alternative x is said to *cover* another alternative y if $(x, y) \in E$ and $(x, z) \in E$, for all z with $(y, z) \in E$. This notion can be extended to weighted tournaments, where an alternative x is said to *weighted cover* another alternative y if $m(x, y) > 0$ and $m(x, z) \geq m(y, z)$, for all $z \in V(T) \setminus \{x, y\}$, i. e., every alternative dominated by y is also dominated by x by at least the same margin.

The **un**weighted *outdegree* of x is denoted by $\delta^+(x) = |\{y \in V \colon (x, y) \in E\}|$, and the **un**weighted *indegree* of x by $\delta^-(x) = |\{y \in V \colon (y, x) \in E\}|$. Equivalently, the weighted *outdegree* of x is denoted

		Borda (BO)	**Split Cycle** (SC)	**weighted Uncovered Set** (wUC)
Computing MoV				
Destructive		P (Thm. 3.1)	P (Thm. 3.4)	P (Thm. 3.7)
Constructive		P (Thm. 3.3)	NP-complete (Thm. 3.5)	NP-complete (Thm. 3.8)
Bounds				
Destructive (upper bound)		$\left\lfloor \frac{n\cdot(m-2)}{2} \right\rfloor + 1$ (Thm. 5.1)	$-\left\lceil \frac{n}{2} \right\rceil \cdot (m-1)$ (Thm. 5.2)	$\left\lfloor \frac{n}{2} \right\rfloor + 1 + \left\lfloor \frac{n\cdot(m-2)}{2} \right\rfloor$ (Thm. 5.3)
Constructive (lower bound)		$-n \cdot (m-2)$ (Thm. 5.1)	$n + \left\lceil \frac{(m-2)}{2} \right\rceil$ (Thm. 5.2)	$-\log_2(m) \cdot (\left\lfloor \frac{n}{2} \right\rfloor + 1)$ (Thm. 5.3)
Structural properties				
monotonicity	tournament solution S	✓ (Prop. 4.2)	✓ (Prop. 4.2)	✓ (Prop. 4.2)
	MoV_S	✓ (Thm. 4.4)	✓ (Thm. 4.4)	✓ (Thm. 4.4)
transfer-monotonicity		✓ (Prop. 4.6)	✗ (Prop. 4.7)	✓ (Prop. 4.6)
cover-consistency		✓ (Thm. 4.10)	✓ (Thm. 4.11)	✓ (Thm. 4.10)
degree-consistency		✗ (Prop. 4.14)	✗ (Prop. 4.14)	✗ (Prop. 4.14)

Table 1: Result overview with references to the corresponding theorems and propositions given in parentheses.

by $\delta^+_{\mathrm{w}}(x) = \sum_{z \in V\setminus\{x\}} w(x,z)$, and the weighted *indegree* of x by $\delta^-_{\mathrm{w}}(x) = \sum_{z \in V\setminus\{x\}} w(z,x)$.

2.1 Weighted Tournament Solutions

A *(weighted) tournament solution* is a function S mapping a tournament $T = (V, E)$, or weighted tournament $T = (V, w)$, to a non-empty subset of its alternatives, referred to as the *winning set* $S(T) \subseteq V$. In this paper we consider the following three weighted tournament solutions:

- The winning set according to *Borda's rule* (BO) are all alternatives with maximum Borda score (weighted outdegree)
$$\mathrm{s}_{\mathrm{BO}}(x,(V,w)) = \sum_{z \in V\setminus\{x\}} w(x,z) = \delta^+_{\mathrm{w}}(x).$$
- The winning set according to *Split Cycle* (SC) [16] are all alternatives which are undominated after the following deletion process: for every directed cycle in the margin graph delete the edges with the smallest margin in the cycle, i. e., the least deserved dominations.
- The *weighted Uncovered Set* (wUC) [9, 10] contains all alternatives that are not weighted covered by any other alternative.

We mainly use the following alternative characterization of wUC via a path definition similar to k-kings for unweighted tournaments, which are all alternatives that can reach every other alternative by a path of length at most k. In the unweighted setting, the uncovered alternatives correspond to 2-kings and, consequently, are exactly those that can reach every other alternative by a path of length at most two [19, Proposition 5.1.3]. In the weighted setting, we get a *decreasing path*. If x beats an alternative v_k via a weighted path, y

Definition 2.1. *Let $T = (V, w)$ be an n-weighted tournament and $x, y \in V$. A* decreasing path *$p = (x, y)$ from x to y of length 1 exists if and only if $m(x,y) \geq 0$. A* decreasing path *from x to y of length $k \in \{2, \ldots, n-1\}$ is a sequence of alternatives $p = (v_1, \ldots, v_{k+1})$, such that $v_1 = x$, $v_{k+1} = y$, and $\min\{w(v_{i-1}, v_i) \colon 1 \leq i \leq k\} > w(y, v_k)$.*

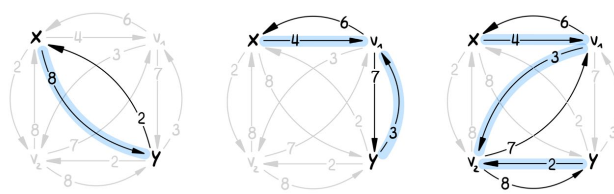


Figure 1: Examples of decreasing paths from x to y.

Refer to Figure 1 for examples of decreasing paths of length one to three. Analogous to the unweighted setting, wUC corresponds precisely to those alternatives which can reach every other alternative with a decreasing path of length 2.

Lemma 2.2. *An alternative x is in $\mathrm{wUC}(T)$ if and only if it can reach every other alternative by a decreasing path of length at most 2.*

Further studying this generalization of wUC for larger k might be an interesting avenue for future work. See the appendix for a proper definition of this generalization.

It is easy to see that BO is always contained in wUC. Further, wUC and SC both define the winner over some weighted path/cycle definition. This opens the question whether wUC $\subseteq$ SC or SC $\subseteq$ wUC. In the following example we show that neither is the case and demonstrate how the three tournament solutions behave.

Example 1. First, consider the 10-weighted tournament T_1 on the left in Figure 2. The Borda scores are $\mathrm{s}_{\mathrm{BO}}(a) = 10$, $\mathrm{s}_{\mathrm{BO}}(b) = 5$ and $\mathrm{s}_{\mathrm{BO}}(a) = 15$ and hence, $\mathrm{BO}(T_1) = \{c\}$. Furthermore, the margin graph is acyclic and thus $\mathrm{SC}(T_1) = \{c\}$. For wUC, we see that no alternative covers the other, and thus $\mathrm{wUC}(T_1) = \{a, b, c\}$. Therefore, wUC is not contained in SC.

On the other hand, in tournament T_2 on the right in Figure 2, every edge is contained in at least one cycle in the margin graph in which all edges have weight 10 and hence, all edges are deleted. As no edge remains, no alternative is dominated and thus, $\mathrm{SC}(T_2) = \{a, b, c, d\}$. For wUC, we notice that alternative a is covered by alternative b, while all other alternatives are not covered. Hence, $\mathrm{wUC}(T_2) = \{b, c, d\}$, and SC is not contained in wUC. For completeness, we note that every element in $\mathrm{wUC}(T_2)$ has a Borda score of 20, while a has a Borda score of 10.

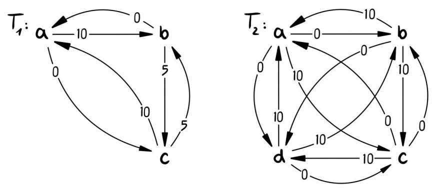


Figure 2: Counterexamples for wUC $\not\subseteq$ SC **on the left and** SC $\not\subseteq$ wUC **on the right.**

2.2 Margin of Victory

The *margin of victory* (MoV) for tournament solutions on unweighted tournaments was formally introduced by Brill et al. [8]. Let $T = (V, E)$ be an **un**weighted tournament and S a tournament solution. Given a winner $a \in S(T)$ regarding S, the $\mathrm{MoV}(a)$ is the minimum number of edges which have to be reversed such that after reversal alternative a is not in the winning set anymore. The corresponding set of edges is called the *destructive reversal set* (DRS). Equivalently, for a non-winner $d \in V \setminus S(T)$ the $MoV(d)$ is the minimum number of edges which have to be reversed such that after reversal d is in the winning set, and the corresponding set of edges is called the *constructive reversal set* (CRS).

In order to generalize this notion for weighted tournaments, we have to extend the notion of reversing edges to reversing a specific amount of weight between any pair of alternatives. We fix the following notation. A *reversal function* is a mapping $R \colon V \times V \longrightarrow \{-n, \dots, n\}$ with $R(x, y) = -R(y, x)$, such that $w(x, y) + R(x, y) \in [0, n]$. Whenever only one direction $R(x, y)$ is specified, we define the corresponding $R(y, x)$ to be set accordingly. Given an n-weighted tournament $T = (V, w)$ and a reversal function R on $V \times V$, we denote by T^R the n-weighted tournament that results from T by reversing the weight given in R:

$$T^R = (V, w(x, y) + R(x, y)).$$

Any reversal function R corresponds to a *weighted destructive reversal set* (wDRS) for $a \in S(T)$ if $a \notin S(T^R)$. Analogously, R corresponds to a *weighted constructive reversal set* (wCRS) for $d \notin S(T)$ if $d \in S(T^R)$. These reversal sets are generally not unique and finding any wDRS or wCRS is usually quite easy. For example, given a Condorcet-consistent tournament solution, i. e., whenever there is a Condorcet winner it is chosen as the only winner of the tournament, a straightforward wCRS for any $d \notin S(T)$ is given by $R(d, y) = n - w(d, y)$, for all $y \in V$. This works, as d is a Condorcet winner in T^R. Using these reversal sets we can now define the MoV.

Definition 2.3. *For an n-weighted tournament $T = (V, w)$ and a tournament solution S, the* margin of victory *(MoV) of a winning alternative $a \in S(T)$ is given by*

$$MoV_S(a, T) = \min \left\{ \sum_{\substack{y,z \in V \\ R(y,z) > 0}} R(y, z) \colon R \text{ is a wDRS for } a \text{ in } T \right\},$$

and for a non-winning alternative $d \notin S(T)$ it is given by

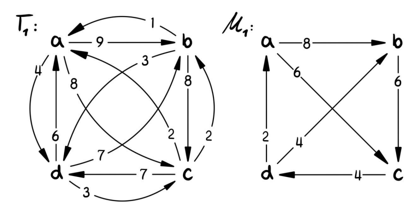
$$MoV_S(d, T) = -\min \left\{ \sum_{\substack{y,z \in V \\ R(y,z) > 0}} R(y, z) \colon R \text{ is a wCRS for } d \text{ in } T \right\},$$

whereas $\sum_{y,z \in V,\ R(y,z) > 0} R(y, z)$ *is called the* size *of* R.

We omit the subscript S, whenever the tournament solution is clear from the context.

Example 2. Consider the 10-weighted tournament T in Figure 3. The set of Borda winners is $\mathrm{BO}(T) = \{a\}$ with a Borda score of $s_{\mathrm{BO}}(a) = 21$. For SC we consider all cycles of the margin graph M. In cycle (b, c, d) the edges (c, d) and (d, b) have the smallest margin, in cycle (a, c, d) the edge with smallest margin is (d, a) and lastly, cycle (a, b, c, d) has as smallest margin edge (d, a). Thus, we delete the edges (c, d), (d, b) and (d, a) from the margin graph, and obtain a and d as the only undominated alternatives, i. e., $\mathrm{SC}(T) = \{a, d\}$. For wUC we check for covering relations in T. The only alternative covering another alternative is a which covers b. Hence, $\mathrm{wUC}(T) = \{a, c, d\}$.

The MoV values and possible weighted reversal sets are given in the table of Figure 3. For instance, $MoV_{\mathrm{SC}}(a) = 2$, since reversing a weight of 2 from a to d would increase the margin of d over a to 6, after which this edge is no longer a minimum weight edge in any cycle. Thus, this edge is not deleted and a is no longer a Split Cycle winner. Similarly, $MoV_{\mathrm{SC}}(b) = -3$, since after strengthening the edge from b to a by a weight of 3, the margin of a over b is 2, which would cause this edge to be deleted and b to be undominated. It is also easy to see, that after any two changes, the edge from a to b cannot be a minimum weight edge in any cycle and hence the bound of -3 cannot be improved to -2.



	a	b	c	d
$MoV_{\mathrm{BO}}(x, T_1)$	3	-5	-5	-3
min. weighted reversal set	$R(d, a) = 3$	$R(b, a) = 5$	$R(c, a) = 5,$	$R(d, a) = 3$
$MoV_{\mathrm{SC}}(x, T_1)$	2	-3	-3	1
min. weighted reversal set	$R(d, a) = 2$	$R(b, a) = 3,$	$R(c, b) = 1,$ $R(c, a) = 1,$ $R(d, a) = 1$	$R(c, d) = 1$
$MoV_{\mathrm{UC}}(x, T_1)$	7	-1	3	2
min. weighted reversal set	$R(d, b) = 2,$ $R(d, c) = 5$	$R(b, c) = 1$	$R(a, d) = 3,$	$R(a, d) = 2$

Figure 3: MoV values of all alternatives $x \in \{a, b, c, d\}$ of the 10-weighted tournament T for each tournament solution BO, SC **and** wUC **together with possible minimum reversal sets.**

3 COMPUTING THE MARGIN OF VICTORY

We now begin the study of the computational complexity for computing the MoV for the three tournament solutions. For each tournament solution, we either give a polynomial-time algorithm for computing the MoV or show that the problem is NP-complete. Whenever we provide a polynomial-time algorithm, the algorithm does not only compute the MoV, but also a corresponding minimum wDRS when considering winners, respectively a corresponding minimum wCRS when considering non-winners.

3.1 Borda

The MoV for BO in weighted tournaments behaves similar to the MoV for Copeland's rule in unweighted tournaments. Note that indeed Copeland's rule and BO coincide on 1-weighted tournaments. Due to the inherent similarities we are able to generalize the algorithms of Brill et al. [8] for determining the MoV of Copeland's rule to also work for BO. First, for determining the MoV of a winning alternative we design a simple greedy algorithm.

Theorem 3.1. *Computing the MoV of a* BO *winner of an n-weighted tournament $T = (V, w)$ can be done in polynomial time.*

Proof. We compute the MoV for $a \in \mathrm{BO}(T)$.

Case 1 ($|\mathrm{BO}(T)| > 1$): As there are other BO winners besides a, lowering $s_{\mathrm{BO}}(a, T)$ by 1 is enough for a to drop out of the winning set. Take any alternative $x \in V \setminus \{a\}$ for which $w(x, a) < n$, i. e., we can reverse up to $w(a, x) \geq 1$ weight from a to x. Such an alternative always exists, as otherwise $s_{\mathrm{BO}}(a, T) = 0$ while $s_{\mathrm{BO}}(y, T) \geq n$ for all $y \in V \setminus \{a\}$, which contradicts a having the highest Borda score. Set $R(a, x) = -1$ and 0 everywhere else. The winning set of T^R is

$$\mathrm{BO}(T^R) = \begin{cases} \mathrm{BO}(T) \setminus \{a\}, & \text{if } s_{\mathrm{BO}}(x, T) < s_{\mathrm{BO}}(a, T) - 1, \\ (\mathrm{BO}(T) \setminus \{a\}) \cup \{x\}, & \text{if } s_{\mathrm{BO}}(x, T) \geq s_{\mathrm{BO}}(a, T) - 1. \end{cases}$$

Therefore, $a \notin \mathrm{BO}(T^R)$ and R is a minimum wDRS for a in T which is computable in $O(|V|)$ time.

Case 2 ($|\mathrm{BO}(T)| = 1$): As there are no other BO winners besides a, at least one of the non-winner alternatives needs to be in the new winning set instead of a. Consider a fixed minimum wDRS R for a in T and let b be the alternative with $s_{\mathrm{BO}}(b, T^R) > s_{\mathrm{BO}}(a, T^R)$. We claim, R reverses weight along edges adjacent to a or b only. Towards contradiction, assume $R(x, y) > 0$ for some $x, y \in V \setminus \{a, b\}$. Reversing that weight did not change the Borda scores of a or b, therefore they stay the same when revoking that reversal, resulting in a wDRS of smaller size. This contradicts R being minimal. This directly implies a simple polynomial-time greedy algorithm to compute the MoV of a and a corresponding minimum wCRS.

Algorithm We iterate over all $b \in V \setminus \{a\}$ and compute a minimum wDRS R such that $s_{\mathrm{BO}}(b, T^R) > s_{\mathrm{BO}}(a, T^R)$. We do so by greedily reversing weight away from a or towards b, starting with reversing along the edge ab directly. Set

$$R(b, a) = \min \left\{ w(a, b), \left\lfloor \frac{s_{\mathrm{BO}}(a, T) - s_{\mathrm{BO}}(b, T)}{2} \right\rfloor + 1 \right\},$$

where the latter is the distance between their two Borda scores, i. e., the necessary amount of weight to be reversed from a to b. If after that reversal $s_{\mathrm{BO}}(a, T^R) \geq s_{\mathrm{BO}}(b, T^R)$ still holds, we greedily set

$$R(x, a) = \min \left\{ w(a, x),\ \ s_{\mathrm{BO}}(a, T^R) - s_{\mathrm{BO}}(b, T^R) + 1 \right\},$$

$$R(b, x) = \min \left\{ w(x, b),\ \ s_{\mathrm{BO}}(a, T^R) - s_{\mathrm{BO}}(b, T^R) + 1 \right\},$$

for $x \in V \setminus \{a, b\}$, until $s_{\mathrm{BO}}(a, T^R) < s_{\mathrm{BO}}(b, T^R)$. Among all possible choices of b, we select one inducing a wDRS of minimum size.

Correctness The correctness of the algorithm directly follows from the following observation: For a fixed b the algorithm terminates when $s_{\mathrm{BO}}(a, T^R) - s_{\mathrm{BO}}(b, T^R) < 0$. Reversing one weight from a to b reduces the difference between their Borda scores by two, and reversing one weight from a to any $x \in V \setminus \{a, b\}$, resp. from any $x \in V \setminus \{a, b\}$ to b, reduces the difference by one. A minimum wDRS thus has to consider reversal between a and b first, and then reverse weight from a, resp. to b, arbitrarily, until $s_{\mathrm{BO}}(a, T^R) < s_{\mathrm{BO}}(b, T^R)$. This is the procedure of the algorithm.

Polynomial runtime The algorithm clearly runs in time $O(|V|)$. □

Turning to BO non-winners, we show that the problem of computing the MoV can be reduced to the Minimum Cost b-Flow problem, see for instance [18]. Our algorithm iterates over all possible Borda scores l, such that d is a BO winner with Borda score l after reversal. To achieve this, we construct a suitable flow network G_l, such that any b-flow of G_l corresponds to a weighted reversal set for d in T of the same weight, and compute a minimum cost b-flow. The range of possible Borda scores is reduced, using the following simple observation concerning a lower bound on a winning Borda score.

Observation 3.2. *Let $T = (V, w)$ be an n-weighted tournament with m alternatives. For any $a \in \mathrm{BO}(T)$, we have*

$$s_{\mathrm{BO}}(a) \geq \left\lceil \frac{n \cdot \frac{m \cdot (m-1)}{2}}{m} \right\rceil = \left\lceil \frac{n}{2} \cdot (m-1) \right\rceil.$$

Theorem 3.3. *Computing the MoV of a* BO *non-winner of an n-weighted tournament $T = (V, w)$ can be done in polynomial time.*

3.2 Split Cycle

Split Cycle is a tournament solution directly dealing with the problem of majority cycles in a tournament. It was introduced by Holliday and Pacuit [15, 17] to combat common problems of tournament solutions like "spoiler effect" and "no show paradox".

The similarities of the Split Cycle problem to the problem of finding weighted paths in the margin graph allow us to reduce the MoV computation of an SC winner to the Minimum Cut problem in graphs, see [18].

Theorem 3.4. *Computing the MoV of an* SC *winner of an n-weighted tournament $T = (V, w)$ can be done in polynomial time.*

For the constructive case we can show NP-completeness by reducing from Dominating Set, again utilizing the alternative path definition of SC.

Theorem 3.5. *Deciding whether there is a wCRS of size k for an* SC *non-winner of an n-weighted tournament is NP-complete.*

3.3 Weighted Uncovered Set

In the **un**weighted setting, computing the MoV of a UC winner, a 3-king, or an $(n-1)$-king can be done in polynomial time as shown by Brill et al. [8, Chapter 3.1.2.], via an l-length bounded a-cut in the tournament for winner a. Unfortunately, simply extending this approach to decreasing paths in the weighted setting does not seem possible. This is mainly due to the fact that instead of just two choices for every edge, i. e., reversing or keeping it, we have $n+1$, i. e., changing the weight on the edge xy to anything from 0 to n.

Luckily, finding a polynomial time algorithm for wUC is quite straightforward. It builds on the following property inherent exclusively to decreasing paths of length at most two.

Proposition 3.6. *Given two alternatives, all decreasing paths of length at most two between them are pairwise distinct aside from their endpoints.*

Given a wUC winner a, the algorithm iterates over all alternatives $d \in V \setminus \{a\}$ and computes the minimum wDRS such that d covers a after reversal. This is equivalent to eliminating every decreasing a-d-path of length at most two. Since those paths pairwise don't intersect, we can process all such paths iteratively, and greedily compute for each the minimum necessary reversals.

Theorem 3.7. *Computing the MoV of a* wUC *winner of an n-weighted tournament $T = (V, w)$ can be done in polynomial time.*

Proof. We compute the MoV for $a \in \mathrm{wUC}(T)$.

Algorithm We iterate over all $d \in V \setminus \{a\}$ and compute the wDRS R for letting d cover a in T^R. We do so by iterating over all $x \in V \setminus \{a\}$ and checking for decreasing a-d-paths via x.

Case 1 ($x = d$): Check for decreasing a-d-path of length one, i. e., $m(a,d) \geq 0$. If so, we need to reverse weight such that $m^R(a,d) < 0$. To get a minimum size wDRS only $m^R(a,d) \in \{-1,-2\}$ is necessary, depending on the parity of n. Set

$$R(d,a) = \left\lfloor \frac{n}{2} \right\rfloor + 1 - w(d,a).$$

Case 2 ($x \neq d$): Check for decreasing a-d-path of length two via x, i. e., $w(a,x) > w(d,x)$. If so, we need to reverse weight such that $w^R(a,x) \leq w^R(d,x)$. To get a minimum size wDRS, we only need $w^R(d,x) = w^R(a,x)$. Set

$$R(d,x) = w(a,x) - w(d,x).$$

Among all possible choices of d, we select the one inducing a wDRS R of minimum size.

Correctness The correctness of the algorithm directly follows from Prop. 3.6: An alternative a is a wUC winner if and only if it can reach every other alternative by a decreasing path of length at most two. Equivalently, if it is not covered by any other alternative. If we reverse weight such that there is at least one alternative which a cannot reach by a decreasing path of length at most two, then $a \notin \mathrm{wUC}(T^R)$. By Prop. 3.6, all decreasing paths of length at most two between two fixed alternatives are pairwise distinct. Therefore, we can reverse weight along all such paths without influencing the other paths.

Polynomial runtime The algorithm clearly runs in $O(|V|^2)$. □

For UC non-winners, [8, Theorem 3.7] showed that the problem of computing the MoV in the unweighted case is equivalent to the Minimum Dominating Set problem on tournaments. Since tournaments always admit a dominating set of size $O(\log(n))$, this reduction implies a $n^{O(\log(n))}$ time algorithm and makes the problem unlikely to be NP-complete. For weighted tournaments, though, we can show that wUC is actually NP-complete by reducing from Set Cover.

Theorem 3.8. *Deciding whether there is a wCRS of size k of a* wUC *non-winner of an n-weighted tournament is NP-complete.*

4 STRUCTURAL RESULTS

Now that we have analyzed the computational complexity of the MoV for the three tournament solutions, we turn to generalize some structural properties of the MoV from the unweighted to the weighted setting.

4.1 Monotonicity

We start by considering the classical structural property of monotonicity. A tournament solution is monotonic if a winner of the tournament stays a winner after being reinforced, i.e., after increasing the margin of a winning alternative over any other alternative, the winning alternative does not drop out of the winning set. We generalize this notion for unweighted tournaments.

Definition 4.1. *A tournament solution S for an n-weighted tournament $T = (V, w)$ is said to be* monotonic, *if for any $a, b \in V$ with $w(a,b) < n$,*

$$a \in S(T) \quad \textit{implies} \quad a \in S(T^R),$$

where the reversal function R is defined as $R(a,b) = 1$ and 0 otherwise.

It is straightforward to show that all three tournament solutions for which we analyzed the MoV are monotonic.

Proposition 4.2. BO, SC *and* wUC *satisfy monotonicity.*

Besides monotonicity of the underlying tournament solution, we can also analyze the monotonicity of the MoV. For this, we say that the MoV of a tournament solution is monotonic if the MoV of an alternative does not decrease after this alternative is reinforced. In the following two definitions we use Q for reversal functions instead of R to ensure readability of the proofs.

Definition 4.3. *Given a tournament solution S, we say MoV_S is* monotonic *if, for any n-weighted tournament $T = (V, w)$ and any alternatives $a, b \in V$ with $w(a,b) < n$,*

$$MoV_S(a, T^Q) \geq MoV_S(a, T),$$

where the reversal function Q is defined as $Q(a,b) = 1$ and 0 otherwise.

We can show that the MoV_S of any monotonic weighted tournament solution S behaves monotonically. The idea is to construct a reversal set R' for a in T^Q from a minimum reversal set R for a in T, or vice versa, depending on whether $a \in S(T)$ or not. In particular, this result implies monotonicity of the MoV for BO, SC and wUC.

Theorem 4.4. *Let S be a weighted tournament solution. If S is monotonic, its margin of victory function MoV_S is monotonic as well.*

As final monotonicity notion, we consider transfer-monotonicity [8]. An unweighted tournament solution is transfer-monotonic if and only if a winning alternative a remains in the winning set, when an alternative c is "transferred" from the dominion of another alternative b to the dominion of a. To generalize this to weighted tournaments, we do not consider the transfer of *an alternative* c from one dominion to another, but the transfer of *weight over an alternative* c from one alternative to another.

Definition 4.5. *A tournament solution S for an n-weighted tournament $T = (V, w)$ is said to be* transfer-monotonic, *if for any $a, b, c \in V$ with $w(b, c) > 0$ and $w(a, c) < n$,*

$$a \in S(T) \quad \textit{implies} \quad a \in S(T^Q),$$

where the reversal function Q is defined as $Q(b, c) = -1$, $Q(a, c) = +1$ and 0 otherwise.

While all tournament solutions studied by Brill et al. [8] are transfer-monotonic, this is not the case for here. While BO and wUC are both transfer-monotonic, SC is not.

Proposition 4.6. BO *and* wUC *satisfy transfer-monotonicity.*

Proof. Let $T = (V, w)$ be an n-weighted tournament, with $S \in \{\mathrm{BO}, \mathrm{wUC}\}$ and $a, b, c \in V$ such that $a \in S(T)$, $w(b, c) > 0$ and $w(a, c) < n$. Further, let Q be defined by $Q(b, c) = -1$, $Q(a, c) = +1$ and 0 otherwise.

Borda Since $a \in \mathrm{BO}(T)$, a has the highest Borda score. The definition of Q implies $s_{\mathrm{BO}}(a, T^Q) = s_{\mathrm{BO}}(a, T) + 1$, $s_{\mathrm{BO}}(b, T^Q) = s_{\mathrm{BO}}(b, T) - 1$ and $s_{\mathrm{BO}}(z, T^Q) = s_{\mathrm{BO}}(z, T)$ for all $z \in V \setminus \{a, b\}$. Thus, a still has the highest Borda score, and $a \in \mathrm{BO}(T^Q)$.

weighted Uncovered Set Since $a \in \mathrm{wUC}(T)$, a is not covered by any alternative, i. e., for every alternative x, there is a decreasing a-x-path of length at most two. The definition of Q implies $w^Q(a, c) = w(a, c) + 1$ and $w^Q(b, c) = w(b, c) - 1$. The only decreasing path from a to some alternative in T that could be different in T^R, is the decreasing path to c or via c. But since the weight on the outgoing edge (a, c) of a increases, any decreasing path in T is still a decreasing path in T^Q. Thus, a has a decreasing path of length at most two to every alternative in T^Q, and $a \in \mathrm{wUC}(T^Q)$. □

While the full proof of Prop. 4.7 can be found in the appendix, refer to Figure 4 for the counterexample discussed in it.

Proposition 4.7. SC *does not satisfy transfer-monotonicity.*

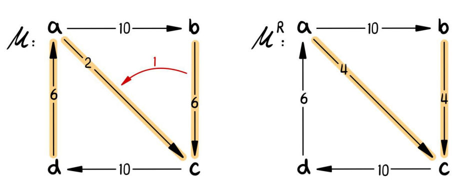


Figure 4: The counterexample used in the proof of Prop. 4.7. Edges ignored by SC **are marked in orange; the weight-transfer from b to a over c is indicated in red. We have $a \in \mathrm{SC}(T)$, but $a \notin \mathrm{SC}(T^Q)$.**

4.2 Cover-Consistency

Next, we turn to cover-consistency. Intuitively, if x covers y, then x should be preferable to y, i. e., $\mathrm{MoV}(x) \geq \mathrm{MoV}(y)$. A MoV_S of any tournament solution S satisfying this condition, is cover-consistent.

Definition 4.8. *Given a weighted tournament solution S, we say that MoV_S is* cover-consistent, *if for any n-weighted tournament $T = (V, w)$ and any alternatives $x, y \in V$,*

$$x \textit{ covers } y \quad \textit{implies} \quad MoV_S(x, T) \geq MoV_S(y, T).$$

For unweighted tournaments, Brill et al. [8] proved that cover-consistency is implied by monotonicity and transfer-monotonicity. For BO and wUC we follow this approach. For SC, which is not transfer-monotonic (see Prop. 4.7), we use a different technique. For the former we provide a proof sketch here, while the full proofs of both results can be found in the appendix.

Theorem 4.9. *If a weighted tournament solution S is monotonic and transfer-monotonic, then MoV_S satisfies cover-consistency.*

Proof Sketch Let S be a monotonic and transfer-monotonic weighted tournament solution and consider an n-weighted tournament $T = (V, w)$ with alternatives $x, y \in V$ such that x covers y. We consider four cases consisting of all combinations of x and y being (not) in the winning set $S(T)$. Case 1 ($x \in S(T), y \notin S(T)$), follows per definition of MoV_S, and Case 2 ($x \notin S(T), y \in S(T)$) is not possible because x covers y.

We continue with a sketch of Case 3 ($x \notin S(T), y \notin S(T)$). We show $\mathrm{MoV}_S(x, T) \geq \mathrm{MoV}_S(y, T)$, which is equivalent to $|\mathrm{MoV}_S(x, T)| \leq |\mathrm{MoV}_S(y, T)|$ since both values are negative, using the following two steps:

(1) Given a minimum wCRS R_y for y in T we construct a reversal function R_x with $|R_x| \leq |R_y|$.
(2) We prove $x \in S(T^{R_x})$, i. e., R_x is a wCRS for x in T. This implies $|\mathrm{MoV}_S(x, T)| \leq |\mathrm{MoV}_S(y, T)|$.

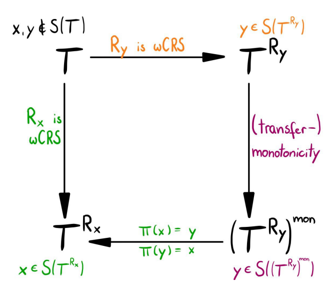


Figure 5: Illustration of Case 3 in the Thm. 4.10 proof sketch.

Since R_y is a wCRS for y in T, we know $y \in S(T^{R_y})$. From T^{R_y}, we create another tournament $(T^{R_y})^{\mathrm{mon}}$ using only monotonic and transfer-monotonic reversals, therefore ensuring $y \in S((T^{R_y})^{\mathrm{mon}})$. Finally, we construct the reversal function R_x such that there is an isomorphism π between $(T^{R_y})^{\mathrm{mon}}$ and T^{R_x} with

$$\pi(x) = y, \quad \pi(y) = x, \quad \text{and} \quad \pi(z) = z \text{ for all } z \in V \setminus \{x, y\}.$$

This means, x in $(T^{R_y})^{\mathrm{mon}}$ corresponds to y in T^{R_x}, and vice versa, while all other alternatives z correspond to themselves. The claim

$x \in S(T^{R_x})$ follows from $y \in S((T^{R_y})^{\text{mon}})$ and the isomorphism of the two tournaments, which we construct in the proof.

Case 4 ($x \in S(T), y \in S(T)$) works analogously.

Prop. 4.2, Thm. 4.4, and Prop. 4.6 together with Thm. 4.10 imply the following:

Theorem 4.10. *MoV_{BO} and MoV_{wUC} satisfy cover-consistency.*

Unfortunately, SC does not satisfy transfer-monotonicity, and as proven in Brill et al. [8, Appendix 1] neither monotonicity nor transfer-monotonicity can be dropped from the condition of the corresponding result [8, Lemma 4] for unweighted tournaments. This implies the same for weighted tournaments.

Nevertheless, SC does satisfy cover-consistency. The proof depends on the fact that all cycles containing x are in strong correlation with cycles containing alternatives covered by x.

Theorem 4.11. *MoV_{SC} satisfies cover-consistency.*

4.3 Degree-Consistency

The last structural property we consider, is degree-consistency. Brill et al. [8, Definition 5.7.] used the notion of degree-consistency to analyze closeness of the ranking naturally induced by the MoV, to the ranking induced by the Copeland scores. For weighted tournaments, we define weighted degree-consistency indicating closeness of the ranking naturally induced by the MoV, to the ranking induced by Borda scores, the weighted extension of Copeland scores.

Definition 4.12. *For a tournament solution S, let $T = (V, w)$ be an n-weighted tournament with alternatives $x, y \in V$. We say MoV_S is w-degree-consistent / equal-w-degree-consistent / strong-w-degree-consistent, if*

$$\delta^+_{\mathrm{w}}(x)\ (> / = / \geq)\ \delta^+_{\mathrm{w}}(y) \ \ \textit{implies}\ \ MoV_S(x,T)\ (> / = / \geq)\ MoV_S(y,T).$$

Lemma 4.13. *Let S be a weighted tournament solution. If MoV_S satisfies w-degree-consistency, MoV_S is cover-consistent.*

Degree-consistency is a property we do not desire for any MoV_S, as it would seem to consider winning many times in total to be the indicator of a winning alternative. Fortunately, none of our three tournament solutions satisfy any weighted degree-consistency, which we show by providing one counterexample each. Those counterexample can be found in the appendix.

Proposition 4.14. *MoV_{BO}, MoV_{SC} and MoV_{wUC} satisfy neither w-degree-consistency nor equal-w-degree-consistency. This implies that they also do not satisfy strong-w-degree-consistency.*

5 BOUNDS ON THE MARGIN OF VICTORY

At last, we give bounds on the MoV, i. e., upper bounds for winners and lower bounds for non-winners. These bounds can give further context to actually obtained MoV values and allow us to compare the innate robustness of our tournament solutions. To avoid case distinctions, we assume $m > 2$. Otherwise, $\lceil \frac{n}{2} \rceil$ reversals are sufficient and necessary for all three studied tournament solutions. In the following, let $T = (V, w)$.

For BO, we show that both the upper and lower bound are in the order of $n \cdot m$. Both values can therefore get arbitrarily large with increasing number of voters/duels or candidates/alternatives.

Theorem 5.1. *For any n-weighted tournament T with m alternatives, BO winner $a \in \mathrm{BO}(T)$ and non-winner $d \in V \setminus \mathrm{BO}(T)$, we have*

$$-(n \cdot (m-2)) \leq MoV(d,T) \leq MoV(a,T) \leq \left\lfloor \frac{n \cdot (m-2)}{2} \right\rfloor + 1.$$

Moreover, both bounds are tight.

The upper and lower bounds for the MoV of an SC winner are also polynomial in n and m, although they do not grow as fast as the upper bound for BO does.

Theorem 5.2. *For any n-weighted tournament T with m alternatives, SC winner $a \in \mathrm{SC}(T)$ and non-winner $d \in V \setminus \mathrm{SC}(T)$, we have*

$$-\left\lceil \frac{n}{2} \right\rceil \cdot (m-1) \leq MoV(d,T) \leq MoV(a,T) \leq n + \left\lceil \frac{(m-2)}{2} \right\rceil.$$

Moreover, both bounds are tight.

Lastly, we analyse the MoV of wUC. The upper bound is similar to the upper bound of the MoV for Borda's rule. It differs from the BO upper bound by an additional $\lceil \frac{n}{2} \rceil$. For wUC the upper bound is also correct for the case $m = 2$.

Theorem 5.3. *For any n-weighted tournament T with m alternatives, wUC winner $a \in \mathrm{wUC}(T)$ and non-winner $d \in V \setminus \mathrm{wUC}(T)$, we have*

$$-\log_2(m) \cdot \left\lceil \tfrac{n+1}{2} \right\rceil \leq MoV(b,T) \leq MoV(a,T) \leq \left\lceil \tfrac{n+1}{2} \right\rceil + \left\lfloor \tfrac{n \cdot (m-2)}{2} \right\rfloor.$$

Moreover, the upper bound is tight.

6 DISCUSSION

The notion of margin of victory (MoV), introduced by Brill, Schmidt-Kraepelin, and Suksompong [8] for unweighted tournaments, provides a generic framework for refining tournament solutions. In this paper, we extended the notion to weighted tournaments. We considered Borda's rule, Split Cycle, and the weighted Uncovered Set, and analyzed the computational complexity of computing the MoV and provided structural insight.

There are several natural weighted tournament solutions whose MoV we did not study, for instance, the Beat Path (or Schulze method)[20] or the Maximin rule [24].

Any connection between the behavior of a tournament solution S, or a class of tournament solutions, and the structural properties of MoV_S could help with understanding both. In our conducted, yet omitted, experiments we stuck to the state of the art using uniform random distributions or transitive preferences. Instead, one might use a *ground truth of strength* of the players presented as a tournament or using systems like Elo ranking or True Skill. Given this input, analyzing the behavior of S and MoV_S or looking for bounds in expectation seems compelling.

One very practical generalization would be to work with partial tournaments, relaxing the requirement of n comparisons between all pairs of alternatives. In many natural scenarios like election, resp. tournament, prognosis or data acquisition settings, in which a group of voters is asked for pairwise comparing only a certain subset of alternatives, we have to work with partial information. An MoV notion for such partial tournaments could be of assistance.

One open question by Brill et al. [8] asks for the number of distinct minimum reversal sets and its meaning. This of course might also be interesting for weighted tournaments.

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