Fairly Allocating (Contiguous) Dynamic Indivisible Items with Few Adjustments

Extended Abstract

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ABSTRACT

We study the problem of dynamically allocating T indivisible items to n agents with the restriction that the allocation is fair all the time. Due to the negative results to achieve fairness when allocations are irrevocable, we allow adjustments to make fairness attainable with the objective to minimize the number of adjustments. For restricted additive or general identical valuations, we show that *envy-freeness up to one item (EF1)* can be achieved with no adjustments. For additive valuations, we give an EF1 algorithm that requires $O(mT)$ adjustments, improving the previous result of $O(nmT)$ adjustments, where m is the maximum number of different valuations for items among all agents.

We further impose the contiguity constraint on items such that items are arranged on a line by the order they arrive and require that each agent obtains a consecutive block of items. We present extensive results to achieve either *proportionality* with an additive approximate factor (PROPa) or EF1, where PROPa is a weaker fairness notion than EF1. In particular, we show that for identical valuations, achieving PROPa requires $\Theta(nT)$ adjustments. Moreover, we show that it is hopeless to make any significant improvement for either PROPa or EF1 when valuations are nonidentical.

Our results exhibit a large discrepancy between the identical and nonidentical cases in both contiguous and noncontiguous settings. All our positive results are computationally efficient.

KEYWORDS

Fair division; Envy-freeness; Online algorithms

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1 INTRODUCTION

Fair division is one of the most fundamental and well-studied topics in Computational Social Choice with much significance and several applications in many real-life scenarios [15, 24]. Generally, there are some resources and our objective is to divide them among a group of competing agents in a fair manner. In our discussion, we assume that the items to be allocated are *goods*, whose valuations are nonnegative. Arguably, the most compelling fairness notion is *envy-freeness*, which is defined as each agent weakly preferring his own bundle to any other agent's bundle.

However, in the indivisible regime, the existence of envy-free solutions is not guaranteed. For instance, if there are two agents but only one item, the agent who receives the item is certainly envied by the other one. One of the natural relaxations of envy-freeness is *envy-freeness up to one item (EF1)* [11]. In an EF1 allocation, every agent i may envy another agent j, but the envy could be eliminated by removing one item from agent j's bundle. EF1 allocations are always guaranteed to exist and can be computed in polynomial time even for general valuations [19].

Imposing some constraints to the model will make the fairness objective less tractable. A series of works focus on the setting where items lie on a line and each agent obtains a consecutive block of items. For monotone valuations, Bilò et al. [8] present polynomial-time algorithms to compute contiguous EF1 allocations for any number of agents with identical valuations or at most three agents, and prove the existence of contiguous EF1 allocations for four agents. More recently, it is shown that contiguous EF1 allocations for any number of agents always exist [18].

Another natural generalization assumes that items arrive online. Furthermore, the types of future items are unknown and decisions have to be made instantly. He et al. [16] investigate this model with the requirement that the allocations returned after the arrival of each item are EF1 and consider additive valuations. However, due to the negative results against adversaries when allocations are irrevocable [7], adjustments are necessary to achieve EF1 deterministically. Notably, the $O(T^2)$ of adjustments is attained trivially by redistributing all items in each round, where T is the number of items. He et al. [16] show that the $\Omega(T)$ of adjustments is inevitable for more than two agents, even if the information of all items is known upfront. On the positive side, they give two algorithms with respectively $O(T^{1.5})$ and $O(nmT)$ adjustments, where n is the number of agents and m denotes the maximum number of distinct valuations for items among all agents.

In this work, we adopt the model proposed by He et al. [16]. Furthermore, we also impose the contiguity constraint on items. More precisely, items are arranged on a line by the order they arrive and each agent obtains a contiguous block of items. To motivate this, consider a library that has several bookshelves with books of the same types being placed together. Moreover, the numbers of books of two different types should not differ by a large amount. With more and more bookshelves being deployed, the library needs to reallocate a consecutive block of bookshelves to each certain type of book and redistribute the books according to the new allocation. In this case, the objective is to minimize the cost of moving the books

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Table 1: Results for the contiguous setting with additive valuations. The result for two agents in the identical case also works for general valuations.

	Identical			Nonidentical	
	PROPa	EF1		PROPa	EF1
		Lower	Upper		
$n = 2$		$\Theta(T)$		$\Theta(T^2)$	
$n > 2$	$\Theta(nT)$	$\Omega(nT)$	$O\left(\frac{R}{L} \cdot n^2 T\right)$	$\Omega(T^2/n)$	$\Theta(T^2)$

or equivalently, the number of adjustments. We study the number of adjustments necessary to achieve some fairness guarantee in both contiguous and noncontiguous settings. The complete proofs of all results can be found in the full version of the paper [26].

2 OUR CONTRIBUTIONS

We first describe our positive results in the noncontiguous setting. We show that if valuations are limited to be restricted additive[1] or general identical, EF1 can be achieved with no adjustments. In addition, we give an EF1 algorithm for additive valuations that requires $O(mT)$ adjustments, improving the previous result of $O(nmT)$ adjustments. Note that if m is a constant, it matches the $\Omega(T)$ lower bound and thus is optimal.

With the contiguity constraint, EF1 is too stringent and we start with a weaker fairness notion. It is known that contiguous $(\frac{n-1}{n} \cdot v^{\max})$-*proportional (PROPa)* allocations can be computed efficiently, where $v^{\max}$ is the maximum valuation of items, and this additive approximate factor is tight in some sense [25]. We first consider identical valuations. We give a PROPa algorithm that requires $O(nT)$ adjustments and then establish the matching lower bound to show that our algorithm is optimal. When it comes to EF1, for two agents with general valuations, we show that EF1 is achievable with $O(T)$ adjustments, which is optimal. If the valuation of each item is assumed to lie in $[L, R]$ such that $0 < L \leq R$, we give an EF1 algorithm that requires $O\left((R/L) \cdot n^2 T\right)$ adjustments. By contrast, in the nonidentical case, we show that it is hopeless to make any significant improvement even for additive valuations. Specifically, we give instances to establish the lower bounds of $\Omega(T^2/n)$ to achieve PROPa and $\Omega(T^2)$ for EF1. Our results in the contiguous setting with additive valuations are summarized in Table 1.

Our results exhibit a large discrepancy between the identical and nonidentical cases in both contiguous and noncontiguous settings. In addition, all the algorithms given in this paper can be implemented in polynomial time.

3 RELATED WORK

Even though both divisible and indivisible models are extensively studied, we only focus on the indivisible setting, which is more relevant to our work.

Dynamic fair division. Our work belongs to the vast literature of *dynamic* or *online* fair division [4]. Under the assumption that valuations are normalized to $[0, 1]$ and items are allocated irrevocably, the maximum envy of $\tilde{O}(\sqrt{T/n})$ can be achieved deterministically and this bound is tight asymptotically [7]. To bypass the negative results, motivated by the notion of *disruptions* [12, 13], He et al. [16] introduce adjustments to achieve EF1 deterministically.

Without the ability to adjust, another popular measure of compromise is assuming that agents' valuations are stochastic. When the valuation of each agent to each item is drawn i.i.d. from some distribution, the algorithm of allocating each item to the agent with the maximum valuation for it is envy-free with high probability and ex-post Pareto optimal [20, 21]. Besides, Bai et al. [5] provide the same guarantee for asymmetric agents, i.e., the valuation of an item for each agent is independently drawn from an agent-specific distribution. Furthermore, assuming that the valuations of different agents for the same item are correlated, Pareto efficiency and fairness are also compatible [27]. More recently, Benadè et al. [6] study the partial-information setting where only the ordinal information is revealed. Another series of works [2, 3] resort to random allocations to achieve ex-ante fairness together with some efficiency and incentive guarantees.

Fair division of contiguous blocks. The online fair division model with the contiguity requirement concerned in our work is a strict extension of the fair division of contiguous blocks problem. In the offline setting, the existence of contiguous EF1 allocations is intensively studied [8, 18, 23] and the approximate versions of proportionality, envy-freeness, and equitability are also considered [25]. More recently, Misra et al. [22] designs an algorithm to compute a contiguous EQ1 allocation with the egalitarian welfare guarantee. Besides, the *price of fairness* of contiguous allocations for goods and chores are respectively established by Suksompong [25] and Höhne et al. [17]. More generally, the connectivity relation among items is allowed to form a graph that possesses some structures with a path being a special case [9, 10].

4 CONCLUSION AND FUTURE WORK

We conclude with some directions for future work.

- Even though we have established almost tight upper and lower bounds to achieve PROPa in the contiguous setting or when $n = 2$, there are still large gaps between the upper and lower bounds to achieve EF1 in both continuous and noncontinuous settings. The first future direction is to tighten these bounds.
- If the types of items are drawn from certain distributions rather than chosen adversely, can we show a better upper bound in expectation or asymptotically in both settings[2]?
- It would be interesting to investigate other fairness notions like EQ1 [22].
- Another promising direction, like always asked in the offline setting [14], is to achieve fairness and Pareto optimality simultaneously. This has been shown in some other online models [3, 27].

[1] The valuations are restricted additive if they are additive and every item has an inherent valuation with every agent being interested in only some items [1].

[2] Similar problems are presented in [25].

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