Distance Hypergraph Polymatrix Coordination Games

Extended Abstract

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ABSTRACT

We propose the new class of *distance hypergraph polymatrix coordination* games, properly generalizing distance polymatrix coordination games, in which each subgame can be played by more than two agents. We modelled it using hypergraphs, where each hyperedge represents a subgame played by its agents.

Moreover, as for distance polymatrix coordination games, the overall utility of a player x also depends on the payoffs of the subgames where the involved players are far, at most, a given distance from x. As for the original model, we discount these payoffs proportionally by factors depending on the distance of the related hyperedges.

We focus on the degradation of the social welfare by resorting to the standard measures of strong Price of Anarchy and Price of Stability for both general and bounded-degree graphs.

KEYWORDS

Polymatrix games; Hypergraphs; Strong Nash equilibrium; Price of anarchy; Price of stability

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1 INTRODUCTION

Polymatrix games were introduced more than forty years ago [20] and have received a lot of attention from many researchers since it is a very general model which can be applied in different real scenarios, and several games (e.g., *hedonic games* [12], *max-cut* [17]) can be derived from it. Some classic papers are [13, 18, 19, 21], while more recent studies are [6, 10, 11, 23], where the authors showed results mainly concerning equilibria and computational issues. Another model close to polymatrix games that is worthy of mention is the *group activity selection problem* [5, 8, 9].

In the subclass of *polymatrix coordination games* [23], the interaction graph is undirected since the outcome of a binary game is the same for both players. An extension of this classic model is presented in [1], where the utility of an agent x does not only depend on the games (edges) where x is involved but also on the games (edges) played by agents which are far, at most, a parameter d from x.

In this work, we introduce and study a new, more general model, *distance hypergraph polymatrix coordination games*, where each local game can concern more than two players and the utility of an agent x can also depend on the games at a distance bounded by d. In this new model, the interaction graph becomes an undirected hypergraph, where every hyperedge is a game played by the player contained in it. Following the idea proposed in [1], the utility of an agent x is the sum of the outcomes of the games she plays plus the outcomes of the games played by other players, which are at a distance at most equal to d from x. Each agent x also gets an additional payoff that is a function of the strategy chosen by x. The idea of obtaining utility from non-neighboring players has been analysed also for *distance hedonic games* [15, 16], which generalise *fractional hedonic games* [3, 4, 7, 14, 22] similarly as distance polymatrix games and our model do with polymatrix coordination games. Our model can also be seen as a generalisation of the *hypergraph hedonic games* [2], introducing preferences and increasing the expressiveness of the weight function.

Our new model can be used to represent real-life scenarios, which are not covered by previous models. On the one hand, extending a local game to more than two players is reasonable because, in many natural social environments (e.g., politics, sports, academia, etc.), people get a payoff from activities that involve more than two players. As an example, in a scientific community a project or a paper often involves more than two researchers and its outcome depends on the choice made by each person. This can be modelled using a hyperedge for each project/paper with a weight (payoff) depending on the participants' strategies. On the other hand, any individual also benefits, albeit to a smaller degree, when her close colleagues succeed in a project or publish a good paper that she is not personally a part of. This is quite obvious when considering the student–advisor relationship, but also noticeable for researchers working at the same university or institution. We can model these indirect relationships by introducing distances and discount factors.

Having formalized our new model, we focus on the degradation of the social welfare where k players can simultaneously change their strategies. We analyze the Price of Anarchy and Stability for k-strong Nash equilibria, providing tight lower and upper bounds.

2 MODEL AND PRELIMINARIES

A *d-distance hypergraph polymatrix coordination game* (d-DHPCG) $\mathcal{G} = (\mathcal{H}, (\Sigma_x)_{x \in V}, (w_e)_{e \in E}, (p_x)_{x \in V}, (\alpha_h)_{h \in [d]})$, is a game based on a hypergraph $\mathcal{H}$ of arity r, and defined as follows:

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Agents: The set of agents is $V = [n]$, i.e., each node corresponds to an agent. We reasonably assume that $n \geq r \geq 2$.

Strategy profile or outcome: For any $x \in V$, Σ_x is a finite set of *strategies* of player x. A *strategy profile or outcome* $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ is a configuration in which each player $x \in V$ plays strategy $\sigma_x \in \Sigma_x$.

Weight function: For any edge $e \in E$, let $w_e : \times_{x \in e} \Sigma_x \longrightarrow \mathbb{R}_{\geq 0}$ be the *weight function* that assigns, to each subset of strategies σ_e played respectively by every $x \in e$, a *weight* $w_e(\sigma_e) \geq 0$. In what follows, for the sake of brevity, given any strategy profile $\boldsymbol{\sigma}$, we will often denote $w_e(\sigma_e)$ simply as $w_e(\boldsymbol{\sigma})$.

Preference function: For any $x \in V$, let $p_x : \Sigma_x \longrightarrow \mathbb{R}_{\geq 0}$ be the *player-preference function* that assigns, to each strategy profile σ_x played by player x, a non-negative real value $p_x(\sigma_x)$, called *player-preference*. In what follows, similarly to the weight function, we will often denote $p_x(\sigma_x)$ as $p_x(\boldsymbol{\sigma})$.

Distance-factors sequence: Let $(\alpha_h)_{h \in [d]}$ be the *distance-factors sequence* of the game, that is a non-negative sequence of real parameters, called *distance-factors*, such that $1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_d \geq 0$.

Utility function: For any $h \in [d]$, let $E_h(x)$ be the set of hyperedges $e \in E$ such that the minimum distance between x and one of the players $v \in e$ is exactly $h - 1$. Then, for any $x \in V$, the *utility function* $u_x : \times_{x \in V} \Sigma_x \longrightarrow \mathbb{R}$ of player x, for any strategy profile $\boldsymbol{\sigma}$ is defined as $u_x(\boldsymbol{\sigma}) := p_x(\boldsymbol{\sigma}) + \sum_{h \in [d]} \alpha_h \sum_{e \in E_h(x)} w_e(\boldsymbol{\sigma})$.

The *social welfare* $\mathsf{SW}(\boldsymbol{\sigma})$ of a strategy profile $\boldsymbol{\sigma}$ is defined as $\mathsf{SW}(\boldsymbol{\sigma}) := \sum_{x \in V} u_x(\boldsymbol{\sigma})$. The *k-strong Price of Anarchy* of a game $\mathcal{G}$ is defined as $\mathsf{PoA}_k(\mathcal{G}) := \max_{\boldsymbol{\sigma} \in \mathsf{NE}_k(\mathcal{G})} \frac{\mathsf{OPT}(\mathcal{G})}{\mathsf{SW}(\boldsymbol{\sigma})}$, where $\mathsf{NE}_k(\mathcal{G})$ is the set of k-strong Nash equilibria of $\mathcal{G}$, and $\mathsf{OPT}(\mathcal{G})$ is the value of an optimum outcome $\boldsymbol{\sigma}^*$. The *k-strong Price of Stability* of game $\mathcal{G}$ is defined as $\mathsf{PoS}_k(\mathcal{G}) := \min_{\boldsymbol{\sigma} \in \mathsf{NE}_k(\mathcal{G})} \frac{\mathsf{OPT}(\mathcal{G})}{\mathsf{SW}(\boldsymbol{\sigma})}$.

3 OUR RESULTS

First, we observe that when the number of deviating agents k is lower than the arity r of the hypergraph, there exists a simple d-DHPG $\mathcal{G}$ with n agents such that PoA_k is unbounded. Therefore, we will only consider the estimation of PoA_k for $k \geq r \geq 2$.

3.1 PoA_k for general hypergraphs

In the first part, we focus on finding tight upper and lower bounds for PoA_k when there is no particular assumption on the underlying hypergraph of the considered game.

Concerning PoA_k, we derive the following upper bound.

$$\mathsf{PoA}_k(\mathcal{G}) \leq \frac{(n-1)_{r-1}}{(k-1)_{r-1}} \left(r + \alpha_2(n-2)\right). \tag{1}$$

We complete the study of PoA_k for general hypergraphs providing the following lower bound.

$$\mathsf{PoA}_k(\mathcal{G}) \geq \frac{(n-1)_{r-1}}{(k-1)_{r-1}} \left(r + \alpha_2(n-r)\right). \tag{2}$$

Results (1) and (2) hold for any $n \geq k \geq r$, $d \geq 1$, and any distance-factors sequence.

3.2 PoS_k for general hypergraphs

We conclude the study of the inefficiency for general hypergraphs by providing a lower bound for the k-strong Price of Stability asymptotically equal to the upper bound for the k-strong Price of Anarchy shown in Eq. (1). This means that we can use the upper bound in (1) also for the k-strong Price of Stability and close our study for general hypergraphs.

The idea is to start from the lower bound instance used for Eq. (2), then transform it into a new instance with the property of having every *near-optimal outcome* unstable. An outcome is said to be near-optimal if its social welfare is close to the optimum. The new instance has the property that all the equilibria have the same social welfare. This leads to the existence of a d-DHPCG $\mathcal{G}$ such that

$$\mathsf{PoS}_k(\mathcal{G}) \geq \frac{n-r}{n-1} \frac{(n-1)_{r-1}}{(k-1)_{r-1}} \frac{(r + \alpha_2(n-r))}{2(1+\alpha_2)} \tag{3}$$

which holds for every $n \geq 6$, $k \geq r$, $d \geq 1$, and any distance-factors sequence.

3.3 PoA_k for bounded-degree hypergraphs

In this section, we analyse the k-strong Price of Anarchy for games whose hypergraphs have a bounded-degree. A hypergraph $\mathcal{H}$ has degree bounded by Δ if the degree of every node x of $\mathcal{H}$ is at most Δ. We also say that a game $\mathcal{G}$ is Δ-bounded degree if the degree of every node in the underlying hypergraph is at most Δ. Here, we will only focus on the cases where $k \geq r$, and $\Delta \geq 2$, since the case when $\Delta = 1$ is encompassed by Section 3.1.

First, we show that for every outcome $\boldsymbol{\sigma}$, (i): $\mathsf{SW}(\boldsymbol{\sigma}) \leq \sum_{x \in V} p_x(\boldsymbol{\sigma}) + r \sum_{h \in [d]} \alpha_h \cdot (\Delta - 1)^{h-1} r^{h-1} \cdot \sum_{e \in E} w_e(\boldsymbol{\sigma})$ holds.

Before continuing, we observe that if an outcome $\boldsymbol{\sigma}$ is a strong equilibrium, and $\boldsymbol{\sigma}^*$ is an optimum outcome, then for every hyperedge e, there exists an agent x for which $u_x(\boldsymbol{\sigma}) \geq p_x(\boldsymbol{\sigma}^*) + w_e(\boldsymbol{\sigma}^*)$ holds. Moreover, every other agent in e gets more in $\boldsymbol{\sigma}$ than her preference value in $\boldsymbol{\sigma}^*$. By summing these inequalities for every hyperedge e, and using (i), we obtain the upper bound

$$\mathsf{PoA}_k(\mathcal{G}) \leq r \sum_{h \in [d]} \alpha_h \cdot \Delta \cdot (\Delta - 1)^{h-1} r^{h-1} \tag{4}$$

which holds for any $n \geq k \geq r$, $d \geq 1$, and any distance-factors sequence. Please note that the upper bound in (4) implies that the k-strong Price of Anarchy of Δ-bounded-degree d-DHPCG, as a function of d, grows at most as $O((\Delta - 1)^d r^d)$.

In the following, we provide a lower bound on the k-strong Price of Anarchy, relying on a nice result from graph theory. For any $k \geq r$, $\Delta \geq 3$, $d \geq 1$, and any distance-factors sequence $(\alpha_h)_{h \in [d]}$, there exists a Δ-bounded-degree d-DHPCG $\mathcal{G}$ such that $\mathsf{PoA}_k(\mathcal{G}) \geq$

$$\frac{\sum_{h \in [d]} \alpha_h \cdot (a+1) \cdot a^{h-1} b^{h-1}}{1 + \sum_{h=1}^{d-1} \alpha_{h+1} (2a^{\lfloor (h+1)/2 \rfloor} b^{\lfloor (h+1)/2 \rfloor - 1} + 2a^{\lfloor h/2 \rfloor - 1} b^{\lfloor h/2 \rfloor})} \tag{5}$$

where $a = (\Delta - 1)$, $b = (r - 1)$. We used a and b instead of $(\Delta - 1)$ and $(r - 1)$ just to make (5) shorter.

Please note that, if all the distance-factors are not lower than a constant $c > 0$, from Eq. (5) we can conclude that the k-strong price of anarchy of the Δ-bounded-degree d-DHPCG, as a function of d, can grow as $\Omega((\Delta - 1)^{d/2} (r - 1)^{d/2})$.

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REFERENCES

[1] Alessandro Aloisio, Michele Flammini, Bojana Kodric, and Cosimo Vinci. 2021. Distance Polymatrix Coordination Games. In *Proc. 30th Intl. Joint Conf. Artif. Intell. (IJCAI)*. 3–9.

[2] Alessandro Aloisio, Michele Flammini, and Cosimo Vinci. 2020. The Impact of Selfishness in Hypergraph Hedonic Games. In *Proc. 34th Conf. Artificial Intelligence (AAAI)*. 1766–1773.

[3] Haris Aziz, Florian Brandl, Felix Brandt, Paul Harrenstein, Martin Olsen, and Dominik Peters. 2019. Fractional Hedonic Games. *ACM Trans. Economics and Comput.* 7, 2 (2019), 6:1–6:29.

[4] Vittorio Bilò, Angelo Fanelli, Michele Flammini, Gianpiero Monaco, and Luca Moscardelli. 2018. Nash Stable Outcomes in Fractional Hedonic Games: Existence, Efficiency and Computation. *J. Artif. Intell. Res.* 62 (2018), 315–371.

[5] Vittorio Bilò, Angelo Fanelli, Michele Flammini, Gianpiero Monaco, and Luca Moscardelli. 2019. Optimality and Nash Stability in Additive Separable Generalized Group Activity Selection Problems. In *Proc. 28th Intl. Joint Conf. Artif. Intell. (IJCAI)*. 102–108.

[6] Yang Cai, Ozan Candogan, Constantinos Daskalakis, and Christos H. Papadimitriou. 2016. Zero-Sum Polymatrix Games: A Generalization of Minmax. *Math. Oper. Res.* 41, 2 (2016), 648–655.

[7] Raffaello Carosi, Gianpiero Monaco, and Luca Moscardelli. 2019. Local Core Stability in Simple Symmetric Fractional Hedonic Games. In *Proc. 18th Conf. Autonomous Agents and Multi-Agent Systems (AAMAS)*. 574–582.

[8] Andreas Darmann, Edith Elkind, Sascha Kurz, Jérôme Lang, Joachim Schauer, and Gerhard J. Woeginger. 2012. Group Activity Selection Problem. In *Proc. 8th Intl. Workshop Internet & Network Economics (WINE)*, Vol. 7695. 156–169.

[9] Andreas Darmann and Jérôme Lang. 2017. Group activity selection problems. In *Trends in computational social choice*. 385–410.

[10] Argyrios Deligkas, John Fearnley, and Rahul Savani. 2020. Tree Polymatrix Games Are PPAD-Hard. In *Proc. 47th Intl. Coll. Autom. Lang. Program. (ICALP)*. 38:1–38:14.

[11] Argyrios Deligkas, John Fearnley, Rahul Savani, and Paul G. Spirakis. 2017. Computing Approximate Nash Equilibria in Polymatrix Games. *Algorithmica* 77, 2 (2017), 487–514.

[12] J. H. Drèze and J. Greenberg. 1980. Hedonic Coalitions: Optimality and Stability. *Econometrica* 48, 4 (1980), 987–1003.

[13] B. Curtis Eaves. 1973. Polymatrix Games with Joint Constraints. *SIAM J. Appl. Math.* 24, 3 (1973), 418–423.

[14] Edith Elkind, Angelo Fanelli, and Michele Flammini. 2020. Price of Pareto Optimality in hedonic games. *Artif. Intell.* 288 (2020), 103357.

[15] Michele Flammini, Bojana Kodric, Martin Olsen, and Giovanna Varricchio. 2020. Distance Hedonic Games. In *Proc. 19th Conf. Autonomous Agents and Multi-Agent Systems (AAMAS)*. 1846–1848.

[16] Michele Flammini, Bojana Kodric, Martin Olsen, and Giovanna Varricchio. 2021. Distance Hedonic Games. In *Proc. 47th Intl. Conf. Current Trends in Theory & Practice of Comput. Sci. (SOFSEM)*. 159–174.

[17] Laurent Gourvès and Jérôme" Monnot. 2009. On Strong Equilibria in the Max Cut Game. In *Internet and Network Economics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 608–615.

[18] Joseph T. Howson. 1972. Equilibria of Polymatrix Games. *Management Science* 18, 5 (1972), 312–318.

[19] Joseph T. Howson and Robert W. Rosenthal. 1974. Bayesian Equilibria of Finite Two-Person Games with Incomplete Information. *Management Science* 21, 3 (1974), 313–315.

[20] E. Janovskaja. 1968. Equilibrium points in polymatrix games. *Lithuanian Mathematical Journal* 8, 2 (1968), 381–384.

[21] Douglas A. Miller and Steven W. Zucker. 1991. Copositive-plus Lemke algorithm solves polymatrix games. *Operations Research Letters* 10, 5 (1991), 285–290.

[22] Gianpiero Monaco, Luca Moscardelli, and Yllka Velaj. 2020. Stable outcomes in modified fractional hedonic games. *Auton. Agents Multi Agent Syst.* 34, 1 (2020), 4.

[23] Mona Rahn and Guido Schäfer. 2015. Efficient Equilibria in Polymatrix Coordination Games. In *Proc. 40th Intl. Symp. Math. Foundations of Computer Science (MFCS)*. 529–541.