MMS Allocations of Chores with Connectivity Constraints: New Methods and New Results

Extended Abstract

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ABSTRACT

We study the problem of allocating indivisible chores to agents under the Maximin share (MMS) fairness notion. The chores are embedded on a graph and each bundle of chores assigned to an agent should be connected. Although there is a simple algorithm for MMS allocations of goods on trees, it remains open whether MMS allocations of chores on trees always exist or not, which is a simple but annoying problem in chores allocation. In this paper, we introduce a new method for chores allocation with connectivity constraints, called the group-satisfied method, that can show the existence of MMS allocations of chores on several subclasses of trees. Even these subcases are non-trivial and our results can be considered as a significant step to the open problem. We also consider MMS allocations of chores on cycles where we get the tight approximation ratio for three agents. Our result was obtained via the linear programming (LP) method, which enables us not only to compute approximate MMS allocations but also to construct tight examples of the nonexistence of MMS allocations without complicated combinatorial analysis. These two proposed methods, the group-satisfied method and the LP method, have the potential to solve more related problems.

KEYWORDS

Fair allocation; Chores; Maximin share; Connectivity constraints

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1 INTRODUCTION

The theory of the *fair allocation* investigates how to allocate items to a set of agents under some fairness notion, where different agents may have different valuation functions on the items. The spirit of the fair allocation problem is to achieve a desired outcome for individuals and the whole community simultaneously, which motivates several important problems in mathematics and computer science. The *goods allocation* problem with positive valuation functions has received extensive studies [7, 8, 11, 12]. However, in some scenarios in real life, the items to be allocated may have disutility, i.e., the valuation functions are negative, such as troublesome jobs, household chores, or debt. For this case, the problem is called *chores allocation* problem [2]. Seemingly, the problem of chores allocation is similar to the well-studied goods allocation problem as we can reduce the former to the latter by setting all valuation functions as their negative counterparts. However, many properties of the problem may change and not be applicable under this reduction. Thus, most results for goods allocation cannot be trivially extended to chores allocation [1–3, 9, 17].

There are several fairness notions for allocations, such as envy-free (EF) [11], proportionality (PROP) [14], maximin share (MMS) [8], and so on. In this paper, we consider the MMS fairness notion where the MMS value is the best possible guarantee that an agent can expect when he divides the items into several bundles but picks after all other agents. Moreover, we study MMS allocations of chores on graphs with connectivity constraints in which chores are embedded on a graph and each bundle assigned to an agent is required to be connected [4–6, 10, 15]. The connectivity requirements for chores capture the scenarios such as the allocation for energy conservation and emission reduction obligations between countries where the feature of geographical adjacency is natural, crop harvesting task allocation problem or clean work arrangement in the street where arranging a task with continuous geographical location is more convenient to set and use tools, and so on. Note that MMS allocation may not always exist. It is also meaningful to study the approximate α-MMS allocation: whether each agent can always get α fraction of his MMS guarantee in the allocation.

In this extended abstract, we formally introduce the problem settings and briefly clarify the core ideas of our two novel techniques. More details can be found in the full paper [16].

2 FAIR ALLOCATION OF CHORES WITH CONNECTIVITY CONSTRAINTS

Let A be a set of n agents and C be a set of m chores embedded on an undirected graph $G = (C, E)$. Each agent $i \in A$ associates with an additive cost function $u_i : C \rightarrow \mathbb{R}_{\leq 0}$ on chores such that $u_i(C') = \sum_{c \in C'} u_i(c)$ holds for any subset of chores $C' \subseteq C$. An *feasible allocation* $\pi : A \rightarrow 2^C$ satisfying connectivity constraints can be represented by a permutation over *feasible partition* $\mathbf{P} = (P_1, P_2, \ldots, P_n)$ such that $\bigcup_{i \in [n]} P_i = C$, $P_i \cap P_j = \emptyset$ for any different $i, j \in [n]$ and the induced subgraph $G[P_i]$ is connected for each $i \in [n]$ where $[n] = \{1, \ldots, n\}$. We use $\mathcal{I} = (G, A, U = (u_i)_{i \in A})$

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to denote an instance for chores allocation with connectivity constraints and $\mathcal{P}(\mathcal{I})$ to denote the set of all feasible partitions.

The *Maximin share* (MMS) fairness notion specifies the Maximin share guarantee for each agent $i \in A$ as

$$mms_i(\mathcal{I}) = \max_{\mathbf{P} \in \mathcal{P}(\mathcal{I})} \min_{j \in [n]} u_i(P_j).$$

For any constant $\alpha \geq 1$, a feasible allocation π is an *α-MMS allocation* if $u_i(\pi(i)) \geq \alpha \cdot mms_i$ for each agent $i \in A$. When $\alpha = 1$, it is simply called an MMS allocation. Understanding the existence of α-MMS allocation of chores on graphs is our primary goal.

3 THE GROUP-SATISFIED METHOD AND ITS APPLICATION ON SUBCLASSES OF TREES

The *last-diminisher method* is an effective way for proportionality cuttings of divisible cake [13] and Bouveret et al. [5] applied its discrete version to show that MMS allocation of goods on trees always exists and can be computed in polynomial time. The last-diminisher method for goods MMS allocation on trees proceeds in a top-down manner: Specify the tree as a rooted tree arbitrarily. An arbitrary agent chooses a rooted subtree (consisting all descendants of the chosen root) and every other agent, in turn, has the option to diminish the prescribed rooted subtree into the minimal ones such that the valuation of any rooted subtree in it is strictly less than his MMS guarantee. The last diminisher gets the prescribed subtree and the procedure repeats for the remaining agents. Two significant facts ensure the correctness of this method where at the end of any intermediate phase: (1) the agents who have been allocated goods are satisfied and the Maxmin share guarantee of every other agent is non-decreasing in the new instance, and (2) the embedding graph of the new instance remains a connected tree. It turns out that the top-down feature of this procedure is crucial for these two facts. If we apply this idea for chores allocations on trees, the last diminisher method will proceed in a bottom-up manner. To be specific, each agent would like to expand the prescribed subtree by including more chores until it fails to form a subtree of any larger satisfied connected tree. At first glance, we can also allocate the last expanded subtree to the last agent who expands it. However, removing the maximal satisfied connected tree will possibly ruin the connectivity of the remaining graph. On the other hand, it may also be too bad for the agent to take the whole larger rooted subtree to keep the remaining graph a connected tree.

Essentially, the last diminisher method corresponds to a widely used methodology in MMS allocation–exploiting the recursive structure of the problem. To be specific, given an instance $\mathcal{I}$ of the fair allocation problem, we construct a partial MMS allocation among some agents and items, and recursively compute the MMS allocation on $\mathcal{I}'$ consisting of the remaining agents and items where $\mathcal{I}'$ is an instance satisfying certain properties such that the MMS value for each remaining agent is non-decreasing. From this perspective, it is natural to generalize the last diminisher method by allowing multiple agents to participate in the partial allocation in each recursion step, which we called the *group-satisfied method*. The group-satisfied method seems necessary for understanding the existence of the MMS allocation of chores on trees in a recursive way. From our previous discussion, the failure to exploit the recursive structure for chores results from the possibility that allocating a whole rooted subtree for an agent to keep the remaining tree connected may be an overburden for the agent. So, instead of allocating a whole subtree to one agent, we try to allocate it to several agents. However, it is not obvious that there always exists an appropriate satisfied rooted subtree. Together with a matching technique in graph theory, we prove the following theorem.

Theorem 1. *MMS allocation of chores on trees with depth 3 and spiders always exists and can be computed in polynomial time.*

4 THE LINEAR PROGRAMMING METHOD AND ITS APPLICATION ON CYCLES

As we mentioned before, extensive work on MMS allocation was built upon the effective recursion method. However, there is a trade-off in this methodology between the complexity of analysis and the quality of the allocation: The more agents the partial allocation involves in each recursion step, the more potential allocations the method captures while the more complicated the analysis becomes. One of our main contributions is a novel method based on linear programming (LP) capturing the optimal approximate MMS chores allocation on cycles for three agents without complicated combinatorial analysis.

To determine the optimal approximate ratio of the MMS allocation, it suffices to enumerate all possible allocations among all instances. However, this trivial idea is impractical and we instead try to characterize the non-existence of the approximate MMS allocation. Specifically, we can easily establish a linear programming (LP_α) parameterized by ratio α where the feasible solution corresponds to the non-existence of the α-MMS allocation among all instances. Then, our objective is to find the infimum of $\{\alpha : \text{LP}_\alpha \text{ is feasible}\}$. However, the constraints in our LP are impractical to list since the number of constraints may even be exponential with respect to n and m. It turns out that we can characterize some clean and simple necessary conditions for the non-existence of α-MMS allocation on cycles for three agents. These conditions can help us to dramatically reduce the number of constraints and variables in LP and then prove the following theorem:

Theorem 2. *$\frac{7}{6}$-MMS allocation of chores on cycles for three agents always exists and can be found in polynomial time. Moreover, $\frac{7}{6}$ is the optimal approximate ratio which means that for any $\alpha < \frac{7}{6}$, α-MMS allocation of chores on cycles for three agents may not exist.*

5 DISCUSSION AND CONCLUSION

For MMS allocations of chores on a tree, we propose the group-satisfied method to solve the problem on two subclasses. Whether MMS allocations of chores on general trees always exist or not is still an open problem. We believe that MMS allocations of chores on trees always exist. We also believe that the proposed group-satisfied method can be used to solve more related problems.

Another contribution of this paper is a novel method based on linear programming (LP) to characterize the optimal approximate MMS allocations without complicated combinatorial analysis. This method could potentially solve more general cases by figuring out simple and clean necessary conditions for the non-existence of α-MMS allocation.

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