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An Introduction to String Theory

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ABSTRACT: This set of notes is based on the course "Introduction to String Theory" which was taught by Prof. Kostas Skenderis in the spring of 2009 at the University of Amsterdam. We have also drawn on some ideas from the books *String Theory and M-Theory* (Becker, Becker and Schwarz), *Introduction to String Theory* (Polchinski), *String Theory in a Nutshell* (McMahon) and *Superstring Theory* (Green, Schwarz and Witten), along with the lecture notes of David Tong, sometimes word-for-word.

Contents

1. Introduction/Overview

1.1 Motivation for String Theory

Presently we understand that physics can be described by four forces: gravity, electromagnetism, the weak force, responsible for beta decays and the strong force which binds quarks into protons and neutrons. We, that is most physicists, believe that we understand all of these forces except for gravity. Here we use the word "understand" loosely, in the sense that we know what the Lagrangian is which describes how these forces induce dynamics on matter, and at least in principle we know how to calculate using these Lagrangians to make well defined predictions. But gravity we only understand partially. Clearly we understand gravity classically (meaning in the $\hbar = 0$ limit). As long as we dont ask questions about how gravity behaves at very short distances (we will call the relevant breakdown distance the Planck scale) we have no problems calculating and making predictions for gravitational interactions. Sometimes it is said that we don't understand how to fuse quantum mechanics and GR. This statement is really incorrect, though for "NY times purposes", it's fine. In fact we understand perfectly well how to include quantum mechanical effects into gravity, as long we we dont ask questions about whats going on at distances, less than the Planck length. This is not true for the other forces. That is, for the other forces we know how to include quantum effects, at all distance scales.

So, while we have a quantum mechanical understanding of gravity, we don't have a complete theory of quantum gravity. The sad part about this is that all the really interesting questions we want to ask about gravity, e.g. what's the "big bang", what happens at the singularity of black hole, are left unanswered. What is it, exactly, that goes wrong with gravity at scales shorter than the Planck length? The answer is, it is not "renormalizable". What does "renormalizable" mean? This is really a technical question which needs to be discussed within the context of quantum field theory, but we can gain a very simple intuitive understanding from classical electromagnetism. So, to begin, consider an electron in isolation. The total energy of the electron is given by

$$E_T \sim \hat{m} + \int d^3x |\vec{E}|^2 \sim \hat{m} + 4\pi \int r^2 dr \frac{e^2}{r^4}. \tag{1.1}$$

Now, this integral diverges at the lower endpoint of $r = 0$. We can reconcile this divergence by cutting it off at some scale Λ and when were done well see if can take the limit where the cut-off goes to zero. So our results for the total energy of an electron is now given by

$$E_T \sim \hat{m} + C \frac{e^2}{\Lambda}. \tag{1.2}$$

Clearly the second term dominates in the limit we are interested in. So apparently even classical electrodynamics is sick. Well not really, the point is that weve been rather sloppy. When we write $\hat{m}$ what do we mean? Naively we mean what we would call the mass of the electron which we could measure say, by looking at the deflection of a moving electron in a magnetic field. But we dont measure $\hat{m}$, we measure E_T, that is the inertial mass should include the electromagnetic self-energy. Thus what really happens is that the physical mass m is given by the sum of the bare mass $\hat{m}$ and the electrons field energy. This means that the "bare" mass is "infinite" in the limit were interested in. Note that we must make a measurement to fix the bare mass. We can not predict the electron mass.

It also means that the bare mass must cancel the field energy to many digits. That is we have two huge numbers which cancel each other extremely precisely! To understand this better, note that it is natural to assume that the cut-off should be, by dimensional analysis, the Planck length (note: this is just a guess). Which in turn means that the self field energy is of order the Planck mass. So the bare mass must have a value which cancels the field energy to within at the level of the twenty second digit! Is this some sort of miracle? This cancellation is sometimes referred to as a "hierarchy problem". This process of absorbing divergences in masses or couplings (an analogous argument can be made for the charge e) is called "renormalization".

Now what happens with gravity (GR)? What goes wrong with this type of renormalization procedure? The answer is nothing really. In fact, as mentioned above we can calculate quantum corrections to gravity quite well as long as we are at energies below the Planck mass. The problem is that when we study processes at energies of order the Planck mass we need more and more parameters to absorb the infinities that occur in the theory. In fact we need an infinite number of parameters to renormalize the theory at these scales. Remember that for each parameter that gets renormalized we must make a measurement! So GR is a pretty useless theory at these energies.

How does string theory solve this particular problem? The answer is quite simple. Because the electron (now a string) has finite extent l_p, the divergent integral is cut-off at $r = l_p$, literally, not just in the sketchy way we wrote above using dimensional analysis. We now have no need to introduce new parameters to absorb divergences since there really are none. Have we really solved anything aside from making the energy mathematically more palatable? The answer is yes, because the electron mass as well as all other parameters are now a prediction (at least in principle, a pipe dream perhaps)! String theory has only one unknown parameter, which corresponds to the string length, which presumably is of order l_p, but can be fixed by the one and only one measurement that string theory necessitates before it can be used to make predictions.

It would seem, however, that we have not solved the hierarchy problem. String

theory would seem to predict that the electron mass is huge, of order of the inverse length of the string, unless there is some tiny number which sits out in front of the integral. It turns out that string theory can do more than just cut off the integral, it can also add an additional integral which cancels off a large chunk of the first integral, leaving a more realistic result for the electron mass. This cancellation is a consequence of "supersymmetry" which, as it turns out, is necessary in some form for string theory to be mathematically consistent.

So by working with objects of finite extent, we accomplish two things. First off, all of our integrals are finite, and in principle, if string theory were completely understood, we would only need one measurement to make predictions for gravitational interactions at arbitrary distances. But also we gain enormous predictive power (at least in principle, its not quite so simple as we shall see). Indeed in the standard model of particle physics, which correctly describes all interactions at least to energies of order 200 GeV, there are 23 free parameters which need to be fixed by experiment, just as the electron mass does. String theory, however, has only one such parameter in its Lagrangian, the string length[‡]! Never forget that physics is a predictive science. The less descriptive and the more predictive our theory is, the better. In that sense, string theory has been a holy grail. We have a Lagrangian with one parameter which is fixed by experiment, and then you are done. You have a theory of everything! You could in principle explain all possible physical phenomena. To say that this a a dramatic simplification would be an understatement, but in principle at least it is correct. This opens a philosophical pandoras' box which should be discussed late at night with friends.

But wait there is more! Particle physics tells us that there a huge number of "elementary particles". Elementary particles can be split into two categories, "matter" and "force carriers". These names are misleading and should only be understood as sounds which we utter to denote a set. The matter set is composed of six quarks u, d, s, c, b, t (up, down, strange, charm, bottom, top) while the force carriers are the photon the "electroweak bosons", Z, $W^{\pm}$, the graviton g and eight gluons responsible for the strong force. There is the also the socalled "Higgs boson" for which we only have indirect evidence at this point. So, in particle physics, we have a Lagrangian which sums over all particle types and distinguishes between matter and force carriers in some way. This is a rather unpleasant situation. If we had a theory of everything all the particles and forces should be unified in some way so that we could write down

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[‡]A warning, this is misleading because to describe a theory we need to know more than just the Lagrangian, we also need to know the ground state, of which there can be many. Perhaps you have heard of the "string theory landscape"? What people are referring to is the landscape of possible ground states, or equivalently "vacau". There are people that are presently trying to enumerate the ground states of string theory.

a Lagrangian for a "master entity", and the particles mentioned would then just be different manifestations of this underlying entity. This is exactly what string theory does! The underlying entity is the string, and different excitations of the string represent different particles. Furthermore, force unification is built in as well. This is clearly a very enticing scenario.

With all this said, one should keep in mind that string theory is in some sense only in its infancy, and, as such, is nowhere near answering all the questions we hoped it would, especially regarding what happens at singularities, though it has certainly led to interesting mathematics (4-manifolds, knot theory....). It can also be said that it has taught us much about the subject of strongly coupled quantum field theories via dualities. There are those who believe that string theory in the end will either have nothing to do with nature, or will never be testable, and as such will always be relegated to be mathematics or philosophy. But, it is hard not to be awed by string theory's mathematical elegance. Indeed, the more one learns about its beauty the more one falls under its spell. To some it has become almost a religion. So, as a professor once said: "Be careful, and always remember to keep your feet on the ground lest you be swept away by the siren that is the string".

1.2 What is String Theory

Well, the answer to this question will be given by the entire manuscript. In the meantime, roughly speaking, string theory replaces point particles by strings, which can be either open or closed (depends on the particular type of particle that is being replaced by the string), whose length, or string length (denoted l_s), is approximately $10^{-33} cm$. Also, in string theory, one replaces Feynman diagrams by surfaces, and wordlines become worldsheets.
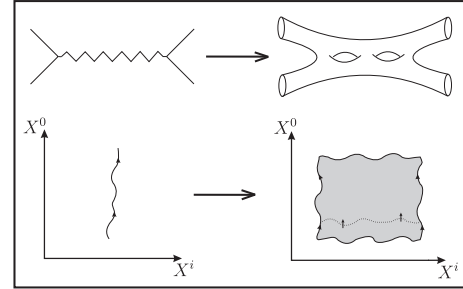
Figure 1: In string theory, Feynman diagrams are replaced by surfaces and worldlines are replaced by worldsheets.

1.2.1 Types of String Theories

The first type of string theory that will be discussed in these notes is that of bosonic string theory, where the strings correspond only to bosons. This theory, as will be shown later on, requires 26 dimensions for its spacetime.

In the mid-80's it was found that there are 5 other consistent string theories (which include fermions):

- Type I

- Type II A

- Type II B

- Heterotic $SO(32)$

- Heterotic $E_8 \times E_8$.

All of these theories use *supersymmetry*, which is a symmetry that relates elementary particles of one type of spin to another particle that differs by a half unit of spin. These two partners are called superpartners. Thus, for every boson there exists its superpartner fermion and vice versa.

For these string theories to be physically consistent they require 10 dimensions for spacetime. However, our world, as we believe, is only 4 dimensional and so one is forced to assume that these extra 6 dimensions are extremely small. Even though these extra dimensions are small we still must consider that they can affect the interactions that are taking place.

It turns out that one can show, non-perturbatively, that all 5 theories are part of the same theory, related to each other through *dualities*.

Finally, note that each of these theories can be extended to D dimensional objects, called D-branes. Note here that $D \leq 10$ because it would make no sense to speak of a 15 dimensional object living in a 10 dimensional spacetime.

1.3 Outline of the Manuscript

We begin we a discussion of the bosonic string theory. Although this type of string theory is not very realistic, one can still get a solid grasp for the type of analysis that goes on in string theory. After we have defined the bosonic string action, or Polyakov action, we will then proceed to construct invariants, or symmetries, for this action. Using Noether's theorem we will then find the conserved quantities of the theory, namely the stress energy tensor and Hamiltonian. We then quantize the bosonic string in the usual canonical fashion and calculate its mass spectrum. This, as will be shown, leads to inconsistencies with quantum physics since the mass spectrum of the bosonic string harbors ghost states - states with negative norm. However, the good news is that we can remove these ghost states at the cost of fixing the spacetime, in which the string propagates, dimension at 26. We then proceed to quantize the string theory in a different way known as light-cone gauge quantization.

The next stop in the tour is conformal field theory. We begin with an overview of the conformal group in d dimensions and then quickly restrict to the case of $d = 2$. Then conformal field theories are defined and we look at the simplifications come with

a theory that is invariant under conformal transformations. This leads us directly into radial quantization and the notion of an operator product expansion (OPE) of two operators. We end the discussion of conformal field theories by showing how the charges (or generators) of the conformal symmetry are isomorphic to the Virasor algebra. With this we are done with the standard introduction to string theory and in the remaining chapters we cover developements.

The developements include scattering theory, BRST quantization and BRST cohomology theory along with RNS superstring theories, dualities and D-branes, effective actions and M-theory and then finally matrix theory.

2. The Bosonic String Action

A string is a special case of a p-brane, where a p-brane is a p dimensional object moving through a D $(D \geq p)$ dimensional spacetime. For example:

- a 0-brane is a point particle,

- a 1-brane is a string,

- a 2-brane is a membrane .

    Before looking at strings, let's review the classical theory of 0-branes, i.e. point particles.

2.1 Classical Action for Point Particles

In classical physics, the evolution of a theory is described by its field equations. Suppose we have a non-relativistic point particle, then the field equations for $X(t)$, i.e. Newton's law $m\ddot{X}(t) = -\partial V(X(t))/\partial X(t)$, follow from extremizing the action, which is given by

$$S = \int dt L, \tag{2.1}$$

where $L = T - V = \frac{1}{2}m\dot{X}(t)^2 - V(X(t))$. We have, by setting the variation of S with respect of the field $X(t)$ equal to zero,

$$0 = \delta S$$

$$= m \int dt \frac{1}{2}(2\dot{X})\delta\dot{X} - \int dt \frac{\partial V}{\partial X}\delta X$$

$$= -m \int dt \ddot{X}\delta X + \underbrace{\text{boundary terms}}_{\text{take } = 0} - \int dt \frac{\partial V}{\partial X}\delta X$$

$$= - \int dt \left(m\ddot{X} + \frac{\partial V}{\partial X} \right) \delta X \,,$$

where in the third line we integrated the first term by parts. Since this must hold for all δX, we have that

$$m\ddot{X}(t) = -\frac{\partial V(X(t))}{\partial X(t)}, \tag{2.2}$$

which are the equations of motion (or field equations) for the field $X(t)$. These equations describe the path taken by a point particle as it moves through Galilean spacetime (remember non-relativistic). Now we will generalize this to include relativistic point particles.

2.2 Classical Action for Relativistic Point Particles

For a relativistic point particle moving through a D dimensional spacetime, the classical motion is given by geodesics on the spacetime (since here we are no longer assuming a Euclidean spacetime and therefore we must generalize the notion of a straight line path). The relativistic action is given by the integral of the infinitesimal invariant length, ds, of the particle's path, i.e.

$$S_0 = -\alpha \int ds, \tag{2.3}$$

where α is a constant, and we have also chosen units in such a way that $c = \hbar = 1$. In order to find the equations that govern the geodesic taken by a relativistic particle we, once again, set the variation of the, now relativistic, action equal to zero. Thus, the classical motion of a relativistic point particle is the path which extremizes the invariant distance, whether it minimizes or maximizes ds depends on how one chooses to parametrize the path.

Now, in order for S_0 to be dimensionless, we must have that α has units of Length^{-1} which means, in our chosen units, that α is proportional to the mass of the particle and we can, without loss of generality, take this constant of proportionality to be unity. Also, we choose to parameterize the path taken by a particle in such a way that the invariant distance, ds, is given by

$$ds^2 = -g_{\mu\nu}(X)dX^\mu dX^\nu, \tag{2.4}$$

where the metric $g_{\mu\nu}(X)$, with $\mu, \nu = 0, 1, ..., D-1$, describes the geometry of the background spacetime[†] in which the theory is defined. The minus sign in (2.4) has been introduced in order to keep the integrand of the action real for timelike geodesics, i.e. particles traveling less than $c = 1$, and thus we see that the paths followed by realistic particles, in this parameterization, are those which maximize ds[‡]. Furthermore, we will

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[†]By background spacetime, also called the target manifold of the theory, we mean the spacetime in which our theory is defined. So, for example, in classical physics one usually takes a Galilean spacetime, whose geometry is Euclidean (flat). While, in special relativity, the spacetime is taken to be a four dimensional flat manifold with Minkowski metric, also called Minkowski space. In general relativity (GR) the background spacetime is taken to be a four dimensional semi-Riemannian manifold whose geometry is determined by the field equations for the metric, Einsteins equations, and thus, in GR the background geometry is not fixed as in classical mechanics and special relativity. Therefore, one says that GR is a background independent theory since there is no a priori choice for the geometry. Finally, note that quantum field theory is defined on a Minkowski spacetime and thus is not a background independent theory, also called a fixed background theory. This leads to problems when one tries to construct quantized theories of gravity, see Lee Smolin "The Trouble With Physics".

[‡]In this parameterization the element ds is usually called the proper time. So, we see that classical paths are those which maximize the proper time.

choose our metrics, unless otherwise stated, to have signatures given by $(p = 1, q = D - 1)$ where, since we are dealing with non-degenerate metrics, i.e. all eigenvalues are non-zero[§], we calculate the signature of the metric by diagonalizing it and then simply count the number of negative eigenvalues to get the value for p and the number of positive eigenvalues to get the value for q. So, consider the metric given by

$$g_{\mu\nu}(X) = \begin{pmatrix} -g_{00}(X) & 0 & 0 & 0 & 0 \\ 0 & -g_{11}(X) & 0 & 0 & 0 \\ 0 & 0 & g_{22}(X) & 0 & 0 \\ 0 & 0 & 0 & g_{33}(X) & 0 \\ 0 & 0 & 0 & 0 & g_{44}(X) \end{pmatrix},$$

where $g_{ii}(X) > 0$ for all $i = 0, 1, ..., 3$. It should be clear that this metric has signature $(2, 3)$.

If the geometry of the background spacetime is flat then the metric becomes a constant function on this spacetime, i.e. $g_{\mu\nu}(X) = c_{\mu\nu}$ for all $\mu, \nu = 0, ..., D - 1$, where $c_{\mu\nu}$ are constants. In particular, if we choose the geometry of our background spacetime to be Minkowskian then our metric can be written as

$$g_{\mu\nu}(X) \mapsto \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.5}$$

This implies that, in a Minkowskian spacetime, the action becomes

$$S_0 = -m \int \sqrt{-(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2}.$$

Note that the signature of the above metric (2.5) is $(1, 3)$.

If we choose to parameterize the particle's path $X^\mu(\tau)$, also called the worldline of the particle, by some real parameter τ, then we can rewrite (2.4) as

$$-g_{\mu\nu}(X)dX^\mu dX^\nu = -g_{\mu\nu}(X)\frac{dX^\mu(\tau)}{d\tau}\frac{dX^\nu(\tau)}{d\tau}d\tau^2, \tag{2.6}$$

which gives for the action,

$$S_0 = -m \int d\tau \sqrt{-g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu}, \tag{2.7}$$

where $\dot{X}^\mu \equiv \frac{dX^\mu(\tau)}{d\tau}$.

[§]They are also smooth and symmetric but that does not concern us here.

An important property of the action is that it is invariant under which choice of parameterization is made. This makes sense because the invariant length ds between two points on a particle's worldline should not depend on how the path is parameterized.

Proposition 2.1 *The action (2.7) remains unchanged if we replace the parameter τ by another parameter $\tau' = f(\tau)$, where f is monotonic.*

Proof By making a change of parameter, or reparameterization, $\tau \mapsto \tau' = f(\tau)$ we have that

$$d\tau \mapsto d\tau' = \frac{\partial f}{\partial \tau} d\tau,$$

and so

$$\frac{dX^\mu(\tau')}{d\tau} = \frac{dX^\mu(\tau')}{d\tau'}\frac{d\tau'}{d\tau} = \frac{dX^\mu(\tau')}{d\tau'}\frac{\partial f(\tau)}{\partial \tau}.$$

Now, plugging τ' into (2.7) gives

$$S_0' = -m \int d\tau' \sqrt{g_{\mu\nu}(X)\frac{dX^\mu(\tau')}{d\tau'}\frac{dX^\nu(\tau')}{d\tau'}}$$

$$= -m \int d\tau' \sqrt{\frac{dX^\mu(\tau')}{d\tau'}\frac{dX_\mu(\tau')}{d\tau'}}$$

$$= -m \int \frac{\partial f}{\partial \tau} d\tau \sqrt{\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}\left(\frac{\partial f}{\partial \tau}\right)^{-2}}$$

$$= -m \int \left(\frac{\partial f}{\partial \tau}\right)\left(\frac{\partial f}{\partial \tau}\right)^{-1} d\tau \sqrt{\frac{dX^\mu(\tau)}{d\tau}\frac{dX_\mu(\tau)}{d\tau}}$$

$$= -m \int d\tau \sqrt{\frac{dX^\mu(\tau)}{d\tau}\frac{dX_\mu(\tau)}{d\tau}}$$

$$= -m \int d\tau \sqrt{-g_{\mu\nu}\frac{dX^\mu(\tau)}{d\tau}\frac{dX_\mu(\tau)}{d\tau}}$$

$$= S_0 \,.$$

Q.E.D.

Thus, one is at liberty to choose an appropriate parameterization in order to simplify the action and thereby simplify the equations of motion which result from variation. This parameterization freedom will now be used to simplify the action given in (2.7).

Since the square root function is a non-linear function, we would like to construct another action which does not include a square root in its argument. Also, what should we do if we want to consider massless particles? According to (2.7) the action of a massless particle is equal to zero, but does this make sense? The claim is that instead of the original action (2.7) we can add an auxiliary field to it and thereby construct an equivalent action which is simpler in nature. So, consider the equivalent action given by

$$\tilde{S}_0 = \frac{1}{2} \int d\tau \left(e(\tau)^{-1} \dot{X}^2 - m^2 e(\tau) \right), \tag{2.8}$$

where $\dot{X}^2 \equiv g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu$ and $e(\tau)$ is some auxiliary field. Before showing that this action really is equivalent to (2.7) we will first note that this action is the simplification we were looking for since there is no longer a square root, and the action no longer becomes zero for massless particles. Now, to see that this new action is equivalent with (2.7), first consider the variation of $\tilde{S}_0$ with respect to the field $e(\tau)$,

$$\delta\tilde{S}_0 = \delta \left(\frac{1}{2} \int d\tau \left(e^{-1} \dot{X}^2 - m^2 e \right) \right)$$

$$= \frac{1}{2} \int d\tau \left(-\frac{1}{e^2} \dot{X}^2 \delta e - m^2 \delta e \right)$$

$$= \frac{1}{2} \int d\tau \frac{\delta e}{e^2} \left(-\dot{X}^2 - m^2 e^2 \right).$$

By setting $\delta\tilde{S}_0 = 0$ we get the field equations for $e(\tau)$,

$$e^2 = -\frac{\dot{X}^2}{m^2} \quad \Longrightarrow \quad e = \sqrt{\frac{-\dot{X}^2}{m^2}}. \tag{2.9}$$

Now, plugging the field equation for the auxiliary field back into the action, $\tilde{S}_0$, we have

$$\tilde{S}_0 = \frac{1}{2} \int d\tau \left[\left(-\frac{\dot{X}^2}{m^2} \right)^{-1/2} \dot{X}^2 - m^2 \left(-\frac{\dot{X}^2}{m^2} \right)^{1/2} \right]$$

$$= \frac{1}{2} \int d\tau \left[\left(-\frac{\dot{X}^2}{m^2} \right)^{-1/2} \left(\dot{X}^2 - m^2 \left(\frac{-\dot{X}^2}{m^2} \right) \right) \right]$$

$$= \frac{1}{2} \int d\tau \left(-\frac{\dot{X}^2}{m^2} \right)^{-1/2} \left(\dot{X}^2 + \dot{X}^2 \right)$$

$$= \frac{1}{2} \int d\tau \left(-\frac{\dot{X}^2}{m^2} \right)^{-1/2} \left(2\dot{X}^2 \right)$$

$$= \frac{1}{2} \int d\tau \left(-\frac{\dot{X}^2}{m^2} \right)^{-1/2} \left(2(-\dot{X}^2) \right) (-1)$$

$$= \frac{1}{2} \int d\tau (-2m) \left(-\dot{X}^2 \right)^{-1/2} \left(-\dot{X}^2 \right)$$

$$= -m \int d\tau \left(-\dot{X}^2 \right)^{-1/2} \left(-\dot{X}^2 \right)$$

$$= -m \int d\tau (-\dot{X}^2)^{1/2}$$

$$= -m \int d\tau \sqrt{ -g_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} }$$

$$= S_0 \, .$$

So, if the field equations hold for $e(\tau)$ then we have that $\tilde{S}_0$ is equivalent to S_0.

2.2.1 Reparameterization Invariance of $\tilde{S}_0$

Another nice property of $\tilde{S}_0$ is that it is invariant under a reparameterization (diffeo-morphism) of τ. To see this we first need to see how the fields $X^\mu(\tau)$ and $e(\tau)$ vary under an infinitesimal change of parameterization $\tau \mapsto \tau' = \tau - \xi(\tau)$. Now, since the fields $X^\mu(\tau)$ are scalar fields, under a change of parameter they transform according to

$$X^{\mu'}(\tau') = X^\mu(\tau),$$

and so, simply writing the above again,

$$X^{\mu'}(\tau') = X^{\mu'}(\tau - \xi(\tau)) = X^\mu(\tau). \tag{2.10}$$

Expanding the middle term gives us

$$X^{\mu'}(\tau) - \xi(\tau)\dot{X}^\mu(\tau) = X^\mu(\tau), \tag{2.11}$$

or that

$$\underbrace{X^{\mu'}(\tau) - X^\mu(\tau)}_{\equiv \delta X^\mu} = \xi(\tau)\dot{X}^\mu. \tag{2.12}$$

This implies that the variation of the field $X^\mu(\tau)$, under a change of parameter, is given by $\delta X^\mu = \xi \cdot X^\mu$. Also, under a reparameterization, the auxiliary field transforms according to

$$e'(\tau')d\tau' = e(\tau)d\tau. \qquad (2.13)$$

Thus,

$$e'(\tau')d\tau' = e'(\tau - \xi)(d\tau - \dot{\xi}d\tau)$$

$$= \left(e'(\tau) - \xi\partial_\tau e(\tau) + \mathcal{O}(\xi^2)\right)(d\tau - \dot{\xi}d\tau)$$

$$= e'd\tau - e'\dot{\xi}d\tau - \xi\dot{e}d\tau + \mathcal{O}(\xi^2)$$

$$= e'(\tau)d\tau - \underbrace{\left(\xi\dot{e}(\tau) + e'(\tau)\dot{\xi}\right)}_{\equiv \frac{d}{d\tau}(\xi e)} d\tau, \qquad (2.14)$$

where in the last line we have replaced by $e'\dot{\xi}$ by $e\dot{\xi}$ since they are equal up to second order in ξ, which we drop anyway. Now, equating (2.14) to $e(\tau)d\tau$ we get that

$$e(\tau) = e'(\tau) - \frac{d}{dt}\left(\xi(\tau)e(\tau)\right)$$

$$\Rightarrow \frac{d}{dt}\left(\xi(\tau)e(\tau)\right) = \underbrace{e'(\tau) - e(\tau)}_{\equiv \delta e},$$

or that, under a reparameterization, the auxiliary field varies as

$$\delta e(\tau) = \frac{d}{dt}\left(\xi(\tau)e(\tau)\right). \qquad (2.15)$$

With these results we are now in a position to show that the action $\tilde{S}_0$ is invariant under a reparametrization. This will be shown for the case when the background spacetime metric is flat, even though it is not hard to generalize to the non-flat case. So, to begin with, we have that the variation of the action under a change in both the fields, $X^\mu(\tau)$ and $e(\tau)$, is given by

$$\delta\tilde{S}_0 = \frac{1}{2}\int d\tau\left(-\frac{\delta e}{e^2}\dot{X}^2 + \frac{2}{e}\dot{X}\delta\dot{X} - m^2\delta e\right).$$

From the above expression for δX^μ we have that

$$\delta\dot{X}\mu = \frac{d}{d\tau}\delta X_\mu = \dot{\xi}\dot{X}_\mu + \xi\ddot{X}_\mu.$$

Plugging this back into $\delta\tilde{S}_0$, along with the expression for δe, we have that

$$\delta\tilde{S}_0 = \frac{1}{2}\int d\tau \left[\frac{2\dot{X}^\mu}{e}(\dot{\xi}\dot{X}_\mu + \xi\ddot{X}_\mu) - \frac{\dot{X}^2}{e^2}(\dot{\xi}e + e\dot{\xi}) - m^2\frac{d(\xi e)}{d\tau}\right].$$

Now, the last term can be dropped because it is a total derivative of something times $\xi(\tau)$, and if we assume that that the variation of $e(\tau)$ vanishes at the τ boundary then $\xi(\tau)$ must also vanish at the τ boundary. Thus, $\xi(\tau)e(\tau) = 0$ at the τ boundary. The remaining terms can be written as

$$\delta\tilde{S}_0 = \frac{1}{2}\int d\tau \frac{d}{d\tau}\left(\frac{\xi}{e}\dot{X}^2\right),$$

which is also the integral of a total derivative of something else times $\xi(\tau)$, and so can be dropped. To recap, we have shown that under a reparameterization the variation in the action vanishes, i.e. $\delta\tilde{S}_0 = 0$, and so this action is invariant under a reparameterization, proving the claim.

This invariance can be used to set the auxiliary field equal to unity, see problem 2.3, thereby simplifying the action. However, one must retain the field equations for $e(\tau)$, (2.9), in order to not lose any information. Also, note that with $e(\tau) = 1$ we have that

$$\frac{\delta}{\delta e}(S_0)\bigg|_{e(\tau)=1} = -\frac{1}{2}(\dot{X}^2 + m^2). \tag{2.16}$$

This implies that

$$\dot{X}^2 + m^2 = 0, \tag{2.17}$$

which is the position representation of the mass-shell relation in relativistic mechanics.

2.2.2 Canonical Momenta

The canonical momentum, conjugate to the field $X^\mu(\tau)$, is defined by

$$P^\mu(\tau) = \frac{\partial L}{\partial \dot{X}^\mu}. \tag{2.18}$$

For example, in terms of the previous Lagrangian (2.7), the canonical momentum is given by

$$P^\mu(\tau) = \dot{X}^\mu(\tau).$$

Using this, we see that the vanishing of the functional derivative of the action $\tilde{S}_0$, with respect to $e(\tau)$ evaluated at $e(\tau) = 1$ (see (2.16) and (2.17)), is nothing more than the mass-shell equation for a particle of mass m,

$$P^\mu P_\mu + m^2 = 0. \tag{2.19}$$

2.2.3 Varying $\tilde{S}_0$ in an Arbitrary Background (Geodesic Equation)

If we choose the parameter τ in such a way that the auxiliary field $e(\tau)$ takes the value $e(\tau) = 1$, then the action $\tilde{S}_0$ becomes

$$\tilde{S}_0 = \frac{1}{2} \int d\tau \Big(g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu - m^2 \Big). \tag{2.20}$$

Now, if we assume that the metric is not flat, and thus depends on its spacetime position, then varying $\tilde{S}_0$ with respect to $X^\mu(\tau)$ results in

$$\delta\tilde{S}_0 = \frac{1}{2} \int d\tau \Big(2g_{\mu\nu}(X)\delta\dot{X}^\mu \dot{X}^\nu + \partial_k g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu \delta X^k \Big)$$

$$= \frac{1}{2} \int d\tau \Big(- 2\dot{X}^k \partial_k g_{\mu\nu}(X)\delta X^\mu \dot{X}^\nu - 2g_{\mu\nu}(X)\delta X^\mu \ddot{X}^\nu + \delta X^k \partial_k g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu \Big)$$

$$= \frac{1}{2} \int d\tau \Big(- 2\ddot{X}^\nu g_{\mu\nu}(X) - 2\partial_k g_{\mu\nu}(X)\dot{X}^k \dot{X}^\nu + \partial_\mu g_{k\nu}(X)\dot{X}^k \dot{X}^\nu \Big)\delta X^\mu .$$

Setting this variation equal to zero gives us the field equations for $X^\mu(\tau)$ in an arbitrary background, namely

$$-2\ddot{X}^\nu g_{\mu\nu}(X) - 2\partial_k g_{\mu\nu}(X)\dot{X}^k \dot{X}^\nu + \partial_\mu g_{k\nu}(X)\dot{X}^k \dot{X}^\nu = 0, \tag{2.21}$$

which can be rewritten as

$$\ddot{X}^\mu + \Gamma^\mu_{kl}\dot{X}^k \dot{X}^l = 0, \tag{2.22}$$

where Γ^μ_{kl} are the Christoffel symbols. These are the geodesic equations describing the motion of a free particle moving through a spacetime with an arbitrary background geometry, i.e. this is the general equation of the motion for a relativistic free particle. Note that the particle's motion is completely determined by the Christoffel symbols which, in turn, only depend on the geometry of the background spacetime in which the particle is moving. Thus, the motion of the free particle is completely determined by the geometry of the spacetime in which it is moving. This is the gist of Einstein's general theory of relativity.

2.3 Generalization to p-Branes

We now want to generalize the notion of an action for a point particle (0-brane), to an action for a p-brane. The generalization of $S_0 = -m \int ds$ to a p-brane in a $D (\geq p)$ dimensional background spacetime is given by

$$S_p = -T_p \int d\mu_p, \tag{2.23}$$

where T_p is the p-brane tension, which has units of mass/vol[‡] , and $d\mu_p$ is the $(p+1)$ dimensional volume element given by

$$d\mu_p = \sqrt{-\det(G_{\alpha\beta}(X))}d^{p+1}\sigma. \tag{2.24}$$

Where $G_{\alpha\beta}$ is the induced metric on the worldsurface, or worldsheet for $p = 1$, given by

$$G_{\alpha\beta}(X) = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}g_{\mu\nu}(X) \qquad \alpha, \beta = 0, 1, ..., p , \tag{2.25}$$

with $\sigma^0 \equiv \tau$ while $\sigma^1, \sigma^2, ..., \sigma^p$ are the p spacelike coordinates[†] for the $p+1$ dimensional worldsurface mapped out by the p-brane in the background spacetime. The metric $G_{\alpha\beta}$ arises, or is induced, from the embedding of the string into the D dimensional background spacetime. Also, note that the induced metric measures distances on the worldsheet while the metric $g_{\mu\nu}$ measures distances on the background spacetime. We can show, see problem 2.5 of Becker, Becker and Schwarz "String Theory and M-Theory", that this action, (6.1), is also invariant under a reparameterization of τ. We will now specialize the p-brane action to the case where $p = 1$, i.e the string action.

2.3.1 The String Action

This is a $(p = 1)$-brane action, describing a string propagating through a D dimensional spacetime. We will parameterize the worldsheet of the string, which is the two dimensional extension of the worldline for a particle, by the two coordinates $\sigma^0 \equiv \tau$ and $\sigma^1 \equiv \sigma$, with τ being time-like and σ being space-like. The embedding of the string into the D dimensional background spacetime is given by the functions (or fields) $X^\mu(\tau, \sigma)$, see figure 2.

Figure 2: Embedding of a string into a background spacetime. Note here that the coordinates, $X^\mu(\tau, \sigma)$, are periodic in some directions. Thus giving a closed string.

Note that if we assume σ is periodic, then the embedding gives a closed string in the spacetime. Also, one should realize that the fields $X^\mu(\tau, \sigma)$, which are parameterized by the worldsheet coordinates, tell how the string propagates and oscillates through the background spacetime, and this propagation defines the worldsheet just as before with the fields $X^\mu(\tau)$, parameterized by the worldline coordinate τ, that described the propagation of the particle through spacetime where the propagation defined the worldline of the particle.

[‡]Note that if T_p has units of mass/vol, then the action S_p is dimensionless, as it should be, since the measure $d\mu_p$ has units vol·length.

[†]Here $G_{\alpha\beta}$ is the metric on the $p + 1$ dimensional surface which is mapped out by the p-brane as it moves through spacetime, while $g_{\mu\nu}$ is the metric on the D dimensional background spacetime. For example, in the case of a string, $p = 1$, $G_{\alpha\beta}$ is the metric on the worldsheet. Note that for the point particle the induced worldline metric is given by (since the worldline is one dimensional there is only one component for the induced metric)

$$-\ 20\ -$$

$$G_{\tau\tau} = g_{\mu\nu}\frac{\partial X^\mu}{\partial\tau}\frac{\partial X^\nu}{\partial\tau},$$

which is none other than the expression for ds^2. Also, mathematically speaking, the induced metric $G_{\alpha\beta}$ is the pull-back of the metric on the background spacetime, $g_{\mu\nu}$.

Now, if we assume that our background spacetime is Minkowski, then we have that

$$G_{00} = \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau} \eta_{\mu\nu} \equiv \dot{X}^2,$$

$$G_{11} = \frac{\partial X^\mu}{\partial \sigma} \frac{\partial X^\nu}{\partial \sigma} \eta_{\mu\nu} \equiv X'^2,$$

$$G_{10} = G_{01} = \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} \eta_{\mu\nu},$$

and thus, we have that

$$G_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & X'^2 \end{pmatrix}. \tag{2.26}$$

From the form of the induced metric (2.26) we see that, in a Minkowski background,

$$det(G_{\alpha\beta}) = (\dot{X}^2)(X'^2) - (\dot{X} \cdot X')^2. \tag{2.27}$$

So our previous action reduces to

$$S_{NG} = -T \int d\tau d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X}^2)(X'^2)}, \tag{2.28}$$

which is known as the Nambu-Goto action. This action can be interpreted as giving the area of the worldsheet mapped out by the string in spacetime. Since the equations of motion follow from minimizing the above action, one can think of the equations of motion for the string as the worldsheet of smallest area mapped out by the string in spacetime.

Now, in order to get rid of the square root, we can introduce an auxiliary field $h_{\alpha\beta}(\tau, \sigma)$ (this really is another metric living on the worldsheet, which is different from the induced metric $G_{\alpha\beta}$)[‡], just like before with the auxiliary field $e(\tau)$. The resulting action is called the string sigma-model, or Polyakov action, and it is given by

$$S_\sigma = -\frac{T}{2} \int d\tau d\sigma \sqrt{-h} \, h^{\alpha\beta} \frac{\partial X^\mu}{\partial \alpha} \frac{\partial X^\nu}{\partial \beta} g_{\mu\nu}, \tag{2.29}$$

where $h \equiv det(h_{\alpha\beta})$. Note that the above expression holds for a general background spacetime since we have not reduced $G_{\alpha\beta}$ for a Minkowski spacetime. Also note that at the classical level, the Polyakov action is equivalent to the Nambu-Goto action, while being better suited for quantization.

Proposition 2.2 *The Polyakov action S_σ is equivalent to the Nambu-Goto action S_{NG}.*

[‡]So, the worldsheet has an induced metric, $G_{\alpha\beta}$, from being embedded into the background spacetime and also the intrinsic metric $h_{\alpha\beta}$, which we put in by hand.

Proof First, note that varying any action with respect to a metric yields the stress energy tensor $T_{\alpha\beta}$, i.e.

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_\sigma}{\delta h^{\alpha\beta}}\,. \tag{2.30}$$

Now, the equations of motion for the field $h^{\alpha\beta}$ follow from setting the variation in the action S_σ with respect to $h^{\alpha\beta}$ equal to zero, $\delta S_\sigma = 0$. When we vary S_σ w.r.t. $h^{\alpha\beta}$ and set it equal to zero we have that

$$\delta S_\sigma \equiv \int \frac{\delta S_\sigma}{\delta h^{\alpha\beta}}\delta h^{\alpha\beta}$$

$$= \underbrace{-\frac{T}{2}\int d\tau d\sigma \sqrt{-h}\,\delta h^{\alpha\beta}T_{\alpha\beta} = 0}_{\text{follows from (2.30)}}\,,$$

which holds if and only if $T_{\alpha\beta} = 0$. So, computing $T_{\alpha\beta}$ and setting the result equal to zero will give the equations of motion for the field $h^{\alpha\beta}$.

Now to compute $T_{\alpha\beta}$ we need to know what δh is. The claim is that

$$\delta h \equiv \delta\big(det(h_{\alpha\beta})\big) = -hh_{\alpha\beta}\delta h^{\alpha\beta}, \tag{2.31}$$

and to see this note that $h = det(h_{\alpha\beta}) = \frac{1}{n!}\epsilon^{\alpha_1\cdots\alpha_n}\epsilon^{\beta_1\cdots\beta_n}h_{\alpha_1\beta_1}\cdots h_{\alpha_n\beta_n}$. So, we have that

$$\delta h = \frac{1}{n!}\epsilon^{\alpha_1\cdots\alpha_n}\epsilon^{\beta_1\cdots\beta_n}h_{\alpha_1\beta_1}\cdots h_{\alpha_n\beta_n} + \frac{1}{(n-1)!}\epsilon^{\alpha_1\cdots\alpha_n}\epsilon^{\beta_1\cdots\beta_n}h_{\alpha_2\beta_2}\cdots h_{\alpha_n\beta_n} = h^{\alpha_1\beta_1}h,$$

which implies that

$$\delta h = h^{\alpha\beta}\delta h_{\alpha\beta}h,$$

and also that

$$\delta h = -h_{\alpha\beta}\delta h^{\alpha\beta}h.$$

Putting these two together gives us,

$$\delta h^{\alpha\beta}h_{\alpha\beta} + h^{\alpha\beta}\delta h_{\alpha\beta} = 0, \tag{2.32}$$

which shows the previous claim, i.e. that $\delta h \equiv \delta\big(det(h_{\alpha\beta})\big) = -hh_{\alpha\beta}\delta h^{\alpha\beta}$.

Now, this gives us that

$$\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h}\delta h^{\alpha\beta}h_{\alpha\beta}, \tag{2.33}$$

and so when we vary the Polyakov action we get

$$\delta S_\sigma = -T\int d\tau d\sigma \sqrt{-h}\,\delta h^{\alpha\beta}\Big(-\frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X + \partial_\alpha X\cdot\partial_\beta X\Big) = 0, \tag{2.34}$$

which gives for the field equations of $h_{\alpha\beta}$,

$$\left(T_{\alpha\beta} \equiv \right) - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X + \partial_\alpha X \cdot \partial_\beta X = 0. \tag{2.35}$$

Thus, we have that

$$\frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X = \underbrace{\partial_\alpha X \cdot \partial_\beta X}_{G_{\alpha\beta}}, \tag{2.36}$$

and taking the square root of minus the determinant of both sides gives that

$$\frac{1}{2}\sqrt{-h}\ h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X = \sqrt{-det(G_{\alpha\beta})}, \tag{2.37}$$

which shows that S_σ is equivalent, classically, to S_{NG}. $\mathcal{Q.E.D.}$

In the next chapter we will look at symmetries, both global and local, that our bosonic string theory possesses along with the field equations for the field $X^\mu(\tau, \sigma)$ and the solutions to these field equations for different boundary conditions, open and closed strings.

2.4 Exercises

Problem 1

Consider a point particle of mass m and charge e moving in a flat background space-time under the influence of an electromagnetic field $A_\mu(X)$. If $X^\mu(\tau)$ is the worldline of the particle, then the dynamics of the point particle in this system can be described by the action:

$$S = -\frac{1}{4}\int d^4X\, F_{\mu\nu}F^{\mu\nu} - m\int \sqrt{-\dot{X}^2}\,d\tau + e\int A_\mu \dot{X}^\mu d\tau, \qquad (2.38)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\dot{X}^2 = \eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu$, with $\eta_{\mu\nu}$ the flat Minkowski metric.

a) Find the equation of motion for $X^\mu(\tau)$.

b) Show that the action (2.38) is invariant under gauge transformations of the electromagnetic field:

$$A_\mu(X) \to A_\mu(X) + \partial_\mu \Lambda(X), \qquad (2.39)$$

where $\Lambda(X)$ is any scalar function which vanishes at infinity.

Problem 2

A relativistic quantum theory that includes gravity, involves three fundamental constants: the speed of light c, Planck's constant $\hbar$ and Newton's gravitational constant G.

a) Determine the mass $[M]$, length $[L]$ and time $[T]$ dimensions of each of these constants using dimensional analysis (namely, using physical relations which involve these quantities, such as Newton's law of gravity).

b) Construct the combination of these constants which has dimension $[L]$, and find its numerical value. This is called the *Planck length* and is roughly the length scale at which we expect the effects of quantum gravity to become important.

c) Find the combination which has dimensions of mass, and thus compute the *Planck mass*.

Problem 3

Consider the action for a point particle:

$$S = -m\int \sqrt{-g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu}\,d\tau. \qquad (2.40)$$

As we saw in this chapter it can be equivalently written as:

$$\tilde{S}_0 = \frac{1}{2}\int d\tau \left(e^{-1}\dot{X}^2 - m^2 e\right), \qquad (2.41)$$

where $e(\tau)$ is an auxiliary field and $\dot{X}^2 = g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu$. The action $\tilde{S}$ is invariant under reparameterizations:

$$\tau \to \tau' = f(\tau), \tag{2.42}$$

provided that we transform $e(\tau)$ appropriately. Show that it is possible to use this reparameterization invariance to choose a gauge in which $e(\tau) = 1$.

Problem 4

Show that the p-brane Polyakov action with the addition of a cosmological constant Λ_p,

$$S_\sigma = -\frac{T_p}{2} \int d^p\tau d\sigma \, \sqrt{-h} \, h^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X + \Lambda_p \int d^p\tau d\sigma \sqrt{-h}. \tag{2.43}$$

is equivalent to the "Nambu–Goto action"

$$S_{NG} = -T_p \int d^p\tau d\sigma \sqrt{-\det \partial_\alpha X \cdot \partial_\beta X}, \tag{2.44}$$

by choosing the "cosmological constant" Λ_p appropriately. (Hint: Solve the equations of motion for the intrinsic metric $h_{\alpha\beta}$.)

3. Symmetries and Field Equations of the Bosonic String

Last week we saw that the action which describes a string propagating in a D dimensional spacetime, with given metric $g_{\mu\nu}$, is given by

$$S_\sigma = -\frac{T}{2} \int d\tau d\sigma \sqrt{-h} \; h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}. \quad (3.1)$$

One advantage of the action is that, usually, it makes it easier to see whether the theory is invariant under a certain transformation or not.
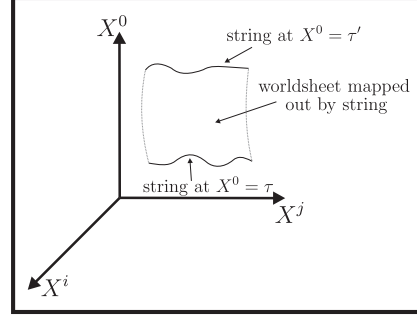
3.1 Global Symmetries of the Bosonic String Theory Worldsheet

A global transformation in some spacetime is a transformation whose parameter(s) do not depend on where in the spacetime the transformation is being performed, i.e the derivative of any parameter with respect to any of the spacetime coordinates vanishes. A local transformation in some spacetime does however, depend on where the transformation is begin performed in the spacetime. Examples of global transformations are rotations about an axis by some parameter θ, translations, etc. Whereas an example of a local transformation would be a rotation where the parameter $\theta(X^\mu)$ does depend on where in spacetime the rotation is being performed. Also, one should note that invariance of a theory under global transformations leads to conserved currents and charges via Noether's theorem, while invariance under local transformations (or gauge transformations) is a sign of absent degrees of freedom in your theory. We will first discuss global transformations, namely the Poincaré transformations, and then, in the next section, we will look at local transformations.

If we take our background spacetime to be Minkowskian then our bosonic string theory[‡], which lives in this space, should have the same symmetries as Minkowski space and, in particular, our theory should be invariant under the Poincaré group.

Poincaré Transformations

These are global transformations of the form,

$$\delta X^\mu(\tau, \sigma) = a^\mu{}_\nu X^\nu(\tau, \sigma) + b^\mu, \quad (3.2)$$

$$\delta h_{\alpha\beta}(\tau, \sigma) = 0, \quad (3.3)$$

[‡]Note that here we are really saying that the string worldsheet should have the same symmetries as the background spacetime, namely the worldsheet should have the same symmetries as a Minkowski space.

where fields $X^\mu(\tau,\sigma)$ are defined on the worldsheet, as is $h^{\alpha\beta}(\tau,\sigma)$ and the $a^\mu{}_\nu$, with both indices lowered, is antisymmetric, i.e. $a_{\mu\nu} = -a_{\nu\mu}$. This is indeed a global symmetry in the eyes of the worldsheet since the transformations do not depend on the worldsheet coordinates, σ and τ. We will now show that the $a^\mu{}_\nu$ generate the Lorentz transformations.

According to Einstein's theory of relativity, the speed of light is the same in all inertial frames, i.e. all inertial observers measure the same value for the speed of light. Thus, if (t, X^i) is the spacetime position of a light ray in one inertial frame and (t', X'^i) in another, then the relation between the two is given by

$$\eta_{\mu\nu}X^\mu X^\nu = -c^2 t^2 + X_i X^i$$

$$= -c^2 t'^2 + X'_i X'^i$$

$$= \eta_{\mu\nu}X'^\mu X'^\nu \, .$$

The linear transformations, denoted by Λ, which preserve this relation are called Lorentz transformations

$$X'^\mu = \Lambda^\mu{}_\nu X^\nu. \tag{3.4}$$

Infinitesimally, we have that the above transformation is given by

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + a^\mu{}_\nu. \tag{3.5}$$

Note that we still have not shown that the $a^\mu{}_\nu$ in (3.5) is equal to the $a^\mu{}_\nu$ in (3.2), we are simply choosing the notation here because we are about to show that they are the same. The infinitesimal form of the Lorentz transformation says that

$$X'^\mu = X^\mu + a^\mu{}_\nu X^\nu,$$

which implies that the variation of X^μ, under the Lorentz transformation, is given by

$$\delta X^\mu = a^\mu{}_\nu X^\nu. \tag{3.6}$$

Now, if we impose that under a Lorentz transformation the spacetime interval vanishes,

$$\delta\Big|_{L.T.} (\eta_{\mu\nu}X^\mu X^\nu) = 0,$$

then we have that

$$\delta\Big|_{L.T.}(\eta_{\mu\nu}X^\mu X^\nu) = 2\eta_{\mu\nu}(\delta X^\mu)X^\nu \tag{3.7}$$

$$= 2\eta_{\mu\nu}(a^\mu{}_k X^k)X^\nu$$

$$= 2a_{k\nu}X^k X^\nu = 0.$$

The most general solution to this is to have $a_{\mu\nu} = -a_{\nu\mu}$, and thus we have that the two $a^\mu{}_\nu$ are equivalent to each other as proposed. Also, note that the first line of the above equation, (3.7), is obtained by noting that $\eta_{\mu\nu}$ is symmetric in its indices, i.e.

$$\delta(\eta_{\mu\nu}X^\mu X^\nu) = \eta_{\mu\nu}(\delta X^\mu)X^\nu + \eta_{\mu\nu}X^\mu(\delta X^\nu) = 2\eta_{\mu\nu}(\delta X^\mu)X^\nu,$$

with the last equality above coming from exchanging μ and ν and noting that $\eta_{\mu\nu} = \eta_{\nu\mu}$.

We will now discuss some examples of Poincaré transformations, namely the rotations and boosts:

1. For the first example of a Lorentz transformation, we will discuss that of a rotation around the X^3 axis by an angle θ. For a four dimensional spacetime, the rotation acts on the remaining two spatial coordinates as (finite transformation)

$$X'^1 = \cos(\theta)X^1 + \sin(\theta)X^2,$$

$$X'^2 = -\sin(\theta)X^1 + \cos(\theta)X^2.$$

Thus, for an infinitesimal θ we have, using the small angle approximations $\cos(\theta) \mapsto 1$ and $\sin\theta \mapsto \theta$, the transformations

$$X'^1 = X^1 + \theta X^2,$$

$$X'^2 = -\theta X^1 + X^2.$$

The infinitesimal transformations give us that $\delta X^1 = \theta X^2$ and $\delta X^2 = -\theta X^1$, which shows that $a^1{}_2 = \theta$ and $a^2{}_1 = -\theta$, while all other $a^\mu{}_\nu$ are equal to zero. Now, to see that $a_{\mu\nu} = -a_{\nu\mu}$ for this particular Lorentz transformation consider,

$$a_{12} = \eta_{1\lambda}a^\lambda{}_2 = \eta_{10}a^0{}_2 + \eta_{11}a^1{}_2 + \eta_{12}a^2{}_2 + \eta_{13}a^3{}_2 = \eta_{11}a^1{}_2 = \theta$$

$$a_{21} = \eta_{2\lambda}a^\lambda{}_1 = \eta_{20}a^0{}_1 + \eta_{21}a^1{}_1 + \eta_{22}a^2{}_1 + \eta_{23}a^3{}_1 = \eta_{22}a^2{}_1 = -\theta,$$

and so $a_{\mu\nu} = -a_{\nu\mu}$.

2. Now we will look at a boost in the X^0 and X^1 directions by φ. The finite transformations are given by

$$X'^0 = \cosh(\varphi)X^0 + \sinh(\varphi)X^1,$$

$$X'^1 = \sinh(\varphi)X^0 + \cosh(\varphi)X^1,$$

and thus for an infinitesimal φ we have, using the small angle approximations $\cosh(\varphi) \mapsto 1$ and $\sinh(\varphi) \mapsto \varphi$, the transformations

$$X'^0 = X^0 + \varphi X^1,$$

$$X'^1 = \varphi X^0 + X^1.$$

The infinitesimal transformations give us that $\delta X^0 = \varphi X^1$ and $\delta X^1 = \varphi X^0$, which shows that $a^0{}_1 = \varphi$ and $a^1{}_0 = \varphi$, while all other $a^\mu{}_\nu$ are equal to zero. Finally, to show that the a^μ_ν is antisymmetric consider,

$$a_{01} = \eta_{0\lambda}a^\lambda{}_1 = \eta_{00}a^0{}_1 + \eta_{01}a^1{}_1 + \eta_{02}a^2{}_1 + \eta_{03}a^3{}_1 = \eta_{00}a^0{}_1 = -\varphi$$

$$a_{10} = \eta_{1\lambda}a^\lambda{}_0 = \eta_{10}a^0{}_0 + \eta_{11}a^1{}_0 + \eta_{12}a^2{}_0 + \eta_{13}a^3{}_0 = \eta_{11}a^1{}_0 = \varphi,$$

and so $a_{\mu\nu} = -a_{\nu\mu}$.

Now that we have seen some concrete examples of Poincaré transformations, the next question to ask is whether our Polyakov action is invariant under them. This would then imply that our bosonic string theory is Poincaré invariant since the $p = 1$ brane action is equivalent with the Polyakov action. To see that the Polyakov action is indeed invariant under Poincaré transformations consider the following,

$$\delta S_\sigma = -T \int d\tau d\sigma \sqrt{-h}\, h^{\alpha\beta}\partial_\alpha(\delta X^\mu)\partial_\beta X^\nu g_{\mu\nu},$$

where we have used the fact that $h^{\alpha\beta}$ is invariant under the transformation, i.e. $\delta h^{\alpha\beta} = 0$, and symmetry of the metric in its indices. Plugging in for the transformation on the coordinates, $\delta X^\mu = a^\mu{}_k X^k + b^\mu$, we get that

$$\delta S_\sigma = -T \int d\tau d\sigma \sqrt{-h}\, h^{\alpha\beta}\partial_\alpha\left(a^\mu{}_k X^k + b^\mu\right)\partial_\beta X^\nu g_{\mu\nu}.$$

This can be further simplified by noting that $a^\mu{}_k$ and b^k are not spacetime dependent and thus we can drop the b^k term and pull the $a^\mu{}_k$ out of the parenthesis to give

$$\delta S_\sigma = -T \int d\tau d\sigma \sqrt{-h}\, h^{\alpha\beta} a^\mu{}_k \partial_\alpha X^k \partial_\beta X^\nu g_{\mu\nu}.$$

Now, we can use the metric $g_{\mu\nu}$ to lower the upper index on $a^\mu{}_k$, doing this we get

$$\delta S_\sigma = -T \int d\tau d\sigma \sqrt{-h}\, \underbrace{[a_{\nu k}]}_{\text{antisym}} \underbrace{[h^{\alpha\beta} \partial_\alpha X^k \partial_\beta X^\nu]}_{\text{sym}},$$

which is the product of an antisymmetric part with a symmetric part, and thus equal to zero. So, we have that, under a Poincaré transformation, the variation in the Polyakov action is zero, $\delta S_\sigma = 0$, which tells us that this action is invariant under these transformations.

3.2 Local Symmetries of the Bosonic String Theory Worldsheet

The next topic to be discussed is that of local symmetries and transformations on the worldsheet, i.e. transformations whose parameters depend on the worldsheet coordinates. What local symmetries does our bosonic string theory actually have?

1. Reparameterization invariance (also known as diffeomorphisms): This is a local symmetry for the worldsheet. The Polyakov action is invariant under the changing of the parameter σ to $\sigma' = f(\sigma)$ since the fields $X^\mu(\tau, \sigma)$ transform as scalars while the auxiliary field $h^{\alpha\beta}(\tau, \sigma)$ transforms as a 2-tensor,

$$X^\mu(\tau, \sigma) = X'^\mu(\tau, \sigma') \quad \text{and} \quad h_{\alpha\beta}(\tau, \sigma) = \frac{\partial f^\gamma}{\partial \sigma^\alpha} \frac{\partial f^\delta}{\partial \sigma^\beta} h'_{\gamma\delta}(\tau, \sigma'), \tag{3.8}$$

   Thus, our bosonic string is invariant under reparameterizations. This, as was mentioned earlier, tells us that we have redundancies in our theory, i.e. we actually have fewer degrees of freedom than we thought. Also, note that these symmetries are called diffeomorphisms, i.e. the transformations and their inverses are infinitely differentiable.

2. Weyl Symmetry: Weyl transformations are transformations that change the scale of the metric,
$$h_{\alpha\beta}(\tau, \sigma) \mapsto h'_{\alpha\beta}(\tau, \sigma) = e^{2\phi(\sigma)} h_{\alpha\beta}(\tau, \sigma), \tag{3.9}$$

   while leaving the scalars, $X^\mu(\tau, \sigma)$, alone or, equivalently, under a Weyl transformation the variation of $X^\mu(\tau, \sigma)$ is zero, $\delta X^\mu(\tau, \sigma) = 0$. Note that this is a local

transformation since the parameter $\phi(\sigma)$ depends on the worldsheet coordinates. To see whether our bosonic string theory is invariant under a Weyl transformation we first need to see how both the quantities $\sqrt{-h}$ and $\sqrt{-h}\, h^{\alpha\beta}$ transform. The transformation of $\sqrt{-h}$ is given by

$$\sqrt{-h'} = \sqrt{-det(h'_{\alpha\beta})}$$

$$= e^{2(2\phi(\sigma))/2}\sqrt{-det(h_{\alpha\beta})}$$

$$= e^{2\phi(\sigma)}\sqrt{-h}\,.$$

While expanding (3.9) in ϕ yields that $h'^{\alpha\beta} = e^{-2\phi}h^{\alpha\beta} = (1 - 2\phi + \cdots)h^{\alpha\beta}$, thus the variation (infinitesimally) of $h^{\alpha\beta}$ is given by $\delta h^{\alpha\beta} = -2\phi h^{\alpha\beta}$. And so, for $\sqrt{-h}\, h^{\alpha\beta}$ we have that

$$\sqrt{-h'}\, h'^{\alpha\beta} = \sqrt{-h}e^{2\phi(\sigma)}e^{-2\phi(\sigma)}h^{\alpha\beta} = \sqrt{-h}\, h^{\alpha\beta}.$$

Thus, under a Weyl transformation S_σ does not change, or is invariant, which implies that the variation of S_σ under a Weyl transformation vanishes. This says that our bosonic string theory is invariant under Weyl transformations. We will now show that since our theory is invariant under Weyl transformations this implies that the stress-energy tensor associated with this theory is traceless, $h^{\alpha\beta}T_{\alpha\beta} = 0$. So, to begin recall that the stress-energy tensor is given by

$$T_{\alpha\beta} \equiv -\frac{2}{T}\frac{1}{\sqrt{h}}\frac{\delta S_\sigma}{\delta h_{\alpha\beta}}, \tag{3.10}$$

which implies that, under a generic transformation of the field $h^{\alpha\beta}$, we can write the variation of S_σ as

$$\delta S_\sigma \equiv \int \frac{\delta S_\sigma}{\delta h^{\alpha\beta}}\delta h^{\alpha\beta} = -\frac{T}{2}\int d\tau d\sigma\sqrt{-h}\,\delta h^{\alpha\beta}T_{\alpha\beta}.$$

Thus, if we now restrict to a Weyl transformation we see that variation of the action becomes

$$\delta S_\sigma = -\frac{T}{2}\int d\tau d\sigma\sqrt{-h}\,\delta h^{\alpha\beta}T_{\alpha\beta}$$

$$= -\frac{T}{2}\int d\tau d\sigma\sqrt{-h}\,(-2\phi)h^{\alpha\beta}T_{\alpha\beta}\,,$$

which must be equal to zero since there is no variation in S_σ under a Weyl transformation. Now, since $\sqrt{-h}$ and ϕ are arbitrary this means that

$$h^{\alpha\beta} T_{\alpha\beta} = 0, \tag{3.11}$$

or that, for a Weyl invariant classical theory, the corresponding stress-energy tensor must be traceless.

Since our theory has local, or gauge, symmetries, we know that the theory has a redundancy in its degrees of freedom,[§] and we can use these symmetries to cope with these redundancies, this is known as gauge fixing. For example, in electrodynamics one has a symmetry under the group of phase transformations, i.e the Lie group $U(1)$ whose parameters are spacetime dependent, $e^{\phi(X)}$. Now, for example, one can fix the gauge by requiring that $\partial_\mu A^\mu = 0$, known as the Lorenz gauge[†], where A^μ is the gauge field associated with the $U(1)$ gauge group. By doing this one is able to remove some[‡] of the redundant degrees of freedom along with simplifying the description of the theory, since in the Lorenz gauge the Maxwell equations reduce to

$$\Box A^\nu = e j^\nu,$$

where e is a constant. Now, lets see how we can fix a gauge in order to simplify our bosonic string theory. In particular, we will now show that if our theory is invariant under diffeomorphisms and Weyl transformations then we can fix a gauge so that our intrinsic metric, $h_{\alpha\beta}$, becomes flat.

First, note that since the metric,

$$h_{\alpha\beta} = \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix}, \tag{3.12}$$

is symmetric it has only three independent components, $h_{00}(X)$, $h_{11}(X)$, and $h_{10}(X) = h_{01}(X)$. Now, a diffeomorphism (or reparameterization) allows us to change two of the

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[§]This is because the worldsheet coordinates have no physical meaning, just like the parameter of the worldline in relativistic physics has no meaning.

[†]Even though Lorentz is typically credited with this gauge choice, it was actually Lorenz who first proposed it.

[‡]The Lorenz gauge is incomplete in the sense that there remains a subspace of gauge transformations which preserve the constraint. These remaining degrees of freedom correspond to gauge functions which satisfy the wave equation

$$\Box \psi = 0,$$

and to obtain a fully fixed gauge, one must add boundary conditions along the light cone of the experimental region.

independent components by using two coordinate transformations, $f^1(X)$ and $f^2(X)$, to set $h_{10}(X) = 0 = h_{01}(X)$ and $h_{00}(X) = \pm h_{11}(X)$ (where the $\pm$ depends on the signature of the metric). Thus, from our theory being diffeomorphism invariant, we see that our two dimensional metric $h_{\alpha\beta}(X)$ is of the form $h(X)\eta_{\alpha\beta}$. Now, we can use a Weyl transformation to remove this function, i.e. we then have that $h_{\alpha\beta}(X) = \eta_{\alpha\beta}$. And so we see that if our theory is invariant under diffeomorphisms and Weyl transformations (there combinations are called conformal transformations), then the two-dimensional intrinsic metric, $h_{\alpha\beta}(X)$, can be "gauged" into the two-dimensional flat metric,

$$h_{\alpha\beta}(X) = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.13}$$

However, one should note that since gauge symmetries are local symmetries this ability to transform the metric $h_{\alpha\beta}(X)$ into in a flat metric is only valid locally and one cannot, in general, extend to the flat $h_{\alpha\beta}(X)$ to the whole worldsheet. Only if the worldsheet is free of topological obstructions, i.e. its Euler characteristic is zero, can the locally flat metric $h_{\alpha\beta}(X)$ be extended to a globally flat metric on the worldsheet[¶].

In terms of the gauge fixed flat metric, the Polyakov action becomes

$$S_\sigma = \frac{T}{2} \int d\tau d\sigma \Big((\dot{X})^2 - (X')^2 \Big), \tag{3.14}$$

where $\dot{X} \equiv dX^\mu/d\tau$ and $X' \equiv dX^\mu/d\sigma$. Finally, note that these two gauges are not the only ones for our theory, i.e. even after we fix the reparameterization gauge and Weyl gauge to construct the flat metric it is still invariant under other local symmetries known as conformal transformations. These transformations will be discussed later on in subsequent lectures.

3.3 Field Equations for the Polyakov Action

Let us now suppose that our worldsheet topology allows for the gauge fixed locally defined flat metric $h^{\alpha\beta}$ to be extended globally. The field equations for the fields $X^\mu(\tau, \sigma)$ on the worldsheet come from setting the variation of S_σ with respect to $X^\mu \mapsto X^\mu + \delta X^\mu$ equal to zero. This leads to

$$\delta S_\sigma = \frac{T}{2} \int d\tau d\sigma \Big(2\dot{X}\delta\dot{X} - 2X'\delta X' \Big),$$

[¶]By being able to extend the locally flat intrinsic metric to the whole worldsheet implies that there exists a flat coordinate system that covers the whole worldsheet. This, in turn, implies that the worldsheet has a flat geometry which implies that the Ricci curvature scalar vanishes. Now, since, in two dimensions, the Euler characteristic of a manifold is proportional to the integral of the Ricci curvature over the manifold, we see that being able to extend the locally flat metric requires for the Euler characteristic of the worldsheet to vanish.

and to proceed we integrate both terms by parts to give

$$T \int d\tau d\sigma \left[\left(-\partial_\tau^2 + \partial_\sigma^2 \right) X^\mu \right] \delta X^\mu + T \int d\sigma \dot{X}^\mu \delta X^\mu \Big|_{\partial\tau}$$
$$- \left[T \int d\tau X' \delta X^\mu \Big|_{\sigma=\pi} + T \int d\tau X' \delta X^\mu \Big|_{\sigma=0} \right].$$

We set the variation of X^μ at the boundary of τ to be zero, i.e. $\delta X^\mu \Big|_{\partial\tau} = 0$, and are left with the field equations for $X^\mu(\tau, \sigma)$ for the Polyakov action,

$$\left(-\partial_\tau^2 + \partial_\sigma^2 \right) X^\mu - T \int d\tau \left[X' \delta X^\mu \Big|_{\sigma=\pi} + X' \delta X^\mu \Big|_{\sigma=0} \right]. \tag{3.15}$$

The σ boundary terms tell us what type of strings we have, either closed or open strings.

- Closed Strings: For closed strings we take σ to have a periodic boundary condition,

$$X^\mu(\tau, \sigma + n) = X^\mu(\tau, \sigma), \tag{3.16}$$

  which implies that the boundary terms appearing in the variation of S_σ vanish since if $X^\mu(\tau, \sigma + n) = X^\mu(\tau, \sigma)$ then $\delta X(\tau, \sigma = 0) = \delta X(\tau, \sigma + n)$ and so subtracting them gives zero. Thus, we are left with the following field equations for the closed string

$$\left(\partial_\tau^2 - \partial_\sigma^2 \right) X^\mu(\tau, \sigma) = 0, \tag{3.17}$$

  with the boundary conditions

$$X^\mu(\tau, \sigma + n) = X^\mu(\tau, \sigma). \tag{3.18}$$

- Open Strings (Neumann Boundary Conditions): For the open string with Neumann boundary conditions we set the derivative of X^μ, by σ, at the σ boundary to vanish, i.e. $\partial_\sigma X^\mu(\tau, \sigma + 0) = \partial_\sigma X^\mu(\tau, \sigma + n) = 0$ (see figure 3). Under these boundary conditions the boundary terms over σ also vanish and thus the field equations become

$$\left(\partial_\tau^2 - \partial_\sigma^2 \right) X^\mu(\tau, \sigma) = 0, \tag{3.19}$$

  with the boundary conditions

$$\partial_\sigma X^\mu(\tau, \sigma + 0) = \partial_\sigma X^\mu(\tau, \sigma + n) = 0. \tag{3.20}$$

Note that the Neumann boundary conditions preserve Poincaré invariance since

$$\partial_\sigma(X'^\mu)\Big|_{\sigma=0,n} = \partial_\sigma\left(a^\mu{}_\nu X^\nu + b^\mu\right)\Big|_{\sigma=0,n}$$

$$= a^\mu{}_\nu \partial_\sigma X^\nu\Big|_{\sigma=0,n}$$

$$= 0\,.$$

- Open Strings (Dirichlet Boundary Conditions): For the Dirichlet boundary conditions we set the value of X^μ to a constant at the σ boundary, $X^\mu(\tau,\sigma+0) = X_0^\mu$ and $X^\mu(\tau,\sigma+n) = X_n^\mu$ where X_0^μ and X_n^μ are constants (see figure 4). This also makes the σ boundary terms vanish and so the field equations are

$$\left(\partial_\tau^2 - \partial_\sigma^2\right)X^\mu(\tau,\sigma) = 0, \tag{3.21}$$

with boundary conditions

$$X^\mu(\tau,\sigma=0) = X_0^\mu, \tag{3.22}$$

and

$$X^\mu(\tau,\sigma=n) = X_n^\mu. \tag{3.23}$$

Whereas the Neumann boundary conditions preserve Poincaré invariance, the Dirichlet boundary conditions do not since

$$(X'^\mu)\Big|_{\sigma=0,n} = \left(a^\mu{}_\nu X^\nu + b^\mu\right)\Big|_{\sigma=0,n}$$

$$= a^\mu{}_\nu X_{0,n}^\nu + b^\mu$$

$$\neq X_{0,n}^\mu\,.$$

Thus, under a Poincaré transformation the ends of the string actually change.

Finally, note that if we have Neumann boundary conditions on $p+1$ of the background spacetime coordinates and Dirichlet boundary conditions on the remaining $D-p+1$ coordinates, then the place where the string ends is a Dp-brane.

So, we can see that under all three boundary conditions the resulting field equations are equivalent, just different boundary conditions. In addition to the above field equations,
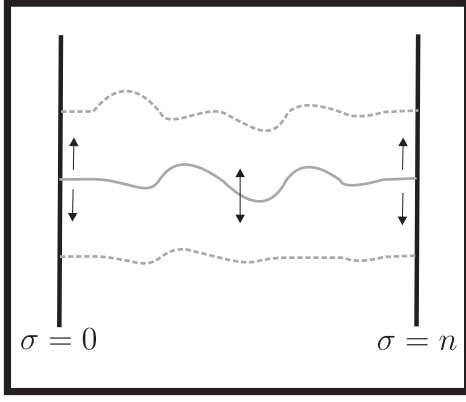
Figure 3: Neumann BC's: The string can oscillate and its endpoints can move along the boundaries as long as thier derivatives vanish at the bondaries.
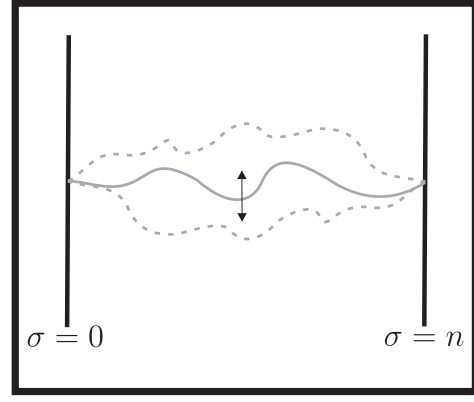
Figure 4: Dirichlet BC's: The string can osciallte but its endpoints are fixed at the boundary.

one must impose the field equations which result from setting the variation of S_σ with respect to $h^{\alpha\beta}$ equal to zero. These field equations are given by (see (2.35))

$$0 = T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X, \tag{3.24}$$

and gauge fixing $h^{\alpha\beta}$ to be flat[‡] we get that the field equations transform into the following two conditions

$$0 = T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2), \tag{3.25}$$

and

$$0 = T_{01} = T_{10} = \dot{X} \cdot X'. \tag{3.26}$$

3.4 Solving the Field Equations

Here again, we have and will assume that we can extend the gauge fixed flat metric to a global flat metric, $h_{\alpha\beta} \mapsto \eta_{\alpha\beta}$, on the worldsheet. Now, we will solve the system of equations by introducing light-cone coordinates for the worldsheet,

$$\sigma^\pm = (\tau \pm \sigma), \tag{3.27}$$

which implies that

$$\tau = \frac{1}{2}(\sigma^+ + \sigma^-),$$

[‡]Recall that we have assumed that the topology on the worldsheet is such that we can extend the local flat metric to a global flat metric and thus we can insert the flat metric into the field equations, which hold globally.

$$\sigma = \frac{1}{2}(\sigma^+ - \sigma^-).$$

The derivatives, in terms of light-cone coordinates, become

$$\partial_+ \equiv \frac{\partial}{\partial\sigma^+} = \frac{\partial\tau}{\partial\sigma^+}\frac{\partial}{\partial\tau} + \frac{\partial\sigma}{\partial\sigma^+}\frac{\partial}{\partial\sigma} = \frac{1}{2}(\partial_\tau + \partial_\sigma),$$

$$\partial_- \equiv \frac{\partial}{\partial\sigma^-} = \frac{\partial\tau}{\partial\sigma^-}\frac{\partial}{\partial\tau} + \frac{\partial\sigma}{\partial\sigma^-}\frac{\partial}{\partial\sigma} = \frac{1}{2}(\partial_\tau - \partial_\sigma),$$

and since the metric transforms as

$$\eta'_{\alpha'\beta'} = \frac{\partial\sigma^\gamma}{\partial\sigma^{\alpha'}}\frac{\partial\sigma^\delta}{\partial\sigma^{\beta'}}\eta_{\gamma\delta},$$

we have that

$$\eta_{++} = -\left(\frac{\partial\tau}{\partial\sigma^+}\right)^2 + \left(\frac{\partial\sigma}{\partial\sigma^+}\right)^2 = -\frac{1}{4} + \frac{1}{4} = 0,$$

$$\eta_{--} = -\left(\frac{\partial\tau}{\partial\sigma^-}\right)^2 + \left(\frac{\partial\sigma}{\partial\sigma^-}\right)^2 = -\frac{1}{4} + \frac{1}{4} = 0,$$

$$\eta_{+-} = -\frac{\partial\tau}{\partial\sigma^+}\frac{\partial\tau}{\partial\sigma^-} + \frac{\partial\sigma}{\partial\sigma^+}\frac{\partial\sigma}{\partial\sigma^-} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2},$$

$$\eta_{-+} = -\frac{\partial\tau}{\partial\sigma^-}\frac{\partial\tau}{\partial\sigma^+} + \frac{\partial\sigma}{\partial\sigma^-}\frac{\partial\sigma}{\partial\sigma^+} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

Thus, in terms of light-cone coordinates, the metric is given by

$$\eta_{\alpha\beta}\Big|_{\text{l-c c.}} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.28}$$

and so,

$$\eta^{\alpha\beta}\Big|_{\text{l-c c.}} = -2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.29}$$

In terms of the light-cone coordinates, the field equations $(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0$ become

$$\partial_+\partial_- X^\mu = 0, \tag{3.30}$$

while the field equations for the intrinsic worldsheet metric, $h_{\alpha\beta}$ become

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0, \tag{3.31}$$

$$T_{--} = \partial_- X^\mu \partial_- X_\mu = 0. \tag{3.32}$$

These are the three equations that we need to solve, (3.30) - (3.32).

The most general solution to the field equations for $X^\mu(\sigma^+, \sigma^-)$, (3.30), is given by a linear combination of two arbitrary functions whose arguments depend only on one of the light-cone coordinates, $X^\mu(\sigma^+, \sigma^-) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+)$. Now that we have the general form of the solution to the field equation in terms of the worldsheet light-cone coordinates, we want to map this solution to the usual worldsheet coordinates, τ and σ. When we do this we see that, since $\sigma^- = \tau - \sigma$ and $\sigma^+ = \tau + \sigma$, the arbitrary functions $X_R^\mu(\sigma^-)$ and $X_L^\mu(\sigma^+)$ can be thought of, in the τ and σ coordinate system, as left and right moving waves,

$$X^\mu = \underbrace{X_R^\mu(\tau - \sigma)}_{\text{right mover}} + \underbrace{X_L^\mu(\tau + \sigma)}_{\text{left mover}}, \tag{3.33}$$

which propagate through space at the speed of light. So, $X^\mu(\tau, \sigma)$ separates into a linear combination of some function of $\tau - \sigma$ only and another function of $\tau + \sigma$ only. Now, we need to apply the boundary conditions.

- Closed String: Applying the closed string boundary conditions $X^\mu(\tau, \sigma + n) = X^\mu(\tau, \sigma)$ gives the particular solution (mode expansion) for the left and right movers as

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2(\tau - \sigma)p^\mu + \frac{i}{2}l_s \sum_{n \neq 0} \frac{1}{n}\alpha_n^\mu e^{-2in(\tau - \sigma)}, \tag{3.34}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2(\tau + \sigma)p^\mu + \frac{i}{2}l_s \sum_{n \neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau + \sigma)}, \tag{3.35}$$

where x^μ is a constant (called the center of mass of the string), p^μ is a constant (called the total momentum of the string), l_s is the string length (also a constant), $T = \frac{1}{2n\alpha'}$ and $\alpha' = \frac{1}{2}l_s^2$. To see that $X^\mu = X_R^\mu + X_L^\mu$ satisfies the boundary conditions consider,

$$X^\mu = X_R^\mu + X_L^\mu = \underbrace{x^\mu + \frac{1}{2}\tau l_s^2 p^\mu}_{\substack{\text{center of mass} \\ \text{motion of string}}} + \underbrace{\frac{i}{2}l_s \sum_{n \neq 0} \frac{1}{n}\left(\alpha_n^\mu e^{2in\sigma} + \tilde{\alpha}_n^\mu e^{-2in\sigma}\right)e^{-2in\tau}}_{\text{oscillations of the string}}, \tag{3.36}$$

and since the first two terms do not depend on σ, and since the second part is periodic in σ, we can see that we have satisfied the periodic boundary conditions. Also, note that the first two terms look like the trajectory of a point particle, satisfying

$$\frac{d^2}{d\tau^2}X^\mu = 0 \implies X^\mu = x^\mu + p^\mu\tau,$$

which we will call the center of mass (c.o.m.) of the string, while the summation part looks like an oscillatory term due to the modes α and $\tilde{\alpha}$. So, the string moves throughout spacetime via the first part and oscillates via the second (summation) part.

Now, since X^μ must be real, i.e. $(X^\mu)^* = X^\mu$, we get that x^μ and p^μ are real along with

$$\alpha^\mu_{-n} = (\alpha^\mu_n)^*,$$

$$\tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^* .$$

Furthermore, from the definition of the canonical momentum,[‡] $P^\mu(\tau, \sigma)$, we can see that the mode expansion of the canonical momentum on the worldsheet is given by

$$P^\mu(\tau, \sigma) = \frac{\delta L}{\delta \dot{X}^\mu} = T\dot{X}^\mu = \frac{\dot{X}^\mu}{\pi l_s^2},$$
$$= \frac{p^\mu}{\pi} + \frac{1}{\pi l_s} \sum_{n \neq 0} \left(\alpha^\mu_n e^{-2in(\tau - \sigma)} + \tilde{\alpha}^\mu_n e^{-2in(\tau + \sigma)} \right) . \tag{3.37}$$

As an aside, we will see later that the canonical momentum is really the 0th component of the conserved current corresponding to a translational symmetry. Now, it can be shown that the field and its canonical momentum satisfy the following Poisson bracket relations

$$\left\{ P^\mu(\tau, \sigma), P^\nu(\tau, \sigma') \right\}_{P.B.} = 0, \tag{3.38}$$

$$\left\{ X^\mu(\tau, \sigma), X^\nu(\tau, \sigma') \right\}_{P.B.} = 0, \tag{3.39}$$

$$\left\{ P^\mu(\tau, \sigma), X^\nu(\tau, \sigma') \right\}_{P.B.} = \eta^{\mu\nu} \delta(\sigma - \sigma') . \tag{3.40}$$

We can plug in the mode expansions (see exercise 2.1) for both the field $X^\mu(\tau, \sigma)$ and its canonical momentum $P^\mu(\tau, \sigma)$ to get the equivalent Poisson bracket relations in

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[‡]Recall that the canonical momentum, conjugate to the field $X^\mu(\tau, \sigma)$, is defined by

$$P^\mu(\tau, \sigma) \equiv \frac{\partial L}{\partial \dot{X}^\mu} .$$

terms of α_n^μ, $\tilde{\alpha}_n^\mu$, x^μ and p^μ, which are given by

$$\left\{\alpha_m^\mu, \alpha_n^\nu\right\}_{P.B.} = \left\{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\right\} = im\eta^{\mu\nu}\delta_{m,-n}, \tag{3.41}$$

$$\left\{\alpha_m^\mu, \alpha_n^\nu\right\}_{P.B.} = 0, \tag{3.42}$$

$$\left\{p^\mu, x^\nu\right\}_{P.B.} = \eta^{\mu\nu}. \tag{3.43}$$

Now that we have solved the field equations for the closed string boundary conditions we study the solutions obeying the open string boundary conditions, both Neumann and Dirichlet.

- Open String (Neumann Boundary Conditions): Recall that the Neumann boundary conditions imply that the derivative, w.r.t. σ, of the field at the σ boundary is zero, $\partial_\sigma X^\mu(\tau, \sigma)\big|_{\sigma=0,\pi} = 0^\ddagger$. Also, the general solution to the field equations is given by

$$X^\mu(\tau, \sigma) = a_0 + a_1\sigma + a_2\tau + a_3\sigma\tau + \sum_{k\neq 0}\left(b_k^\mu e^{ik\sigma} + \tilde{b}_k^\mu e^{-ik\sigma}\right)e^{-ik\tau}, \tag{3.44}$$

where a_i $(i = 1, 3)$, b_k and $\tilde{b}_k$ are constants and the only restraint on k in the summation is that it cannot equal to zero, i.e. it could be anything, a real number, complex number, etc., just so long as it does not take the value $k = 0$. Now, when we apply the Neumann boundary conditions, see problem 4.2 for a hint of how it works (this is for the Dirichlet b.c. but you get the feel for how to do it for any b.c.), we get the specific solution, here we have introduced new constants to resemble the closed string mode expansion,

$$X^\mu(\tau, \sigma) = \underbrace{x^\mu + l_s\tau p^\mu}_{\substack{\text{c.o.m. motion} \\ \text{of the string}}} + il_s\underbrace{\sum_{m\neq 0}\frac{1}{m}\alpha_m^\mu e^{-im\tau}\cos(m\sigma)}_{\text{oscillation of string}}. \tag{3.45}$$

- Open String (Dirichlet Boundary Conditions): The Dirichlet boundary conditions say that at the σ boundary the field assumes the value of a constant, $X^\mu(\tau, \sigma = 0) = X_0^\mu$ and $X^\mu(\tau, \sigma = \pi) = X_\pi^\mu$. The solution to the field equation obeying the Dirichlet boundary conditions is given by, see problem 4.2,

$$X^\mu(\tau, \sigma) = \underbrace{x_0^\mu + \frac{\sigma}{\pi}(x_\pi^\mu - x_0^\mu)}_{\substack{\text{c.o.m. motion} \\ \text{of the string}}} + \underbrace{\sum_{m\neq 0}\frac{1}{m}\alpha_m^\mu e^{-im\tau}\sin(m\sigma)}_{\text{oscillation of string}}. \tag{3.46}$$

$\ddagger$Note that here we have set $n = \pi$ for the boundary conditions.

This ends the discussion of the solutions to the $X^\mu(\tau, \sigma)$ field equations. In the next chapter we will take a deeper look at the symmetries of our theory and how to construct currents and charges from them along with the classical mass formula, the Witt algebra and the (canonical) quantization of our bosonic string theory.

3.5 Exercises

Problem 1

In this problem we want to derive the mode expansion of the field, which is a solution to the wave equation with closed string boundary conditions, $X^\mu(\tau, \sigma)$ ((3.34)-(3.35)) and the Poisson brackets satisfied by the modes ((3.41)-(3.43)).

Mode expansion:

As we showed in this chapter, if we pick a gauge in which $h_{\alpha\beta} = \eta_{\alpha\beta}$, then the field $X^\mu(\tau, \sigma)$ satisfies the wave equation:

$$\Box X^\mu(\tau, \sigma) = \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu(\tau, \sigma) = 0. \tag{3.47}$$

The wave equation is a separable partial linear differential equation in terms of the variables σ, τ, so it has solutions of the form:

$$X^\mu(\tau, \sigma) = g(\tau)f^\mu(\sigma). \tag{3.48}$$

a) Applying this ansatz into the wave equation, show that the two functions must satisfy:

$$\frac{\partial^2 f^\mu(\sigma)}{\partial\sigma^2} = cf^\mu(\sigma), \qquad \frac{\partial^2 g(\tau)}{\partial\tau^2} = cg(\tau), \tag{3.49}$$

where c is an arbitrary constant.

b) Because the σ direction is compact, we have to make sure that the solution (3.48) satisfies the correct boundary condition:

$$X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma). \tag{3.50}$$

Write the most general solution of the first equation in (3.49) and impose the boundary condition to show that the constant c must take the values:

$$c = -4m^2, \qquad m \in \mathbb{Z}. \tag{3.51}$$

c) By taking linear combinations of solutions of the form (3.48) we can construct the most general solution of the wave equation. Show that it takes the form:

$$
\begin{aligned}
X^\mu(\tau, \sigma) &= X_R^\mu(\tau - \sigma) + X_L^\mu(\tau + \sigma), \\
X_R^\mu &= \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau - \sigma) + \frac{i}{2}l_s \sum_{n \neq 0} \frac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma)}, \\
X_L^\mu &= \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau + \sigma) + \frac{i}{2}l_s \sum_{n \neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)},
\end{aligned} \tag{3.52}
$$

where we have introduced the factors of l_s for convenience.

d) Notice that to find the mode expansion of the wave equation, we had to solve a differential equation of the form:

$$\frac{\partial^2 f}{\partial \sigma^2} = cf(\sigma), \tag{3.53}$$

with certain boundary conditions:

$$f(0) = f(\pi). \tag{3.54}$$

This is a special case of the more general eigenvalue problem:

$$\mathcal{L}f(\sigma) = cf(\sigma), \tag{3.55}$$

where $\mathcal{L}$ is a linear operator acting on the space of functions satisfying the boundary conditions (3.54). In our case the linear operator was $\mathcal{L} = \frac{\partial^2}{\partial \sigma^2}$ and the constant c plays the role of the eigenvalue. One can show that if such an operator is hermitian (which is true for the case $\mathcal{L} = \frac{\partial^2}{\partial \sigma^2}$), then its eigenfunctions,

$$\mathcal{L}f_n(\sigma) = c_n f_n(\sigma), \tag{3.56}$$

constitute a good basis for all (smooth enough) functions satisfying the boundary conditions (3.54). What this means is that we can normalize the eigenfunctions to be orthonormal, that is

$$\int_0^\pi d\sigma\, f_m^*(\sigma) f_n(\sigma) = \delta_{mn}. \tag{3.57}$$

Also, the basis is complete, which means that the general function $g(\sigma)$ satisfying (3.54) can be written as:

$$g(\sigma) = \sum_n c_n f_n(\sigma). \tag{3.58}$$

The coefficients c_n can be easily computed by projecting both sides of the equation on the basis of eigenfunctions and using the orthogonality relation (3.57):

$$c_n = \int_0^\pi d\sigma\, f_n^*(\sigma) g(\sigma). \tag{3.59}$$

It is not difficult to see that completeness is expressed by the fact that we can write the Dirac $\delta(\sigma)$ function in terms of the eigenfunctions $f_n(\sigma)$.

$$\delta(\sigma - \sigma') = \sum_n f_n(\sigma) f_n^*(\sigma'). \tag{3.60}$$

For the special case where $\mathcal{L} = \frac{\partial^2}{\partial \sigma^2}$, find the normalized eigenfunctions and use the previous formula to find the following representation of the Dirac function

$$\delta(\sigma - \sigma') = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} e^{2in(\sigma - \sigma')}. \tag{3.61}$$

Poisson brackets:

(Note: In this problem we will drop the notation $\{\cdot, \cdot\}_{P.B.}$ in favor of the simpler $\{\cdot, \cdot\}$) Before, we showed that the field $X^\mu(\tau, \sigma)$ has the expansion in terms of modes (3.52). The canonical momentum to $X^\mu(\tau, \sigma)$ is given by:

$$P^\mu(\tau, \sigma) = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = \frac{1}{\pi l_s^2} \dot{X}^\mu, \tag{3.62}$$

where $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$.

In this problem we want to start with the classical Poisson brackets for the field $X^\mu(\tau, \sigma)$ and its canonical momentum $P^\mu(\tau, \sigma)$:

$$\begin{aligned} \{P^\mu(\tau, \sigma), P^\nu(\tau, \sigma')\} &= \{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} = 0, \\ \{P^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} &= \eta^{\mu\nu} \delta(\sigma - \sigma'), \end{aligned} \tag{3.63}$$

and derive the Poisson brackets for the modes:

$$\begin{aligned} \{\alpha_m^\mu, \alpha_n^\nu\} &= \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\} = im\eta^{\mu\nu}\delta_{m+n,0}, \\ \{\alpha_m^\mu, \tilde{\alpha}_n^\nu\} &= 0, \\ \{x^\mu, p^\nu\} &= i\eta^{\mu\nu}. \end{aligned} \tag{3.64}$$

e) Using (3.52) and (3.62) write the expansion of $P^\mu(\tau, \sigma)$ in terms of the modes $\alpha_m^\mu, \tilde{\alpha}_m^\mu$.

f) Substitute the mode expansions for $X^\mu(\tau, \sigma)$ and $P^\mu(\tau, \sigma)$ in the Poisson brackets (3.63). For simplicity set $\sigma' = 0$ and project on the eigenfunction basis of the operator $\frac{\partial^2}{\partial \sigma^2}$ (in other words, perform the Fourier transform over σ on both sides of the equations).

g) From your results in f) it should be easy to solve for the Poisson brackets of the modes $\alpha_m^\mu, \tilde{\alpha}_m^\mu$ to get the final expressions (3.64).

4. Symmetries (Revisited) and Canonical Quantization

Associated to any global symmetry of a system, in our case the worldsheet, there exists a conserved current, j^μ, and a conserved charge, Q, i.e.

$$\partial_\alpha j^\alpha = 0, \tag{4.1}$$

$$\frac{d}{d\tau}Q = \frac{d}{d\tau}\left(\int d\sigma j^0\right) = 0, \tag{4.2}$$

where the integral in the expression for the charge is taken over the spacelike coordinates, which in our case is just σ. The reason we know that these two objects exists is due to Emmy Noether and her remarkable theorem. However, we can use her theorem to construct an algorithm for finding these currents and charges for any symmetry. That is the topic of the next section of this lecture.

4.1 Noether's Method for Generating Conserved Quantities

As was just mentioned, due to Noether's theorem, associated to any global symmetry there exists a conserved current which then gives rise to a conserved charge. Now, to verify that the charge, defined as the spatial integral of the zeroth component of the current, is indeed conserved consider the following,

$$\frac{d}{d\tau}\left(\int d\sigma j^0\right) = \int d\sigma \frac{d}{d\tau}(j^0)$$

$$= -\int d\sigma \partial_\sigma j^\sigma$$

$$= -j^\sigma \Big|_{\sigma=0}^{\pi}$$

$$= 0,$$

where substitution in the second line with $-\partial_\sigma j^\sigma$ comes from the current being conserved, the third line follows from Stokes' theorem, and the last line is due to the boundary conditions. Now that we have seen that the charge is indeed conserved the next question to ask is how do we actually construct a current from a given symmetry?

In general (not necessarily string theory), suppose we have a field ϕ, then if there exists a global symmetry for the theory in question, under this transformation, $f : \phi \mapsto \phi + \delta\phi$ where $\delta\phi = \epsilon f(\phi)$ and ϵ is infinitesimal, the equations of motion do not change,

i.e the variation in the action δS is equal to zero. To construct the corresponding current to this transformation the Noether method says to proceed as follows.

First, consider ϵ to be a local parameter, i.e. its derivative with respect to spacetime coordinates does not vanish. Now, since ϵ is infinitesimal, the only contributing part to the variation of S will be linear in ϵ. Then the transformation, $\delta\phi = \epsilon f(\phi)$ leads to a variation in the action S which is given by

$$\delta S = \int d\tau d\sigma \left(\partial_\alpha \epsilon\right) j^\alpha. \tag{4.3}$$

Integrating this by parts gives

$$\delta S = -\int d\tau d\sigma \epsilon \left(\partial_\alpha j^\alpha\right). \tag{4.4}$$

If this transformation is a symmetry then this variation vanishes for all ϵ and thus we have just shown that $\partial_\alpha j^\alpha = 0$. Not only have we shown that the current is conserved, but we have, in fact, shown how to construct this current. We simply plug the variation of the transformation, $\epsilon f(\phi)$, into the variation of the action and then the current will be given by all the terms which multiply the $\partial_\alpha \epsilon$ term. In order to further solidify the previous developments lets consider some examples of how to construct currents and charges from symmetries, also see problem 4.1.

1. Poincaré Transformations

   - Translations: For translations we have that $\delta X^\mu = \epsilon(\sigma^\alpha) = b^\mu(\sigma^\alpha)$[¶] and, in a Minkowski background, we have the action

$$S_\sigma = -\frac{T}{2} \int d\tau d\sigma \ h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$$= -\frac{T}{2} \int d\tau d\sigma \ \partial_\alpha X^\mu \partial^\alpha X_\mu .$$

   Thus,

$$\delta S_\sigma = -T \int d\tau d\sigma \ \partial_\alpha \left(b^\mu(\sigma^\alpha)\right) \partial^\alpha X_\mu,$$

   and so we have that

$$j^\alpha_\mu = -T \partial^\alpha X_\mu, \qquad \left(\text{or } j^\mu_\alpha = -T \partial_\alpha X_\mu\right)$$

[¶] Remember, we are performing the Noether method for calculating the current associated to a global symmetry and thus we must assume that the translation is local, i.e. $\delta X^\mu = b^\mu(\sigma^\alpha)$ where the parameter $b^\mu(\sigma^\alpha)$ depends on its spacetime position.

which implies that the corresponding current is given by

$$j^{\alpha\nu} = -T\partial^\alpha X_\mu \eta^{\nu\mu} = -T\partial^\alpha X^\nu. \tag{4.5}$$

Thus, we get a total of ν currents j^α, corresponding to the ν degrees of freedom. To see that this current is indeed conserved consider,

$$\partial_\alpha j^{\alpha\nu} = \partial^\alpha j_\alpha^\nu = -T\partial^\alpha \partial_\alpha X^\nu = T(\partial_\tau^2 - \partial_\sigma^2)X^\mu,$$

which is equal to zero on-shell, i.e. the current is conserved when the field equations hold[‡]. The corresponding charge Q is given by

$$p^\nu = \int d\sigma\ j^{0\nu}$$

$$= -\int_0^\pi d\sigma\ T\partial^0 X^\nu$$

$$= \int_0^\pi d\sigma\ T\partial_0 X^\nu$$

$$= \int_0^\pi d\sigma\ T\dot{X}^\nu$$

$$= \int_0^\pi d\sigma P^\mu\,,$$

where P^μ is the canonical momentum, conjugate to the field X^μ. This charge, p^μ, is called the total momentum (of the string) and is the same as the term p^ν appearing in the mode expansion of the string, but now we see that it follows from the action being invariant under translations. Also, as was previously mentioned, the canonical momentum, P^μ, is really the 0th component of the current associated to translational symmetry.

- Lorentz Transformations: For a Lorentz transformation we have seen that $\delta X^\mu = a^\mu{}_k X^k$ and so

$$\delta S_\sigma = -T \int d\tau d\sigma \partial_\alpha \left(a^\mu{}_k X^k\right) \partial^\alpha X^\nu \eta_{\mu\nu}$$

$$= -T \int d\tau d\sigma \left[\left(\partial_\alpha a^\mu{}_k\right) X^k \partial^\alpha X^\nu \eta_{\mu\nu} + a^\mu{}_k \partial_\alpha X^k \partial^\alpha X^\nu \eta_{\mu\nu}\right]$$

[‡]When the field equations hold we say that we are working on-shell and if something holds on-shell it means that the result is valid only when the field equations hold.

$$= -T \int d\tau d\sigma \left[\left(\partial_\alpha a^\mu{}_k \right) X^k \partial^\alpha X^\nu \eta_{\mu\nu} + \underbrace{a_{\nu k}}_{\substack{\text{anti-}\\\text{symm}}} \underbrace{\partial_\alpha X^k \partial^\alpha X^\nu}_{\text{symmetric}} \right]$$

$$= -T \int d\tau d\sigma \left(\partial_\alpha a^\mu{}_k \right) X^k \partial^\alpha X^\nu \eta_{\mu\nu}.$$

Note that the second term in the third line drops out because it is a product of an antisymmetric part and a symmetric part. Now, we can further simplify this by using the metric $\eta_{\mu\nu}$ to lower the μ index of $\partial_\alpha a^\mu{}_k$, which can be done since the metric commutes with the derivative. Doing this we arrive at

$$\delta S_\sigma = -T \int d\tau d\sigma \left(\partial_\alpha a_{\nu k} \right) X^k \partial^\alpha X^\nu,$$

and we can simply read off the current. But wait! Be careful, because now the term $a_{\nu k}$ is antisymmetric and when we define the corresponding current we need to take this into account. Thus, the current $j_\alpha^{\mu\nu}$ is given by

$$j_\alpha^{\mu\nu} = -\frac{T}{2} \left(X^\mu \partial_\alpha X^\nu - X^\nu \partial_\alpha X^\mu \right), \tag{4.6}$$

which is clearly antisymmetric. To see that this current is conserved consider,

$$\partial^\alpha j_\alpha^{\mu\nu} = -\frac{T}{2} \left(\partial^\alpha X^\mu \partial_\alpha X^\nu + X^\mu \partial_\alpha \partial^\alpha X^\nu - \partial^\alpha X^\nu \partial_\alpha X^\mu - X^\nu \partial^\alpha \partial_\alpha X^\mu \right)$$

$$= -\frac{T}{2} \left(X^\mu \underbrace{\partial^\alpha \partial_\alpha X^\nu}_{\substack{=0 \\ \text{(on-shell)}}} - X^\nu \underbrace{\partial^\alpha \partial_\alpha X^\mu}_{\substack{=0 \\ \text{(on-shell)}}} \right)$$

$$= 0 \quad \text{(on-shell)}.$$

4.2 The Hamiltonian and Energy-Momentum Tensor

In physics, the time evolution of a system is generated by the Hamiltonian and in string theory this is no different. Worldsheet time evolution is generated by the Hamiltonian which is defined by

$$H = \int_{\sigma=0}^{\pi} d\sigma \left(\dot{X}_\mu P^\mu - \mathcal{L} \right), \tag{4.7}$$

where P^μ is the canonical momentum that was defined earlier, (3.37), and $\mathcal{L}$ is the Lagrangian. In the case of the bosonic string theory we have that the canonical momentum is given by $P^\mu = T\dot{X}^\mu$ while the Lagrangian is $\mathcal{L} = \frac{1}{2}(\dot{X}^2 \cdot X'^2)$. So, plugging

this into (4.7) gives the bosonic string Hamiltonian. Namely,

$$H = T \int_{\sigma=0}^{\pi} d\sigma \left(\dot{X}^2 - \frac{1}{2}(\dot{X}^2 \cdot X'^2) \right)$$

$$= \frac{T}{2} \int_{\sigma=0}^{\pi} d\sigma \left(\dot{X}^2 + X'^2 \right),$$

where $\dot{X}^2 \equiv \dot{X}_\mu \dot{X}^\mu$.

Now, this expression for the Hamiltonian holds for our bosonic theory and thus for both open and closed strings. To express the Hamiltonian in terms of an open or closed string we need to expand the above in terms of the mode expansions for the fields $X^\mu(\tau, \sigma)$. So, for a closed string theory the Hamiltonian becomes

$$H = \sum_{n=-\infty}^{\infty} \left(\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \right), \tag{4.8}$$

where we do not have indicies since we are dealing with dot products and also we have defined $\alpha_0^\mu = \tilde{\alpha}_0^\mu = 1/2 l_s p^\mu$. While for open strings we have that

$$H = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot \alpha_n, \tag{4.9}$$

with $\alpha_0^\mu = \tilde{\alpha}_0^\mu = l_s p^\mu$. Note that H is conserved, i.e. $\frac{d}{d\tau}(H) = 0$, since neither α or $\tilde{\alpha}$ depend on $\tau^{\ddagger}$. Also, these results only hold in the classical theory, when we quantize the theory we will have order ambiguities when we promote the modes to operators and thus need to be careful how to resolve these problems.

Now that we have studied the Hamiltonian and its mode expansion we will see the mode expansion for the stress-energy tensor in terms of a closed string theory, while the open string version follows analogously. So, we have seen that the components of the stress-energy tensor are given by

$$T_{--} = (\partial_- X_R^\mu)^2,$$

$$T_{++} = (\partial_+ X_L^\mu)^2,$$

$$T_{-+} = T_{+-} = 0.$$

---

‡The conservation of the Hamiltonian follows from the fact that it is the conserved charge corresponding to the conserved current given by the stress-energy tensor, see page 127 of Ryder "Quantum Field Theory"

And, for a closed string we can plug in the mode expansions for X_R^μ and X_L^μ to get

$$T_{--} = (\partial_- X_R^\mu)^2$$

$$= l_s^2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n e^{-2im(\tau-\sigma)}$$

$$= 2l_s^2 \sum_{m=-\infty}^{\infty} L_m e^{-2im(\tau-\sigma)} , \tag{4.10}$$

where we have defined L_m as

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n . \tag{4.11}$$

While for T_{++}, in terms of the closed string expansion, we get

$$T_{++} = (\partial_+ X_L^\mu)^2$$

$$= l_s^2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n e^{-2im(\tau+\sigma)}$$

$$= 2l_s^2 \sum_{m=-\infty}^{\infty} \tilde{L}_m e^{-2im(\tau-\sigma)} , \tag{4.12}$$

where we have defined $\tilde{L}_m$ as

$$\tilde{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n . \tag{4.13}$$

Note that, once again, these expressions only hold for the classical case and must be modified when we quantize our bosonic string theory. Also, note that we can write the Hamiltonian in terms of the newly defined quantities L_m and $\tilde{L}_m$; for a closed string we have that

$$H = 2(L_0 + \tilde{L}_0), \tag{4.14}$$

while for an open string

$$H = L_0. \tag{4.15}$$

Now that we have an expression for the Hamiltonian and the stress-energy tensor in terms of the modes we can use them to derive a mass formula for both the classical open and closed string theories.

4.3 Classical Mass Formula for a Bosonic String

We have seen that, classically, all the components of the stress-energy tensor vanish[‡] , $T_{\alpha\beta} = 0$, which, only classically, corresponds to having $L_m = 0$ and $\tilde{L}_m = 0$ for all m. Also, recall the mass-energy relation,

$$M^2 = -p^\mu p_\mu. \tag{4.16}$$

Now, for our bosonic string theory we have that

$$p^\mu = \int_{\sigma=0}^{\pi} d\sigma\, P^\mu = T \int_{\sigma=0}^{\pi} d\sigma\, \dot{X}^\mu = \begin{cases} \frac{2\alpha_0^\mu}{l_s} & \text{for a closed string,} \\ \frac{\alpha_0^\mu}{l_s} & \text{for an open string,} \end{cases} \tag{4.17}$$

and so

$$p^\mu p_\mu = \begin{cases} \frac{2\alpha_0^2}{\alpha'} & \text{for a closed string,} \\ \frac{\alpha_0^2}{2\alpha'} & \text{for an open string,} \end{cases} \tag{4.18}$$

where $\alpha' = l_s^2/2$. Combining all of this we get, for the open string,

$$0 = L_0 = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot \alpha_n$$

$$= \frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha_{-n} \cdot \alpha_n + \frac{1}{2} \alpha_0 \cdot \alpha_0$$

$$= \frac{1}{2} \left(\sum_{n=-\infty}^{-1} \alpha_{-n} \cdot \alpha_n + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \right) + \underbrace{\frac{1}{2} \alpha_o^2}_{\alpha' p^\mu p_\mu}$$

$$= \frac{1}{2} \left(\sum_{m=1}^{\infty} \alpha_m \cdot \alpha_{-m} + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \right) + \alpha' p^\mu p_\mu$$

$$= \frac{1}{2} \left(\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \right) + \alpha' p^\mu p_\mu$$

$$= \frac{1}{2} (2) \left(\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \right) + \underbrace{\alpha' p^\mu p_\mu}_{=M^2},$$

[‡]This follows from the field equations for the auxiliary field $h^{\alpha\beta}(\tau,\sigma)$, i.e. the field equations are given by setting $\frac{\delta S_\sigma}{\delta h^{\alpha\beta}}$ equal to zero, which implies that $T_{\alpha\beta} = 0$, see proposition 2.2

where in the fourth line we relabeled the first sum by letting $n \mapsto -m$. So, for an open string we have the following mass formula:

$$M^2 = \frac{1}{\alpha'} \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n . \tag{4.19}$$

For the closed string we have to take into account both left and right movers and thus we must use both the conditions $L_m = 0$ and $\tilde{L}_m = 0$. When one does this they find that the closed string mass formula is given by

$$M^2 = \frac{2}{\alpha'} \sum_{n=1}^{\infty} \left(\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \right). \tag{4.20}$$

These are the mass-shell conditions for open and closed strings and they tell you the mass corresponding to a certain classical string state. They are only valid classically since the expressions for $T_{\alpha\beta}$ and H, in which they were derived, are only valid classically. In the quantized theory they will get altered a bit.

4.4 Witt Algebra (Classical Virasoro Algebra)

The set of elements $\{L_m\}$ forms an algebra whose multiplication is given by

$$\{L_m, L_n\}_{P.B.} = i(m - n)L_{m+n}, \tag{4.21}$$

where $\{\cdot, \cdot\}_{P.B.}$ is the Poisson bracket. To see this simply mode expand L_m and L_n and then use the fact that the operation $\{\cdot, \cdot\}$ is linear along with the Poisson bracket relations for the modes α and $\tilde{\alpha}$, see problem 4.3. This algebra is called the Witt algebra or the classical Virasoro algebra. A good question to ask is what is the physical meaning of the L_m's?

Last week we gauge fixed the metric $h_{\alpha\beta}$ to the flat metric $\eta_{\alpha\beta}$. However, as was already mentioned, this does not completely gauge fix the diffeomorphism and Weyl symmetries. For instance, consider the transformations given by

$$\delta_D \eta^{\alpha\beta} = -(\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha),$$

$$\delta_W \eta^{\alpha\beta} = \Lambda \eta^{\alpha\beta} ,$$

where ξ^α is an infinitesimal parameter of reparameterization, Λ is an infinitesimal parameter for Weyl rescaling, $\delta_D \eta^{\alpha\beta}$ gives the variation of the metric under reparameterization and $\delta_W \eta^{\alpha\beta}$ give the variation under a Weyl rescaling. If we combine these two transformations we get

$$(\delta_D + \delta_W)\eta^{\alpha\beta} = \left(-\partial^\alpha \xi^\beta - \partial^\beta \xi^\alpha + \Lambda \eta^{\alpha\beta} \right). \tag{4.22}$$

Now, what is the most general solution for ξ and Λ such that the above equation is zero? If we can find these then it means that we have found additional symmetries for our system, which correspond to reparametrizations which are also Weyl rescalings, i.e. conformal transformations.

To solve the above equation for ξ and Λ we will use light-cone coordinates,

$$\xi^{\pm} = \xi^0 \pm \xi^1,$$

$$\sigma^{\pm} = \tau \pm \sigma.$$

In terms of the light-cone coordinates, the equation to be solved becomes

$$\partial^{\alpha}\xi^{\beta} + \partial^{\beta}\xi^{\alpha} = \Lambda\eta^{\alpha\beta}. \tag{4.23}$$

To solve this we need to find the solutions when $\alpha = \beta = +$, $\alpha = \beta = -$ and $\alpha = +, \beta = -$, while the solution when $\alpha = -, \beta = +$ follows from symmetry of $\eta_{\alpha\beta}$. So, consider:

1. $\alpha = \beta = +$: Noting that $\eta^{++} = 0$ we have to solve

$$\partial^+\xi^+ + \partial^+\xi^+ = \Lambda\eta^{++},$$

$$\implies 2\partial^+\xi^+ = 0,$$

$$\implies \partial^+\xi^+ = 0.$$

The solution to the above is given by some arbitrary function whose argument is only a function of σ^-, which we denote as $\xi^-(\sigma^-)$.

2. $\alpha = \beta = -$: In this case we have, since $\eta^{--} = 0$,

$$\partial^-\xi^- + \partial^-\xi^- = \Lambda\eta^{--},$$

$$\implies 2\partial^-\xi^- = 0,$$

$$\implies \partial^-\xi^- = 0,$$

which has as its solution some arbitrary function of σ^+ only. This solution is denoted by $\xi^+(\sigma^+)$.

3. $\alpha = +, \beta = -$: For this case we have that

$$\partial^+\xi^- + \partial^-\xi^+ = \Lambda\eta^{-+},$$

$$\implies \partial^+\xi^- + \partial^-\xi^+ = -2\Lambda.^{\S}$$

So, local transformations which satisfy

$$\delta\sigma^+ = \xi^+(\sigma^+),$$

$$\delta\sigma^- = \xi^-(\sigma^-),$$

$$\Lambda = \partial^-\xi^+ + \partial^+\xi^-,$$

leave our theory invariant. Thus, we have found another set of gauge transformations that our bosonic string theory is invariant under and, as before, they can be used to further fix the form of the metric $h^{\alpha\beta}$. Note that the infinitesimal generators for the transformations $\delta\sigma^\pm = \xi^\pm$ are given by

$$V^\pm = \frac{1}{2}\xi^\pm(\sigma^\pm)\frac{\partial}{\partial\sigma^\pm},$$

and a complete basis for these transformations is given by

$$\xi_n^\pm(\sigma^\pm) = e^{2in\sigma^\pm}, \qquad n \in \mathbb{Z}.$$

The corresponding generators $V_n^\pm$ give two copies of the Witt algebra, while in the case of the open string there is just one copy of the Witt algebra, and the infinitesimal generators are given by

$$V_n^\pm = e^{in\sigma^+}\frac{\partial}{\partial\sigma^+} + e^{in\sigma^-}\frac{\partial}{\partial\sigma^-}, \qquad n \in \mathbb{Z}.$$

Now that we have studied the classical bosonic string theory and all of its properties and structure we will look at the quantized theory.

4.5 Canonical Quantization of the Bosonic String

We will first quantize the bosonic string theory in terms of canonical quantization, while later we will look at the light-cone gauge quantization of the theory.

So, in the canonical quantization procedure, we quantize the theory by changing Poisson brackets to commutators,

$$\{\cdot, \cdot\}_{P.B.} \mapsto i[\cdot, \cdot], \tag{4.24}$$

and we promote the field X^μ to an operator in our corresponding Hilbert space. This is equivalent to promoting the modes α, the constant x^μ and the total momentum p^μ to operators. In particular, for the modes α_m^μ, we have that (here and usually in the sequel we are dropping the i factor)

$$[\hat{\alpha}_m^\mu, \hat{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m,-n},$$

$$[\hat{\tilde{\alpha}}_m^\mu, \hat{\tilde{\alpha}}_n^\nu] = m\eta^{\mu\nu}\delta_{m,-n},$$

$$[\hat{\alpha}_m^\mu, \hat{\tilde{\alpha}}_n^\nu] = 0\,,$$

where the $\hat{\alpha}$'s on the RHS are realized as operators in a Hilbert space, while the α's on the LHS are just the modes. If we define new operators as $\hat{a}_m^\mu \equiv \frac{1}{\sqrt{m}}\hat{\alpha}_m^\mu$ and $\hat{a}_m^{\mu\dagger} \equiv \frac{1}{\sqrt{m}}\hat{\alpha}_{-m}^{\mu\dagger}$, for $m > 0$, then they clearly satisfy[§]

$$[\hat{a}_m^\mu, \hat{a}_n^{\nu\dagger}] = [\hat{\tilde{a}}_m^\mu, \hat{\tilde{a}}_n^{\nu\dagger}] = \eta^{\mu\nu}\delta_{m,n} \qquad \text{for } m, n > 0. \tag{4.25}$$

This looks like the same algebraic structure as the algebra constructed from the creation/annihilation operators of quantum mechanics, except that for $\mu = \nu = 0$ we get a negative sign, due to the signature of the metric,

$$[\hat{a}_m^0, \hat{a}_n^{0\dagger}] = \eta^{00}\delta_{m,n} = -\delta_{m,n}. \tag{4.26}$$

We will see later that this negative sign in the commutators leads to the prediction of negative norm physical states, or ghost states, which is incorrect.

Next, we define the ground state[‡], which is denoted by $|0\rangle$, as the state which is annihilated by all of the lowering operators $\hat{a}_m^\mu$,

$$\hat{a}_m^\mu|0\rangle = 0 \qquad \text{for } m > 0. \tag{4.27}$$

Also, physical states are states that are constructed by acting on the ground state with the raising operators $\hat{a}_m^{\mu\dagger}$,

$$|\phi\rangle = \hat{a}_{m_1}^{\mu_1\dagger}\hat{a}_{m_2}^{\mu_2\dagger}\cdots\hat{a}_{m_n}^{\mu_n\dagger}|0; k^\mu\rangle, \tag{4.28}$$

[§]Note that we are also assuming that $(\hat{a}_m^{\mu\dagger})^\dagger = \hat{\alpha}_m^\mu$.

[‡]The fact that there exists a ground state follows from the Stone-von Neumann theorem, see Altland and Simons "Condensed Matter Field Theory" page 45-46 info block.

which are also eigenstates of the momentum operator $\hat{p}^\mu$,

$$\hat{p}^\mu|\phi\rangle = k^\mu|\phi\rangle. \tag{4.29}$$

It should be pointed out that this is first quantization, and all of these states, including the ground state, are one-particle states.

To prove the claim of negative norm states, consider the state $|\psi\rangle = \hat{a}_m^{0\dagger}|0; k^\mu\rangle$, for $m > 0$, then we have that

$$\||\psi\rangle\|^2 = \langle 0|\hat{a}_m^0 \hat{a}_m^{0\dagger}|0\rangle$$

$$= \langle 0|[\hat{a}_m^0, \hat{a}_m^{0\dagger}]|0\rangle$$

$$= -\langle 0|0\rangle .$$

So, if we define $\langle 0|0\rangle$ to be positive then we can see that we will get some negative norm states, while if we define $\langle 0|0\rangle$ to be negative then there will be other states which have negative norm, in particular any state not of the same form as $|\psi\rangle$. These negative norm states are a problem because they are unphysical and we don't want our string theory to predict unphysical states. The good news is that we can remove these negative norm states but the bad news is that it will put a constraint on the number of dimensions of the background spacetime in which our theory is defined. This will be shown in the next lecture.

4.6 Virasoro Algebra

We have seen that when we quantize our bosonic string theory the modes α become operators. This then implies that the generators L_m will also become operators since they are constructed from the α's. However, one must be careful because we simply cannot just say that $\hat{L}_m$ is given by

$$\hat{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \hat{\alpha}_{m-n} \cdot \hat{\alpha}_n, \qquad \text{(wrong!)}$$

but we must, as in QFT, normal order[‡] the operators and thus we define $\hat{L}_m$ to be

$$\hat{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} : \hat{\alpha}_{m-n} \cdot \hat{\alpha}_n : \ . \tag{4.31}$$

Note that normal ordering ambiguity only arises for the case when $m = 0$, i.e for the operator $\hat{L}_0$. In normal ordering, we have that $\hat{L}_0$ is given by

$$\hat{L}_0 = \frac{1}{2} \hat{\alpha}_0^2 + \sum_{n=1}^{\infty} \hat{\alpha}_{-n} \cdot \hat{\alpha}_n. \tag{4.32}$$

This can be seen as follows,

$$\hat{L}_0 \equiv \frac{1}{2} \sum_{-\infty}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n :$$

$$= \frac{1}{2} \hat{\alpha}_0^2 + : \frac{1}{2} \sum_{n=-\infty}^{-1} \hat{\alpha}_{-n} \cdot \hat{\alpha}_n : + : \frac{1}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{-n} \cdot \hat{\alpha}_n :$$

$$= \frac{1}{2} \hat{\alpha}_0^2 + \frac{1}{2} \sum_{n=-\infty}^{-1} \hat{\alpha}_n \cdot \hat{\alpha}_{-n} + \frac{1}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{-n} \cdot \hat{\alpha}_n$$

$$= \frac{1}{2} \hat{\alpha}_0^2 + \frac{1}{2} \sum_{m=1}^{\infty} \hat{\alpha}_{-m} \cdot \hat{\alpha}_m + \frac{1}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{-n} \cdot \hat{\alpha}_n$$

$$= \frac{1}{2} \hat{\alpha}_0^2 + \sum_{n=1}^{\infty} \hat{\alpha}_{-n} \cdot \hat{\alpha}_n \, ,$$

where in the second to last line we relabeled $n \mapsto -m$ and in the last line we relabeled $m \mapsto n$. We introduce normal ordering due to the fact that there is an ordering ambiguity arising from the commutation relations of the operators $\hat{\alpha}$ and $\hat{\tilde{\alpha}}$. When we commute the operators past each other we pick up extra constants. So, how do we know what order to put the operators in? The answer is that we do not. We simply take the correct ordering to be normal ordering. Note that due to normal ordering we expect to

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[‡]Normal ordering is defined to be

$$: \alpha_i \cdot \alpha_j : \ = \begin{cases} \alpha_i \cdot \alpha_j & \text{when } i \leq j \ , \\ \alpha_j \cdot \alpha_i & \text{when } i > j \ , \end{cases} \tag{4.30}$$

which says that we put an operator with a lower index to the left of an operator with a higher index, which is equivalent, in our case, to saying that we put all lowering operators to the left of raising operators.

pick up extra constants due to moving creation modes to the left of annihilation modes (commutation relations), and so, we should expect to see these constants when we look at expressions concerning $\hat{L}_0$. See, for example section 4.7.

Also, in terms of the commutation relations for the operators $\hat{\alpha}$, we get that the commutation relations for the operators $\hat{L}_m$ are given by

$$[\hat{L}_m, \hat{L}_n] = (m - n)\hat{L}_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m,-n}, \tag{4.33}$$

where c is called the central charge. We will see that, in the bosonic string theory, c is equal to the dimension of the spacetime where the theory lives, and in order to no longer have non-negative norm states it must be that $c = 26$. Also, note that for $m = -1, 0, 1$ the c term drops out and we get a subalgebra of $SL(2, \mathbb{R})$, i.e the set $\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1\}$ along with the relations

$$[\hat{L}_m, \hat{L}_n] = (m - n)\hat{L}_{m+n},$$

becomes an algebra which is isomorphic to $SL(2, \mathbb{R})$.

Now we will give a more concrete definition of physical states in terms of the Virasoro operators.

4.7 Physical States

Classically we have seen that $L_0 = 0$ since to the vanishing of the stress-energy tensor implies that $L_m = 0$ for all m, but when we quantize the theory we cannot say that $\hat{L}_0 = 0$, or equivalently $\hat{L}_0|\phi\rangle = 0$ for all physical states, follows from this as well because when we quantize the theory we have to normal order the operator $\hat{L}_0$ and so we could have some arbitrary constant due to this normal ordering. Thus, after quantizing we can at best say that for an open string the vanishing of the L_0 constraint transforms to

$$(\hat{L}_0 - a)|\phi\rangle = 0, \tag{4.34}$$

where a is a constant. This is called the mass-shell condition for the open string. While for a closed string we have that

$$(\hat{L}_0 - a)|\psi\rangle = 0, \tag{4.35}$$

$$(\hat{\bar{L}}_0 - a)|\psi\rangle = 0, \tag{4.36}$$

where $\hat{\bar{L}}$ is the operator corresponding to the classical generator $\tilde{L}$.

Normal ordering also adds correction terms to the mass formula. For an open string theory, the mass formula becomes

$$\alpha' M^2 = \frac{1}{\alpha'} \sum_{n=1}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n : -a = \hat{N} - a, \tag{4.37}$$

where we have defined the number operator $\hat{N}$ as

$$\hat{N} = \sum_{n=1}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n : = \sum_{n=1}^{\infty} n : \hat{a}_n^\dagger \cdot \hat{a}_n : . \tag{4.38}$$

We can use the number operator to compute the mass spectrum,

$$\alpha' M^2 = -a \qquad \text{(ground state n =0)}$$

$$\alpha' M^2 = -a + 1 \qquad \text{(first excited state n =1)}$$

$$\alpha' M^2 = -a + 2 \qquad \text{(second excited state n =2)}$$

$$.$$
$$.$$
$$.$$

For a closed string we have the mass formula

$$\frac{4}{\alpha'} M^2 = \sum_{n=1}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n : - a = \sum_{n=1}^{\infty} : \hat{\tilde{\alpha}}_{-n} \cdot \hat{\tilde{\alpha}}_n : - a, \tag{4.39}$$

or,

$$\hat{N} - a = \hat{\tilde{N}} - a, \tag{4.40}$$

where $\hat{N}$ is the number operator for right movers and $\hat{\tilde{N}}$ is the number operator for left movers. Also, note that if we subtract the left moving physical state condition,(4.35), from the right moving physical state condition, (4.36), we get that

$$\left(\hat{L}_0 - a - \hat{\tilde{L}} + a \right) |\phi\rangle = 0, \tag{4.41}$$

which implies that

$$\left(\hat{L}_0 - \hat{\tilde{L}}_0 \right) |\phi\rangle = 0, \tag{4.42}$$

which in turn implies that

$$\hat{N} = \hat{\tilde{N}}. \tag{4.43}$$

This is known as the level matching condition of the bosonic string and it is the only constraint that relates the left moving and right moving modes.

Virasoro Generators and Physical States

Classically we have that $L_m = 0$ for all m, which we know does not hold for $\hat{L}_0$, but what about the operators $\hat{L}_m$ for $m \neq 0$? Well, if $\hat{L}_m|\phi\rangle = 0$ for all $m \neq 0$ then we would have that (if we take n in such a way that $n + m \neq 0$)

$$[\hat{L}_m, \hat{L}_n]|\phi\rangle = 0. \tag{4.44}$$

But when we plug in the commutation relations we get

$$(m - n)\hat{L}_{n+m}|\phi\rangle + \frac{c}{12}m(m^2 - 1)\delta_{m,-n}|\phi\rangle = 0 \tag{4.45}$$

and since the first term vanishes (because we are assuming $\hat{L}_m|\phi\rangle = 0$ for all $m \neq 0$) we see that if $c \neq 0$ then it must be that either $m = -1$, $m = 0$ or $m = 1$. Thus, if we want to have $\hat{L}_m|\phi\rangle = 0$ for all m then we must restrict our Virasoro algebra to only $\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1\}$. Instead of doing this we will only impose that $\hat{L}_m|\phi\rangle = 0 = \langle\phi|\hat{L}_m^\dagger$ for $m > 0$. Physical states are then characterized by

$$\hat{L}_{m>0}|\phi\rangle = 0 = \langle\phi|\hat{L}_{m>0}^\dagger, \tag{4.46}$$

and the mass-shell condition

$$(\hat{L}_0 - a)|\phi\rangle = 0. \tag{4.47}$$

Equivalently, one could replace (4.46) by $\langle\phi|\hat{L}_m = 0 = \hat{L}_m^\dagger|\phi\rangle = 0$ for all $m < 0$.

As an aside:

Classically, the Lorentz generators (or charges), $Q^{\mu\nu}$, are given by the spatial integral of the time component of the current corresponding to the Lorentz transformations, $j_\alpha^{\mu\nu}$, i.e.

$$Q^{\mu\nu} = \int_{\sigma=0}^{\pi} d\sigma\, j_0^{\mu\nu}. \tag{4.48}$$

We can rewrite this as

$$Q^{\mu\nu} = T \int_{\sigma=0}^{\pi} d\sigma \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right), \tag{4.49}$$

which can be expanded into modes as

$$Q^{\mu\nu} = \left(x^\mu p^\nu - x^\nu p^\mu\right) - 1 \sum_{m=1}^{\infty} \frac{1}{m}\left(\alpha_{-m}^\mu \alpha_m^\nu - \alpha_{-m}^\nu \alpha_m^\mu\right), \tag{4.50}$$

and since this does not have any normal ordering ambiguities we can quantize the expression as

$$\hat{Q}^{\mu\nu} = \left(\hat{x}^\mu \hat{p}^\nu - \hat{x}^\nu \hat{p}^\mu\right) - 1 \sum_{m=1}^{\infty} \frac{1}{m}\left(\hat{\alpha}_{-m}^\mu \hat{\alpha}_m^\nu - \hat{\alpha}_{-m}^\nu \hat{\alpha}_m^\mu\right). \tag{4.51}$$

Now that we have an operator expression for the Lorentz generators, we can compute the commutator with them and the Virasoro generators, and when we do this we see that $[\hat{L}_m, \hat{Q}^{\mu\nu}] = 0$ which implies that physical states (defined in terms of the Virasoro generators) appear in complete Lorentz multiplets[‡]. This will allow us to calculate representations of the Virasoro algebra, or states, and then be able to relate them to representations of the Lorentz group to see what type of particle our physical state corresponds to. But then a question arises: Shouldn't we relate them to representations of the Poincaré group since they describe the fundamental particles?

In the next chapter we will define spurious states and then use them to show that if we are to have a theory which is free of negative norm states then the constant a appearing in the mass formula is fixed at unity while the central charge c of the Virasoro algebra, or equivalently the dimension of the spacetime, is fixed at 26.

[‡]This is because the vanishing of the commutator implies that the physical state condition is invariant under Lorentz transformations.

4.8 Exercises

Problem 1

a) Consider a closed string with the sigma model action in conformal gauge

$$S = \frac{T}{2} \int d^2\sigma (\dot{X}^2 - X'^2), \tag{4.52}$$

where $d^2\sigma \equiv d\tau d\sigma$. Show that in the world-sheet light-cone coordinates $\sigma^\pm = \tau \pm \sigma$ the action takes the form

$$S = T \int d\sigma^+ d\sigma^- \partial_+ X^\mu \partial_- X_\mu. \tag{4.53}$$

b) Show that this action is invariant under the infinitesimal transformation

$$\delta X^\mu = a_n e^{2in\sigma^-} \partial_- X^\mu, \tag{4.54}$$

where a_n is a constant infinitesimal parameter.

c) As we showed in class for any continuous symmetry of the Lagrangian there is a corresponding conserved current, called the Noether current. Starting from (4.53), use the Noether method to show that the Noether current corresponding to the symmetry (4.54) is given by

$$j^+ = T(\partial_- X^\mu \partial_- X_\mu)e^{2in\sigma^-}, \qquad j^- = 0. \tag{4.55}$$

In other words, make the parameter a_n in (4.54) local and show that the variation of the action (4.53) is proportional to $\partial_\pm a_n$. The coefficients of $\partial_\pm a_n$ are the components $j^\pm$ of the Noether current in light cone coordinates. Verify that the current (4.55) is conserved when the field equations hold.

d) The Noether current transforms as a vector under coordinate changes. Use this to write the component $j^0 = j^\tau$ in terms of j^+, j^-.

e) The Noether charge is defined by

$$Q = \int d\sigma \, j^0. \tag{4.56}$$

Using your result from d), show that the Noether charge corresponding to the symmetry (4.54) is equal to the Virasoro generator L_n

$$\int d\sigma \, j^0 = L_n. \tag{4.57}$$

In deriving this relation, you can use the fact that the current j^+ in (4.55) is related to the stress-energy tensor T_{--} in BBS (2.36), and the fact thtat T_{--} can be expanded in L_m as in BBS (2.73).

Problem 2

Problem 2.3 of Becker, Becker, Schwarz "String Theory and M-Theory".

Clarification: in part (i), you are supposed to compute spacetime momentum current $P_\alpha^{25} = T\partial_\alpha X^{25}$ defined in BBS (2.67) and see if it is conserved or not because of the modified boundary condition.

Problem 3

For this problem:

1. Prove equations (2.73) and (2.74) in BBS, starting from the expressions for the energy-momentum tensor in terms of the field X^μ (2.36)–(2.37) and using the mode expansion (2.40)–(2.41).

2. Using the Poisson brackets (2.51)–(2.52) derive the Virasoro algebra (2.84).

5. Removing Ghost States and Light-Cone Quantization

Last week we quantized our bosonic string theory by promoting the terms x^μ, p^μ, α, and $\tilde{\alpha}$ into operators, as well as the Witt generators L_m and $\tilde{L}_m$, and by replacing the Poisson brackets with commutators $\{\cdot, \cdot\} \mapsto i[\cdot, \cdot]$, which gave us the commutation relations for the newly defined operators, for example

$$[\hat{x}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu}, \qquad [\hat{\alpha}_m^\mu, \hat{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m,-n}. \qquad (5.1)$$

The physical Hilbert space for the theory is given by taking the Fock space that is generated by vectors $|\psi\rangle$ of the form

$$|\psi\rangle = \hat{a}_{m_1}^{\mu_1\dagger}\hat{a}_{m_2}^{\mu_2\dagger}\cdots\hat{a}_{m_n}^{\mu_n\dagger}|0; k^\mu\rangle,$$

and then moding out by the subspace formed from the constraint $\hat{L}_{m>0}|\psi\rangle = 0$. More precisely, we had that the physical states $|\phi\rangle$ were defined as states, $|\phi\rangle = \hat{a}_{m_1}^{\mu_1\dagger}\hat{a}_{m_2}^{\mu_2\dagger}\cdots \hat{a}_{m_n}^{\mu_n\dagger}|0; k^\mu\rangle$, which obeyed the following two constraints

$$\left(\hat{L}_0 - a\right)|\phi\rangle = 0, \qquad (5.2)$$

$$\hat{L}_{m>0}|\phi\rangle = 0. \qquad (5.3)$$

We also saw that there were certain physical states whose norm was less than zero, a trait that no physical state can have. However, as was already mentioned, we can get rid of these negative norm physical states by constraining the constant a, appearing in (5.2), and by also constraining the central charge of the Virasoro algebra. Note that since the value of the central charge of the Virasoro algebra is equivalent to the number of dimensions of the background spacetime in which our theory is

Figure 5: This is a pictorially representation of the physical Hilbert space for our theory. Note that the boundary of the circle is where certain positively normed states turn negative and thus we get an increase in the amount of states with zero norm at this boundary.

defined, we will also get a constraint on the allowed dimensions for the background. Pictorially, our physical Hilbert space looks like the one in figure 5. As an aside, it turns out that the boundary of the circle, see figure 5, is related to physical principles, gauge symmetries, and thus is an important area of study, see AdS/CFT.

In order to construct a theory which is free of negative norm physical states we should study physical states of zero norm which satisfy the physical state conditions. Thus we need to introduce spurious states.

5.1 Spurious States

A state, $|\psi\rangle$, is said to be spurious if it satisfies the mass-shell condition,

$$\left(\hat{L}_0 - a\right)|\phi\rangle = 0,$$

and is orthogonal to all other physical states,

$$\langle\phi|\psi\rangle = 0, \qquad \forall \text{ physical states } |\phi\rangle.$$

We can think of the set of all spurious states as the vacuum state. This is due to the fact that the vacuum state is an orthogonal subspace to the space of all physical states.

In general, a spurious state can be written as

$$|\psi\rangle = \sum_{n=1}^{\infty} \hat{L}_{-n}|\chi_n\rangle, \tag{5.4}$$

where $|\chi_n\rangle$ is some state which satisfies the, now modified, mass-shell condition given by

$$\left(\hat{L}_0 - a + n\right)|\chi_n\rangle = 0. \tag{5.5}$$

This follows from the definition of a spurious state, since if

$$\langle\phi|\psi\rangle = 0,$$

then

$$\hat{L}_0|\psi\rangle - a|\psi\rangle = 0,$$

$$\Rightarrow \hat{L}_0 \left(\sum_{n=1}^{\infty} \hat{L}_{-n}|\chi_n\rangle\right) - a|\psi\rangle = 0,$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\hat{L}_0\hat{L}_{-n}|\chi_n\rangle\right) - a|\psi\rangle = 0,$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\left[\hat{L}_0, \hat{L}_{-n}\right] + \hat{L}_{-n}\hat{L}_0\right)|\chi_n\rangle - a|\psi\rangle = 0,$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(n\hat{L}_{-n} + \hat{L}_{-n}\hat{L}_0\right)|\chi_n\rangle - a|\psi\rangle = 0, \qquad \left(\text{from (4.33)}\right),$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(n\hat{L}_{-n} + \hat{L}_{-n}\hat{L}_0 \right)|\chi_n\rangle - \sum_{n=1}^{\infty} a\hat{L}_{-n}|\chi_n\rangle = 0,$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\hat{L}_{-n}n + \hat{L}_{-n}\hat{L}_0 - \hat{L}_{-n}a \right)|\chi_n\rangle = 0,$$

$$\Rightarrow \sum_{n=1}^{\infty} \hat{L}_{-n} \left(\hat{L}_0 - a + n \right)|\chi_n\rangle = 0 \,,$$

which holds for all states $|\chi_n\rangle$, and thus

$$\Rightarrow \left(\hat{L}_0 - a + n \right)|\chi_n\rangle = 0.$$

Now, since any $\hat{L}_{-n}$, for $n \geq 1$, can be written as a combination of $\hat{L}_{-1}$ and $\hat{L}_{-2}$ the general expression for a spurious state (5.4) can be simplified to

$$|\psi\rangle = \hat{L}_{-1}|\chi_1\rangle + \hat{L}_{-2}|\chi_2\rangle, \qquad (5.6)$$

where $|\chi_1\rangle$ and $|\chi_2\rangle$ are called level 1 and level 2 states, respectively, and they satisfy the mass-shell conditions $\left(\hat{L}_0 - a + 1 \right)|\chi_1\rangle = 0$ and $\left(\hat{L}_0 - a + 2 \right)|\chi_2\rangle = 0$, respectively. For example, consider the level 3 state given by $|\psi\rangle = \hat{L}_{-3}|\chi_3\rangle$. We have that $\left(\hat{L}_0 - a + 3 \right)|\chi_3\rangle = 0$ as well as $\hat{L}_{-3} = \left[\hat{L}_{-1}, \hat{L}_{-2} \right]$ which gives us that

$$\hat{L}_{-3}|\chi_3\rangle = \left[\hat{L}_{-1}, \hat{L}_{-2} \right]|\chi_3\rangle$$

$$= \hat{L}_{-1}\hat{L}_{-2}|\chi_3\rangle - \hat{L}_{-2}\hat{L}_{-1}|\chi_3\rangle$$

$$= \hat{L}_{-1}\left(\hat{L}_{-2}|\chi_3\rangle \right) + \hat{L}_{-2}\left(\hat{L}_{-1}| - \chi_3\rangle \right),$$

which is of the form $\hat{L}_{-1}|\chi_1\rangle + \hat{L}_{-2}|\chi_2\rangle$ for some level 1 state $|\chi_1\rangle$ $(= \hat{L}_{-2}|\chi_3\rangle)$ and some level 2 state $|\chi_2\rangle$ $(= \hat{L}_{-1}| - \chi_3\rangle)$. Now, we just need to check that

$$\left(\hat{L}_0 - a + 1 \right)|\chi_1\rangle = 0,$$

and

$$\left(\hat{L}_0 - a + 2 \right)|\chi_2\rangle = 0.$$

First, note that $\hat{L}_{-n}$ raises the eigenvalue of the operator $\hat{L}_0$ by the amount n. To see this let $|h\rangle$ be a state such that $\hat{L}_0|h\rangle = h|h\rangle$, then

$$\hat{L}_0\left(\hat{L}_{-n}|h\rangle\right) = \left(\left[\hat{L}_0, \hat{L}_{-n}\right] + \hat{L}_{-n}\hat{L}_0\right)|h\rangle$$

$$= \left(n\hat{L}_{-n} + h\hat{L}_{-n}\right)|h\rangle$$

$$= \left(n + h\right)\hat{L}_{-n}|h\rangle,$$

and so $\hat{L}_{-n}|h\rangle$ is an eigenvector of $\hat{L}_0$ with eigenvalue $n+h$. Now, if $\left(\hat{L}_0 - a + 3\right)|\chi_3\rangle = 0$ then we have that $\hat{L}_0|\chi_3\rangle = (a-3)|\chi_3\rangle$ and so

$$\hat{L}_0\left(\hat{L}_{-2}|\chi_3\rangle\right) = \left(a - 3 + 2\right)\hat{L}_{-2}|\chi_3\rangle,$$

which, by the previous claim, implies that

$$\left(\hat{L}_0 - a + 1\right)\hat{L}_{-2}|\chi_3\rangle = 0.$$

But remember that $|\chi_1\rangle = \hat{L}_{-2}|\chi_3\rangle$, giving us

$$\left(\hat{L}_0 - a + 1\right)|\chi_1\rangle = 0.$$

Similarly,

$$\hat{L}_0\left(\hat{L}_{-1}|\chi_3\rangle\right) = \left(a - 3 + 1\right)\hat{L}_{-1}|\chi_3\rangle,$$

which implies that

$$\left(\hat{L}_0 - a + 2\right)\hat{L}_{-1}|\chi_3\rangle = 0,$$

or that,

$$\left(\hat{L}_0 - a + 2\right)|\chi_2\rangle = 0.$$

Thus, we have just shown that the level 3 state can be written as the linear combination $\hat{L}_{-1}|\chi_1\rangle + \hat{L}_{-2}|\chi_2\rangle$. The proof for level n states, for all n, follows by similar arguments.

To see that a spurious state $|\psi\rangle$ is orthogonal to any physical state $|\phi\rangle$ consider,

$$\langle\phi|\psi\rangle = \sum_{n=1}^{\infty}\langle\phi|\hat{L}_{-n}|\chi_n\rangle$$

$$= \sum_{n=1}^{\infty}\left(\langle\chi_n|\hat{L}_n|\phi\rangle\right)^*$$

$$= \sum_{n=1}^{\infty}\left(\langle\chi_n|0|\phi\rangle\right)^*$$

$$= 0\,,$$

where the second line follows from the fact that $\hat{L}^{\dagger}_{-n} = \hat{L}_n$ and the third line follows from the fact that since $|\phi\rangle$ is a physical state it is annihilated by all $\hat{L}_{n>0}$.

Since a spurious state $|\psi\rangle$ is perpendicular to all physical states, if we require that $|\psi\rangle$ also be a physical state then by definition it is perpendicular to itself, i.e. $|\psi\rangle$ has zero norm

$$\||\psi\rangle\|^2 = \langle\psi|\psi\rangle = 0, \tag{5.7}$$

since $|\psi\rangle$ is perpendicular to all physical states and $|\psi\rangle$ is itself a physical state.

Thus, we have succeeded in constructing physical states whose norm is zero and these are precisely the states we need to study in order to get rid of the negative norm physical states in our bosonic string theory.

5.2 Removing the Negative Norm Physical States

We want to study physical spurious states in order to determine the values of a and c that project out the negative norm physical states, also called ghost states. So, in order to find the corresponding a value we should start with a level 1 physical spurious state,

$$|\psi\rangle = \hat{L}_{-1}|\chi_1\rangle, \tag{5.8}$$

with $|\chi_1\rangle$ satisfying $(\hat{L}_0 - a + 1)|\chi_1\rangle = 0$ and $\hat{L}_{m>0}|\chi_1\rangle = 0$, where the last relation comes because we have assumed $|\psi\rangle$ to be physical. The reason why we look at a level 1 spurious state is because we want to fix a and so therefore we need an expression that has a in it, and also, to make our life easier, we take the simplest expression that has an a in it, which is a level 1 state. Now, if $|\psi\rangle$ is physical, which we have assumed, then it must satisfy the mass-shell condition for physical states,

$$\left(\hat{L}_0 - a\right)|\psi\rangle = 0, \tag{5.9}$$

along with the condition

$$\hat{L}_{m>0}|\psi\rangle = 0. \tag{5.10}$$

So, if $\hat{L}_{m>0}|\psi\rangle = 0$ then this holds for, in particular, the operator $\hat{L}_1$, i.e. $\hat{L}_1|\psi\rangle = 0$, which implies that $0 = \hat{L}_1\big(\hat{L}_{-1}|\chi_1\rangle\big) = \big([\hat{L}_1, \hat{L}_{-1}] + \hat{L}_{-1}\hat{L}_1\big)|\chi_1\rangle = [\hat{L}_1, \hat{L}_{-1}]|\chi_1\rangle = 2\hat{L}_0|\chi_1\rangle = 2(a-1)|\chi_1\rangle$. And thus, since $2(a-1)|\chi_1\rangle$ must be zero (since $|\psi\rangle$ is physical), we have that $a = 1$. This tells us that if $|\psi\rangle$ is to be a physical spurious state then we need to have that $a = 1$ and so $a = 1$ is part of the boundary between positive and negative norm physical states, see figure 5.

Next, in order to determine the appropriate value of c for spurious physical states we need to look at a level 2 spurious state. Note that a general level 2 spurious state is given by

$$|\psi\rangle = \Big(\hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1}\Big)|\chi_2\rangle, \tag{5.11}$$

where γ is a constant, that will be fixed to insure that $|\psi\rangle$ has a zero norm (i.e. physical), and $|\chi_2\rangle$ obeys the relations,

$$\Big(\hat{L}_0 - a + 2\Big)|\chi_2\rangle = 0, \tag{5.12}$$

and

$$\hat{L}_{m>0}|\chi_2\rangle = 0. \tag{5.13}$$

Now, if $|\psi\rangle$ is to be physical, and thus have zero norm, then it must satisfy $\hat{L}_{m>0}|\psi\rangle = 0$ and thus it must satisfy, in particular, $\hat{L}_1|\psi\rangle = 0$. This implies that

$$\Rightarrow \hat{L}_1\Big(\hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1}\Big)|\chi_2\rangle = 0,$$

$$\Rightarrow \Big([\hat{L}_1, \hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1}] + (\hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1})\hat{L}_1\Big)|\chi_2\rangle = 0,$$

$$\Rightarrow \Big([\hat{L}_1, \hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1}]\Big)|\chi_2\rangle = 0,$$

$$\Rightarrow \Big(3\hat{L}_{-1} + 4\gamma\hat{L}_{-1}\hat{L}_0 + \gamma 2\hat{L}_{-1}\Big)|\chi_2\rangle = 0,$$

$$\Rightarrow \Big(3\hat{L}_{-1} - 4\gamma\hat{L}_{-1} + \gamma 2\hat{L}_{-1}\Big)|\chi_2\rangle = 0,$$

$$\Rightarrow \Big(3 - 4\gamma + \gamma 2\Big)\hat{L}_{-1}|\chi_2\rangle = 0,$$

$$\Rightarrow 3 - 2\gamma = 0,$$

$$\Rightarrow \gamma = \frac{3}{2},$$

where the fourth line from the bottom follows from the fact that if $a = 1$ then the condition $(\hat{L}_0 - a + 2)|\chi_2\rangle$ implies that $\hat{L}_0|\chi_2\rangle = -|\chi_2\rangle$. Thus, if a level 2 spurious state is to be physical then we must have $\gamma = 3/2$. With the previous results, any general level 2 physical spurious state is of the form

$$|\psi\rangle = \left(\hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1}\right)|\chi_2\rangle. \tag{5.14}$$

Now, since $\hat{L}_{m>0}|\psi\rangle = 0$, we have that, in particular, $\hat{L}_2|\psi\rangle = 0$ which yields

$$\Rightarrow \hat{L}_2\left(\hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1}\right)|\chi_2\rangle = 0,$$

$$\Rightarrow \left([\hat{L}_2, \hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1}] + (\hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1})\hat{L}_2\right)|\chi_2\rangle = 0,$$

$$\Rightarrow \left([\hat{L}_2, \hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1}]\right)|\chi_2\rangle = 0,$$

$$\Rightarrow \left(13\hat{L}_0 + 9\hat{L}_{-1}\hat{L}_1 + \frac{c}{2}\right)|\chi_2\rangle = 0,$$

$$\Rightarrow \left(-13 + \frac{c}{2}\right)|\chi_2\rangle = 0,$$

$$\Rightarrow c = 26\,.$$

Thus, if we are to have that $|\psi\rangle$ is both spurious and physical then we must have that $c = 26$, which is the other part of the boundary between positive and negative norm physical states, see figure 5.

So, if we want to project out the negative norm physical states (ghost states) then we must restrict the values of a, γ and c to $1, 3/2$ and 26, respectively. Also, note that since the central charge c is equivalent to the dimension of the background spacetime for our bosonic string theory, then our theory is only physically acceptable for the case that it lives in a space of 26 dimensions. The $a = 1$, $c = 26$ bosonic string theory is called critical, and the critical dimension is 26. Finally, there can exist bosonic string theories with non-negative norm physical states for $a \leq 1$ and $c \leq 25$, which are called non-critical.

Previously we showed how to remove the negative norm physical states from our quantized bosonic string theory, at the cost of introducing constraints on the constants a and c. Now, we will quantize the theory in a different manner that will no longer have negative norm physical states at the cost of not being manifestly Lorentz invariant. We can fix this however, at the cost of, once again, constraining the constants a and c.

5.3 Light-Cone Gauge Quantization of the Bosonic String

Instead of the canonical quantization of the bosonic string theory, which was previously carried out, we can quantize the bosonic string theory in another way. Before when we quantized the sting theory we did so in a way that left the theory manifestly Lorentz invariant but it predicted the existence of negative-norm states. In order to get rid of these states we were forced to set $a = 1$ and $c = 26$ as constraints of the Virasoro algebra. Now we will quantize the theory in such a way that it does not predict negative-norm states but it is no longer manifest Lorentz invariance. When we impose that our theory be Lorentz invariant we will see that this forces $a = 1$ and $c = 26$. Also, we will no longer use the hat overtop of operators, i.e. we will write A for $\hat{A}$, unless there is chance for confusion.

As was discussed earlier in section 4.4, even after we choose a gauge such that the spacetime metric $h^{\alpha\beta}$ becomes Minkowskian the bosonic string theory still has residual diffeomorphism symmetries, which consists of all the conformal transformations. In terms of the worldsheet light-cone coordinates σ^+ and σ^-, this residual symmetry corresponds to being able to reparameterize these coordinates as

$$\sigma^\pm \mapsto \sigma'^\pm = \xi^\pm(\sigma^\pm), \tag{5.15}$$

without changing the theory, i.e. the bosonic string action is invariant under these reparameterizations. Therefore, there is still the possibility of making an additional gauge choice, and if one chooses a particular noncovariant gauge, the light-cone gauge (see below), it is then possible to describe a Fock space which is manifestly free of negative-norm states. To proceed we will first define the light-cone coordinates for the background spacetime in which our bosonic string is moving[‡].

In general, one defines light-cone coordinates for a spacetime by taking linear combinations of the temporal coordinate along with another transverse, or spacelike, coordinate. There is no preference on which choice to take for the transverse coordinate, but in our case we will pick the $D - 1$ spacetime coordinate. Thus, the light-cone

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[‡]Note that before when we defined the light-cone coordinates $\sigma^\pm$ they were coordinates for the worldsheet, mapped out by the string as it moved through the background spacetime, and now we are defining light-cone coordinates for the actual background spacetime itself.

coordinates for the background spacetime, X^+ and X^-, are defined as

$$X^+ \equiv \frac{1}{\sqrt{2}}\left(X^0 + X^{D-1}\right),$$

$$X^- \equiv \frac{1}{\sqrt{2}}\left(X^0 - X^{D-1}\right).$$

So, the spacetime coordinates become the set $\{X^-, X^+, X^i\}_{i=1}^{D-2}$.

In this light-cone coordinate system the inner product of two vectors V and W is given by

$$V \cdot W = -V^+W^- - V^-W^+ + \sum_{i=1}^{D-2} V^i W^i.$$

While raising/lowering of indices goes as

$$V_+ = -V^-,$$

$$V_- = -V^+,$$

$$V_i = V^i.$$

Note that since we are treating two coordinates of spacetime differently from the rest, namely X^0 and X^{D-1}, we have lost manifest Lorentz invariance and so our Lorentz symmetry $SO(1, D-1)$ becomes $SO(D-2)$.

What simplifications to our bosonic string theory can we make by using the residual gauge symmetry? We know that the residual symmetry corresponds to being able to reparameterize $\sigma^\pm$ as $\sigma^\pm \mapsto \xi^\pm(\sigma^\pm)$ and still have the same theory. This implies that we can reparameterize τ and σ, since they are given by linear combinations of σ^+ and σ^-, as

$$\tau \mapsto \tilde{\tau} = \frac{1}{2}(\tilde{\sigma}^+ + \tilde{\sigma}^-) = \frac{1}{2}(\xi^+(\sigma^+) + \xi^-(\sigma^-)),$$

$$\sigma \mapsto \tilde{\sigma} = \frac{1}{2}(\tilde{\sigma}^+ - \tilde{\sigma}^-) = \frac{1}{2}(\xi^+(\sigma^+) - \xi^-(\sigma^-)).$$

From the form of $\tilde{\tau}$, we can see that it is a solution to the massless wave equation, i.e.

$$\partial_+\partial_-\tilde{\tau} = \left(\partial_\tau^2 - \partial_\sigma^2\right)\tilde{\tau} = 0. \tag{5.16}$$

So, we can pick a $\tilde{\tau}$ which makes our theory simpler, however it must satisfy (5.16). Also, note that once we pick this $\tilde{\tau}$ then we have also fixed $\tilde{\sigma}$.

Now, in the gauge choice which sends the spacetime metric $h_{\alpha\beta}$ to the Minkowski metric in D dimensions, we have also seen that the spacetime coordinates $X^\mu(\tau, \sigma)$ satisfy the massless wave equation since this equation was the field equations for $X^\mu(\tau, \sigma)$, i.e.

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu(\tau, \sigma) = 0.$$

Thus, we can pick a reparameterization of τ such that $\tilde{\tau}$ is related to one of the $X^\mu(\tau, \sigma)$'s. The light-cone gauge corresponds to choosing a reparameterization such that

$$\tilde{\tau} = \frac{X^+}{l_s^2 p^+} + x^+, \tag{5.17}$$

where x^+ is some arbitrary constant. This implies that, in the light-cone gauge, we have

$$X^+ = x^+ + l_s^2 p^+ \tilde{\tau} \tag{5.18}$$

So, to reiterate, the light-cone gauge uses the residual gauge freedom to make the choice $X^+(\tilde{\tau}, \tilde{\sigma}) = x^+ + l_s^2 p^+ \tilde{\tau}$.

If we look at the mode expansion of $X^+(\tau, \sigma)$ for the open string, here dropping the tildes,

$$X^+(\tau, \sigma) = x^+ + l_s^2 p^+ \tau + \sum_{n \neq 0} \frac{1}{n} \alpha_n^+ e^{-in\tau} \cos(n\sigma),$$

then we can see that this gauge corresponds to setting $\alpha_n^+ = 0$ for all $n \neq 0$. Similarly, by inspecting the mode expansion of X^+ for the closed string one sees that the light-cone gauge corresponds to setting both $\alpha_n^+ = 0$ and $(\alpha_n^+)^\dagger = 0$ for all $n \neq 0$.

Now that we have seen that the light-cone gauge eliminates the oscillator modes of X^+ a good question to ask is what happens to the oscillator modes of X^- in the light-cone gauge. To answer this we use the fact that the Virasoro constraints,

$$0 = T_{01} = T_{10} = \dot{X}X',$$

$$0 = T_{00} = T_{11} = \frac{1}{2}\left(\dot{X}^2 + (X')^2\right),$$

or equivalently,

$$\left(\dot{X} \pm X'\right)^2 = 0, \tag{5.19}$$

must still hold. In terms of the light-cone coordinates, these constraints, (5.19), become

$$\dot{X}^- \pm X'^- = -2(\dot{X}^+ \pm (X^+)')(\dot{X}^- \pm (X^-)') + \sum(\dot{X}^i \pm (X^i)')(\dot{X}^i \pm (X^i)')$$

$$= \frac{1}{2l_s^2 p^+}(\dot{X}^i \pm (X^i)')^2, \tag{5.20}$$

where $i = 1, ..., D - 2$. We can solve these equations to determine X^- (see problem 5.3), and for an open string with Neumann boundary conditions we have that, again dropping the tildes,

$$X^-(\tau, \sigma) = x^- + l_s^2 p^- \tau + \sum_{n \neq 0} \frac{1}{n} \alpha_n^- e^{-in\tau} \cos(n\sigma).$$

Plugging this expression into (5.20) gives that

$$\alpha_n^- = \frac{1}{p^+ l_s} \left(\frac{1}{2} \sum_{i=1}^{D-2} \sum_{m=-\infty}^{\infty} : \alpha_{n-m}^i \alpha_m^i : - \underbrace{a\delta_{n,0}}_{\text{from n.o.}} \right), \tag{5.21}$$

and so only the zero mode survives for X^- as was the case for X^+. Thus, one can express the bosonic string theory in terms of transverse oscillators only and so a (critical) string only has transverse oscillations, just as massless particles only have transverse polarizations.

5.3.1 Mass-Shell Condition (Open Bosonic String)

In light-cone coordinates the mass-shell condition is given by

$$-p^\mu p_\mu = M^2 = 2p^+ p^- - \sum_{i=1}^{D-2} p^i p^i. \tag{5.22}$$

Also, for n=0 we have that, by expanding P^μ in modes (see problem 5.3),

$$p^- l_s \equiv \alpha_0^- = \frac{1}{p^+ l_s} \left(\frac{1}{2} \sum_{i=1}^{D-2} \sum_{m=-\infty}^{\infty} : \alpha_{-m}^i \alpha_m^i : -a \right)$$

$$= \frac{1}{p^+ l_s} \left(\frac{1}{2} (\alpha_0^i)^2 + \underbrace{\sum_{m>0} : \alpha_{-m}^i \alpha_m^i :}_{\equiv N} -a \right),$$

which implies that

$$p^- l_s = \frac{1}{p^+ l_s} \left(\frac{1}{2} (\alpha_0^i)^2 + N - a \right)$$

$$\Rightarrow l_s^2 p^+ p^- = \frac{1}{2} (p^i)^2 l_s^2 + (N - a)$$

$$\Rightarrow \underbrace{2p^+ p^- - (p^i)^2}_{-P^\mu P_\mu \text{ in l-c coord.}} = \frac{2}{l_s^2} (N - a)$$

$$- 74 -$$

$$\Rightarrow M^2 = \frac{2}{l_s^2}(N - a).$$

So, in the light-cone gauge we have that the mass-shell formula for an open string is given by

$$M^2 = \frac{2}{l_s^2} \sum_{i=1}^{D-2} \sum_{n=-\infty}^{\infty} : \alpha_{-n}^i \alpha_n^i : -a = \frac{2}{l_s^2}(N - a). \tag{5.23}$$

5.3.2 Mass Spectrum (Open Bosonic String)

First, note that in the light-cone gauge all of the excitations are generated by transverse oscillators (α_n^i) where as before, in the canonical quantization, we had to include all the oscillators in the spectrum which, by the commutation relations

$$[\alpha_m^\mu, (\alpha_n^\nu)^\dagger] = \eta^{\mu\nu} \delta_{m,n} ,$$

lead to negative norm states. Now the commutator of the transverse oscillations no longer have the negative value coming from the 00 component of the metric and so we do not have negative norm states in the light-cone gauge quantization!

The first excited state, which is given by

$$\alpha_{-1}^i |0; k^\mu\rangle ,$$

belongs to a $(D-2)$-component vector representation of the rotation group $SO(D-2)$ in the transverse space. As a general rule, Lorentz invariance implies that physical states form a representation of $SO(D-1)$ for massive states and $SO(D-2)$ for massless states. Thus, since $\alpha_{-1}^i |0; k^\mu\rangle$ belongs to a representation of $SO(D-2)$ it must correspond to a massless state if our bosonic string theory is to be Lorentz invariant. To see what this implies consider the result of acting on the first excited state by the mass operator (squared),

$$M^2(\alpha_{-1}^i |0; k^\mu\rangle) = \frac{2}{l_s^2}(N - a)(\alpha_{-1}^i |0; k^\mu\rangle)$$

$$= \frac{2}{l_s^2}(1 - a)(\alpha_{-1}^i |0; k^\mu\rangle) .$$

And so, in order to have an eigenvalue of 0 for the mass operator, and thus to be in agreement with Lorentz invariance, we must impose that $a = 1$.

Now that we have a value for a we want to determine the spacetime dimension D (or c)[§]. Let us try to calculate the normal ordering constant a directly. Recall that the

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[§]One should note that this is a heuristic argument at best.

normal ordering constant a arose when we had to normal the expression

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty}\alpha^i_{-m}\alpha^i_m \ ,$$

see (5.21) with $n = 0$. So, when we normal order this expression we get

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty}\alpha^i_{-m}\alpha^i_m = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} :\alpha^i_{-m}\alpha^i_m: + \frac{(D-2)}{2}\sum_{m=1}^{\infty}m \ , \qquad (5.24)$$

since $[\alpha^i_m, \alpha^i_{-m}] = m\delta_{ij}$. The second sum on the right hand side is divergent and we will use Riemann ζ-function regularization to take care of this problem. So, first consider the sum

$$\zeta(s) = \sum_{m=1}^{\infty}m^{-s},$$

which is defined for any $s \in \mathbb{C}$. For $Re(s) > 1$ this sum converges to the Riemann zeta function $\zeta(s)$. The zeta function has a unique analytic continuation to $s = -1$ (which would correspond to our sum), for which it takes the value $\zeta(-1) = -1/12$. Thus, plugging this into (5.24) gives us that the second term becomes

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} :\alpha^i_{-m}\alpha^i_m: - \frac{(D-2)}{24}.$$

But, this is equal to (inserting the normal ordering constant back in, as in (5.24) with $n = 0$)

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} :\alpha^i_{-m}\alpha^i_m: - a,$$

which gives us that

$$\frac{(D-2)}{24} = a,$$

and since we have already found out that we must have $a = 1$, for Lorentz invariance, we get that $D = 26$.

So, to recap, we first quantized our theory canonically which had manifest Lorentz invariance but also negative norm states. We got rid of these states by imposing the constraints that $a = 1$ and $c = D = 26$. Here, in the light-cone quantization, we started with a theory that was free of negative norm states but no longer possessed manifest Lorentz invariance. We were able to recover Lorentz invariance at the cost of once again imposing the constraints $a = 1$ and $c = D = 26$.

5.3.3 Analysis of the Mass Spectrum

Open Strings

For the first few mass levels, the physical states of the open string are as follows:

- For $N = 0$ there is a tachyon (imaginary mass) $|0; k^\mu\rangle$, whose mass is given by $\alpha' M^2 = -1$, where $\alpha' = l_s^2/2$.

- For $N = 1$ there is a vector boson $\alpha^i_{-1} |0; k^\mu\rangle$ which, due to Lorentz invariance, is massless. This state gives a vector representation of $SO(24)$.

- For $N = 2$ we have the first state with a positive mass. The states are $\alpha^i_{-2} |0; k^\mu\rangle$ and $\alpha^i_{-1}\alpha^j_{-1} |0; k^\mu\rangle$ with $\alpha' M^2 = 1$. These have 24 and $24 \times 25/2$ states, respectively. Thus, the total number of states is 324, which is the dimensionality of the symmetric traceless second-rank tensor representation of $SO(25)$. So, in this sense, the spectrum consists of a single massive spin-two state at the $N = 2$ mass level.

Closed String

For the closed string one must take into account the level matching condition since there are both left-moving and right-moving modes. The spectrum of the closed string can be deduced from that of the open string since a closed string state is a tensor product of a left-moving state and a right-moving state, each of which has the same structure as open string states. The mass of the states in the closed string spectrum is given by

$$\alpha' M^2 = 4(N - 1) = 4(\tilde{N} - 1). \tag{5.25}$$

The physical states of the closed string at the first two mass levels are:

- The ground state $|0; k^\mu\rangle$ has mass $\alpha' M^2 = -4$ and is again a tachyon.

- For the $N = 1$ level there is a set of $24^2 = 576$ states of the form

$$\left|\Omega^{ij}\right\rangle = \alpha^i_{-1}\tilde{\alpha}^j_{-1} |0; k^\mu\rangle,$$

  corresponding to the tensor product of two massless vectors, one left mover and one right mover. The part of $|\Omega^{ij}\rangle$ that is symmetric and traceless in i and j transforms under $SO(24)$ as a massless spin-2 particle, the graviton. The trace term $\delta_{ij} |\Omega^{ij}\rangle$ is a massless scalar, which is called the dilaton and the antisymmetric part $|\Omega^{ij}\rangle = - |\Omega^{ji}\rangle$ transforms under $SO(24)$ as an antisymmetric second-rank tensor.

Note that all of the states for open strings and all of the states for closed strings either fall into multiplets of $SO(24)$ or $SO(25)$ depending upon whether the state is massless or massive, respectively. This is because for a massive state we can Lorentz transform to a frame such that in this frame the state takes the form

$$|E, \underbrace{0, 0, \cdots, 0}_{25\text{times}}\rangle.$$

Now, the set of all transformations that leave this state unchanged is the set of rotations in 25 dimensions, i.e. the Little group (the group that doesn't change the velocity of a state) is given by $SO(25)$, and so the massive state corresponds to some representation of the rotation group, $SO(25)$. While for a massless state the best we can do with a Lorentz transformation is to transform into a frame such that the state in this frame takes the form

$$|E, E, \underbrace{0, 0, \cdots, 0}_{24\text{times}}\rangle.$$

and so massless states have a Little group given by $SO(24)$. Thus giving us that the massless states corresponds to some representation of $SO(24)$.

This concludes the discussion of the bosonic string theory for now. In the next chapter we will start the study of the conformal group and conformal field theory.

5.4 Exercises

Problem 1

The Virasoro algebra reads:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}. \tag{5.26}$$

We take a so-called "highest-weight state" of the Virasoro algebra, which is by definition a state $|\phi\rangle$ that obeys $L_0|\phi\rangle = h|\phi\rangle$ and $L_m|\phi\rangle = 0$ for $m > 0$. The number h is called the conformal weight. In order to find zero-norm states we can proceed as in the notes, or we could also do the following: define the two states

$$|a\rangle = L_{-2}|\phi\rangle, \qquad |b\rangle = L_{-1}^2|\phi\rangle, \tag{5.27}$$

and form the two-by-two matrix of inner products:

$$\Delta = \det \begin{pmatrix} \langle a|a\rangle & \langle a|b\rangle \\ \langle b|a\rangle & \langle b|b\rangle \end{pmatrix}. \tag{5.28}$$

i) Find Δ as a function of c, h.

ii) Take $h = -1$. For which values of c does Δ vanish? Did you expect this result? Why?

iii) Take $c = 1/2$. Find the three values of h for which Δ vanishes.

Note the following:

- These three values play a prominent role in the field-theoretical description of the Ising model at the critical temperature.

- A highest weight state (representation) of the Virasoro algebra is the same as being a physical state.

Problem 2

Problem 2.4 of BBS (page 55).

Problem 3

Using the mode expansion for the fields X^-, X^i, prove that the solution of the constraints (5.20) is given by (5.21).

6. Introduction to Conformal Field Theory

Conformally invariant quantum field theories describe the critical behavior of systems at second order phase transitions. For example, the two-dimensional Ising model has a disordered phase at high temperature and an ordered phase at low temperature. These two phases are related to each other by a duality of the model and there is a second order phase transition at the self-dual point, i.e. fixed points of the RG mapping. At the phase transition, typical configurations have fluctuations on all length scales, so the field theory, at its critical point, is invariant under changes of scale. It turns out that the theory is invariant under the complete conformal group. In dimensions of three or more the conformal invariance doesn't give much more information than scale invariance, but in two dimensions the conformal algebra becomes infinite dimensional, leading to significant restrictions on two dimensional conformally invariant theories. Thus, perhaps leading to a classification of possibly critical phenomena in two dimensions.

Also, two dimensional conformal field theories provide the dynamical variable in string theory. In this context, conformal invariance turns out to give constraints on the allowed spacetime dimension, i.e. it restricts the central charge of the Virasoro algebra c, and the possible internal degrees of freedom. Therefore, a classification of two dimensional conformal field theories would provide useful information on the classical solution space of string theory.

6.1 Conformal Group in d Dimensions

Before we begin the study of conformal field theories we will first discuss the conformal group and its algebra. Roughly speaking, the conformal group is the group that preserves angles and maps light cones to light cones. More precisely, if one has a metric $g_{\alpha\beta}(x)$ in d dimensional spacetime then under a coordinate change $x \mapsto x'$ (such that $x^\mu = f^\mu(x'^\nu)$) we have that, since the metric is a 2-tensor,

$$g_{\mu\nu}(x) \mapsto g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(f(x')). \tag{6.1}$$

The conformal group is defined to be the subgroup of coordinate transformations that leave the metric unchanged, up to a scale factor $\Omega(x)$,

$$g_{\mu\nu}(x) \mapsto g'_{\mu\nu}(x') = \Omega(x)g_{\mu\nu}(x). \tag{6.2}$$

The transformation in (6.2) is called a conformal transformation and, roughly speaking, a conformal field theory is a field theory which is invariant under these transformations. Note that this implies that a conformal field theory has no notion of length scales and

thus, only cares about angles. Later on we will expand upon this definition to make it more precise, see section 6.4.

A transformation of the form (6.2) has a different interpretation depending on whether we are considering a fixed background metric or a dynamical background metric. When the metric is dynamical, the transformation is a diffeomorphism; this is a gauge symmetry. When the background is fixed, the transformation should be thought of as an honest, physical symmetry, taking a point to another point, which is now a global symmetry with its corresponding conserved currents. We will see later that the corresponding charges for this current are the Virasoro generators.

We will work with a flat background metric, i.e. we will assume that there exists coordinates such that $g_{\mu\nu} = \eta_{\mu\nu}$. In two dimensions this is not really a restriction on the theory since, as we have already seen in the case of the Polyakov action, if the theory has conformal symmetry then one can use this symmetry to fix the metric to be flat. However, in dimensions other than two, we do not have this freedom, and thus the flat metric assumption is really a restriction to non-gravitational theories in flat space.

We can find the infinitesimal generators of the conformal group by looking at an infinitesimal coordinate transformation that leaves the metric unchanged, up to the scale factor. So, if we take $x^\mu \mapsto f^\mu(x^\nu) = x^\mu + \epsilon^\mu$ then we have that

$$g'_{\mu\nu}(x^\mu + \epsilon^\mu) = g_{\mu\nu} + (\partial_\mu \epsilon^\mu + \partial_\nu \epsilon^\nu) g_{\mu\nu}$$

$$= g_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \,,$$

which must be equal to (6.1) for a conformal transformation. Thus, we have that

$$\Omega(x) g_{\mu\nu} = g_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu,$$

$$\Rightarrow (\Omega(x) - 1) g_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \,. \tag{6.3}$$

Now, in order to determine the scaling term, $\Omega(x) - 1$, we trace both sides of (6.3) with $g^{\mu\nu}$ and get that

$$\Omega(x) - 1 = \frac{2}{d}(\partial \cdot \epsilon).$$

Thus, plugging this all back into (6.3) we see that if ϵ is an infinitesimal conformal transformation then it must obey the following equation

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu}, \tag{6.4}$$

which is known as the conformal Killing vector equation. So, solutions to the conformal Killing vector equation correspond to infinitesimal conformal transformations.

For $d > 2$ we get the following solutions (see problem 6.1):

(a) $\epsilon^\mu = a^\mu$ where a^μ is a constant. These correspond to translations.

(b) $\epsilon^\mu = \omega^\mu{}_\nu x^\nu$ where $\omega^\mu{}_\nu$ is an antisymmetric tensor, which is in fact the infinitesimal generator of the Lorentz transformations. Note that the first two solutions, (a) and (b), correspond to the infinitesimal Poincaré transformations.

(c) $\epsilon^\mu = \lambda x^\nu$ where λ is a number. These correspond to scale transformations.

(d) $\epsilon^\mu = b^\mu x^2 - 2x^\mu b \cdot x$ which are known as the special conformal transformations.

Integrating the infinitesimal generators to finite conformal transformations we find:

(a) $\epsilon^\mu = a^\mu \mapsto x' = x + a$

(b) $\epsilon^\mu = \omega^\mu_\nu x^\nu \mapsto x' = \Lambda x$ where $\Lambda \in SO(1, d)$. The transformation in (a) and this one corresponds to the Poincaré group as expected.

(c) $\epsilon^\mu = \lambda x^\nu \mapsto x' = \lambda x$

(d) $\epsilon^\mu = b^\mu x^2 - 2x^\mu b \cdot x \mapsto x' = \frac{x + bx^2}{1 + 2b\cdot + b^2 x^2}$

The collection of these transformations forms the conformal group in d dimensions, which is isomorphic to $SO(2, d)$ as can be seen by defining the generators for (a) through (d) to be

$$P_\mu = \partial_\mu,$$

$$M_{\mu\nu} = \frac{1}{2}(x_\mu \partial_\nu - x_\nu \partial_\mu),$$

$$D_\mu = x^\mu \partial_\mu,$$

$$k_\mu = x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu,$$

and then calculating their commutation relations as well as the commutation relations for the generators of $SO(2, d)$ and then finding the trivial isomorphism between the two groups.

6.2 Conformal Algebra in 2 Dimensions

Now we will take $d = 2$ and $g_{\mu\nu} = \delta_{\mu\nu}$, where $\delta_{\mu\nu}$ is the two-dimensional Euclidean metric[‡]. With these considerations the equation we need to solve to construct generators of the conformal algebra in 2 dimensions, (6.4), becomes

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = (\partial \cdot \epsilon) \delta_{\mu\nu}. \tag{6.5}$$

This equation reduces to, for different values of μ and ν,

- For $\mu = \nu = 1$ we have that (6.5) becomes $2\partial_1 \epsilon_1 = \partial_1 \epsilon_1 + \partial_2 \epsilon_2 \implies \partial_1 \epsilon_1 = \partial_2 \epsilon_2$.

- For $\mu = \nu = 2$ we have that (6.5) becomes $2\partial_2 \epsilon_2 = \partial_1 \epsilon_1 + \partial_2 \epsilon_2 \implies \partial_2 \epsilon_2 = \partial_1 \epsilon_1$.

- For $\mu = 1$ and $\nu = 2$ (this also covers the case $\mu = 2$ and $\nu = 1$ since our equation is symmetric with respect to the exchange of μ and ν) we have that (6.5) becomes $\partial_1 \epsilon_2 + \partial_2 \epsilon_1 = 0 \implies \partial_1 \epsilon_2 = -\partial_2 \epsilon_1$.

Thus, in the two dimensional case, the conformal Killing vector equation reduces to nothing more than the Cauchy-Riemann equations,

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2,$$

$$\partial_1 \epsilon_2 = -\partial_2 \epsilon_1.$$

So, in two dimensions, infinitesimal conformal transformations are functions which obey the Cauchy-Riemann equations.

In terms of the coordinates for the two dimensional complex plane, $z, \overline{z} = x^1 \pm ix^2$, we can write $\epsilon = \epsilon^1 + i\epsilon^2$ and $\overline{\epsilon} = \epsilon^1 - i\epsilon^2$ which, if ϵ and $\overline{\epsilon}$ are infinitesimal conformal transformations, implies that $\partial_{\overline{z}} \epsilon = 0$ and $\partial_z \overline{\epsilon} = 0$. Thus, in two dimensions, conformal transformations coincide with the holomorphic and anti-holomorphic coordinate transformations given by,

$$z \mapsto f(z)$$

$$\overline{z} \mapsto \overline{f}(\overline{z}).$$

[‡]The two-dimensional Euclidean metric is given by

$$\delta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

One can immediately see that the conformal algebra in two dimensions is infinite dimensional since there are an infinite number of these coordinate transformations.

We can write a change of coordinates $z, \overline{z} \mapsto z', \overline{z}'$ in an infinitesimal form as

$$z \mapsto z' = z + \epsilon(z),$$

and

$$\overline{z} \mapsto \overline{z}' = \overline{z}' + \overline{\epsilon}(\overline{z}).$$

Also, we can expand $\epsilon(z)$ and $\overline{\epsilon}(\overline{z})$ in terms of basis functions as

$$\epsilon(z) = \sum_{n \in \mathbb{Z}} -z^{n+1},$$

and

$$\overline{\epsilon}(\overline{z}) = \sum_{n \in \mathbb{Z}} -\overline{z}^{n+1}.$$

Now, the corresponding infinitesimal generators that generate these symmetries are given by

$$\ell_n = -z^{n+1}\partial_z, \tag{6.6}$$

$$\overline{\ell}_n = -\overline{z}^{n+1}\partial_{\overline{z}}. \tag{6.7}$$

The set $\{\ell_n, \overline{\ell}_m\}_{n,m \in \mathbb{Z}}$ becomes an algebra, via the usual commutator, with the algebraic structure given by

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}, \tag{6.8}$$

$$[\overline{\ell}_m, \overline{\ell}_n] = (m-n)\overline{\ell}_{m+n}, \tag{6.9}$$

$$[\ell_m, \overline{\ell}_n] = 0. \tag{6.10}$$

There are several things that should be noted. First, as one can see from the commutation relations, the two dimensional conformal algebra is isomorphic to the Witt algebra. Second, when one quantizes the 2-d conformal field theory, these commutation relations will need to be modified a bit. Specifically, one will have to include an extra term which is proportional to a central charge c, i.e. one needs to add a central extension to the above algebra, and when this is done it will turn out that this new algebra, the old algebra plus the central extension, is isomorphic with the Virasoro algebra (just

as one can construct the Virasoro algebra by adding a central extension to the Witt algebra). Third, since the generators, ℓ_m and $\overline{\ell}_n$, commute $\forall m, n \in \mathbb{Z}$, we see that the two dimensional conformal algebra decomposes into the direct sum of two sub-algebras $\{\ell\}_{m\in\mathbb{Z}} \oplus \{\overline{\ell}\}_{n\in\mathbb{Z}}$ whose algebraic structure is given by (6.8) and (6.9), respectively. Finally, this really should be called the local conformal algebra since not all of the generators are well-defined globally on the Riemann sphere $S^2 = \mathbb{C} \cup \infty$. To see which of the generators are well-defined globally we proceed as follows.

Holomorphic conformal transformations are generated by the vector fields

$$v(z) = -\sum_n a_n \ell_n = \sum_n a_n z^{n+1} \partial_z.$$

Non-singularity of $v(z)$ as $z \mapsto 0$ allows $a_n \neq 0$ only for $n \geq -1$. To investigate the behavior of $v(z)$ as $z \mapsto \infty$, we perform the transformation $z = -1/w$,

$$v(z) = \sum_n a_n \left(\frac{-1}{w}\right)^{n+1} \left(\frac{dz}{dw}\right)^{-1} \partial_w$$

$$= \sum_n a_n \left(\frac{-1}{w}\right)^{n-1} \partial_w .$$

Non-singularity as $w \mapsto 0$ allows $a_n \neq 0$ only for $n \leq 1$. And so, we see that only the conformal transformations generated by $a_n \ell_n$ for $n = -1, 0, 1$ are globally well-defined. Similarly, the same type of analysis leads to the fact that only the transformations generated by $\overline{a}_n \overline{\ell}_n$ for $n = -1, 0, 1$ are globally defined on the Riemann sphere. We can now use these globally well-defined generators to construct the (global) conformal group in two dimensions.

6.3 (Global) Conformal Group in 2 Dimensions

In two dimensions, the two dimensional (global) conformal group is defined to be the set of all conformal transformations that are well-defined and invertible on the Riemann sphere with composition as the group multiplication. The group is thus generated by the infinitesimal generators

$$\{\ell_{-1}, \ell_0, \ell_1\} \cup \{\overline{\ell}_{-1}, \overline{\ell}_0, \overline{\ell}_1\}.$$

From (6.6) and (6.7) and the expressions for the finite conformal transformations, see section 6.1, one has that:

- $\ell_{-1} = -\partial_z$ and $\overline{\ell}_{-1} = -\partial_{\overline{z}}$ are the generators of translations.

- $i(\ell_0 - \bar{\ell}_0) = i(-z\partial_z + \bar{z}\partial_{\bar{z}})$ are the generators of rotations.

- $\ell_0 + \bar{\ell}_0 = -z\partial_z - \bar{z}\partial_{\bar{z}}$ are the generators of dilatations.

- ℓ_1 and $\bar{\ell}_1$ are the generators of the special conformal transformations.

The finite form of these transformations are given by

$$z \mapsto \frac{az + b}{cz + d}, \tag{6.11}$$

and

$$\bar{z} \mapsto \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}, \tag{6.12}$$

where $a, b, c, d, \bar{a}, \bar{b}, ... \in \mathbb{C}$ and $ad - bc = 1$. Note that the collection of these transformations is isomorphic with the group $SL(2, \mathbb{C})/\mathbb{Z}_2$, also known as the group of projective conformal transformations. We mod out by the group $\mathbb{Z}_2$, i.e. the group $\{1, -1\}$ with ordinary multiplication, above because every element in the group of transformations given by (6.11) and (6.12) is invariant under sending all the coefficients to minus themselves, i.e. $a \mapsto -a, b \mapsto -b$, and so on. Thus, in two dimensions the conformal group is isomorphic to $SL(2, \mathbb{C})/\mathbb{Z}_2$.

Now that we have investigated conformal transformations and the algebra and group formed by them, we will turn to conformal field theories.

6.4 Conformal Field Theories in d Dimensions

We now give a precise definition of a conformal field theory. A conformal field theory (CFT), or a field theory with conformal invariance, is a field theory that is invariant under conformal transformations, (6.2), which also satisfies the following properties:

(1) There is a set of fields $\{A_i\}$, where the index i specifies the different fields, that contains, in particular, all derivatives of each A_i.

(2) There is a subset of fields $\{\phi_j\} \subseteq \{A_i\}$, called quasi-primary fields which, under global conformal transformations $x \mapsto x'$, transform according to

$$\phi_j(x) \mapsto \left|\frac{\partial x'}{\partial x}\right|^{\triangle_j/d} \phi_j(x'), \tag{6.13}$$

where $\triangle_j$ is called the conformal weight (or dimension of the field ϕ_j) and d is from the dimension of the spacetime in which the theory lives. For example, for dilatations $x \mapsto \lambda x$ we have that

$$\left|\frac{\partial x'}{\partial x}\right| = \lambda^d,$$

and so, $\phi_j(x) \mapsto \lambda^{\triangle_j} \phi_j(\lambda x)$. Also, note that (6.13) implies that correlation functions of quasi-primary fields transform according to

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle \mapsto \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\triangle_1/d} \cdots \left| \frac{\partial x'}{\partial x} \right|_{x=x_n}^{\triangle_n/d} \langle \phi_1(x'_1) \cdots \phi_n(x'_n) \rangle. \qquad (6.14)$$

(3) The rest of the fields, i.e. the collection of fields in the set $\{A_i\}$ which do not belong to the set $\{\phi_j\}$, can be expressed as linear combinations of the quasi-primary fields and their derivatives.

(4) There exists a vacuum state $|0\rangle$ which is invariant under the global conformal group.

One should note that even though conformal field theories are a subset of quantum field theories, the language used to describe them is a little different. This is partly out of necessity. Invariance under the transformation (6.2) can only hold if the theory has no preferred length scale. But this means that there can be nothing in the theory like a mass or a Compton wavelength. In other words, conformal field theories only support massless excitations. The questions that we ask are not those of particles and S-matrices. Instead we will be concerned with correlation functions and the behavior of different operators under conformal transformations.

Having a field theory which is conformally invariant, i.e. a CFT, imposes many more constraints on the theory than that of a field theory which is not conformally invariant. These constraints will now be discussed

6.4.1 Constraints of Conformal Invariance in d Dimensions

The transformation property of the n-point correlation function of quasi-primary fields,

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle \mapsto \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\triangle_1/d} \cdots \left| \frac{\partial x'}{\partial x} \right|_{x=x_n}^{\triangle_n/d} \langle \phi_1(x'_1) \cdots \phi_n(x'_n) \rangle, \qquad (6.15)$$

under the conformal group imposes severe restrictions on 2- and 3-point functions of quasi-primary fields.

To identify the conformal invariants on which n-point correlation functions might depend, we construct some invariants of the conformal group in d-dimensions.

First, translational invariance implies that the correlation functions of quasi-primary fields can only depend on the difference of the coordinates, $x_i - x_j$, rather than the coordinates themselves. Next, if we consider spinless objects, then rotational invariance implies that the correlation functions can only depend on the distances

$$r_{ij} = |x_i - x_j|.$$

Scale invariance only allows the correlation functions to depend on the ratios r_{ij}/r_{kl}. Finally, under the special conformal transformation (6.1), we have that

$$|x_1' - x_2'|^2 = \frac{|x_1 - x_2|^2}{(1 + 2b \cdot x_1 + b^2 x_1^2)(1 + 2b \cdot x_2 + b^2 x_2^2)},$$ (6.16)

and so ratios of the form r_{ij}/r_{kl} are in general not invariant under the global conformal group, but ratios of the form (called cross-ratios)

$$\frac{r_{ij}r_{kl}}{r_{ik}r_{jl}},$$

are invariant under the global conformal group and so the correlation functions can have dependence on these cross-ratios. Note that in general there are $N(N-3)/2$ independent cross-ratios formed from N coordinates.

Applying the above to the 2-point function of two quasi-primary fields ϕ_1 and ϕ_2 we have that

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \begin{cases} \frac{c_{12}}{r_{12}^{2\triangle}}, & \text{if } \triangle_1 = \triangle_2 = \triangle , \\ 0, & \text{if } \triangle_1 \neq \triangle_2 , \end{cases}$$ (6.17)

where c_{12} is a constant. This can be seen as follows. From (6.15) we know that the 2-point function must satisfy

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\triangle_1/d} \left| \frac{\partial x'}{\partial x} \right|_{x=x_2}^{\triangle_2/d} \langle \phi_1(x_1')\phi_2(x_2') \rangle,$$

since ϕ_1 are ϕ_2 are quasi-primary fields. Now, invariance under translations and rotations implies that the 2-point function only depends on the difference $|x_1 - x_2|$, while dilatation invariance, $x \mapsto \lambda x$, implies that

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \frac{c_{12}}{r_{12}^{\triangle_1+\triangle_2}},$$

where c_{12} is some constant, which is determined by the normalization of the fields, $\triangle_1$ is the dimension of ϕ_1 and $\triangle_2$ is the dimension of ϕ_2. Finally, from the special conformal transformations, we see that $\triangle_1 = \triangle_2$ when $c_{12} \neq 0$ and thus we recover (6.17).

The same kind of analysis leads to the 3-point of the quasi-primary fields ϕ_1, ϕ_2 and ϕ_3 being given by

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{c_{123}}{\left(r_{12}^{\triangle_1+\triangle_2-\triangle_3} \right) \left(r_{23}^{\triangle_2+\triangle_3-\triangle_1} \right) \left(r_{13}^{\triangle_1+\triangle_3-\triangle_2} \right)},$$ (6.18)

where c_{123} is another constant. Thus, conformal invariance restricts the 2-point and 3-point correlation functions of quasi-primary fields to be dependent only on constants.

This is not true, however, for 4-point functions since they begin to have dependencies on functions of the cross-ratios.

Also, note that the stress-energy tensor of a classical conformal field theory has a vanishing trace (we show this explicitly in 7.2). This is no suprise since the set of conformal transformations is a subset of the Weyl transformations and we have already seen that any theory which is invariant to Weyl transformations must have a trace-less stress-energy tensor, see (3.11). However, one must remember that the vanishing of the trace of the stress-energy tensor only holds for quantum conformal field theories if the space is flat, this is known as the Weyl anomaly (see (9.53)).

6.5 Conformal Field Theories in 2 Dimensions

We are now going to translate the ideas from before to the case of a two dimensional spacetime. Here we will take a Euclidean metric

$$ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2,\tag{6.19}$$

which, in terms of the coordinates $z = x^1 + ix^2$ and $\overline{z} = x^1 - ix^2$, can be rewritten as

$$ds^2 = dzd\overline{z}.\tag{6.20}$$

Also, under a general coordinate transformation, $z \mapsto f(z)$ and $\overline{z} \mapsto \overline{f}(\overline{z})$, we have that ds^2 transforms as[§]

$$ds^2 \mapsto \frac{\partial f}{\partial z}\frac{\partial \overline{f}}{\partial \overline{z}}ds^2,\tag{6.21}$$

i.e. it transforms like a 2-tensor as it should.

We will call a field $\Phi(z,\overline{z})$ that transforms, under a general coordinate change $z \mapsto f(z)$ and $\overline{z} \mapsto \overline{f}(\overline{z})$, as

$$\Phi(z,\overline{z}) \mapsto \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}} \Phi\big(f(z),\overline{f}(\overline{z})\big),\tag{6.22}$$

where h and $\overline{h}$ are constants that need not be related, a primary field. If we have fields that do not transform as (6.22) then theses fields are called secondary fields. Note that primary fields are automatically quasi-primary while secondary fields may or may not be quasi-primary.

[§]This comes from having $dz \mapsto d(f(z)) = \frac{\partial f}{\partial z}dz$ and also $d\overline{z} \mapsto d(\overline{f}(\overline{z})) = \frac{\partial \overline{f}}{\partial \overline{z}}d\overline{z}$ and so,

$$ds^2 \mapsto \frac{\partial f}{\partial z}\frac{\partial \overline{f}}{\partial \overline{z}}ds^2.$$

Under an infinitesimal transformation, $f(z) \mapsto z + \epsilon(z)$ and $\overline{f}(\overline{z}) \mapsto \overline{z} + \overline{\epsilon}(\overline{z})$, we have that

$$\left(\frac{\partial f}{\partial z}\right)^h \mapsto (1 + \partial_z \epsilon(z))^h = 1 + h\partial_z \epsilon(z) + \mathcal{O}(\epsilon^2),$$

$$\left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}} \mapsto (1 + \partial_{\overline{z}} \overline{\epsilon}(\overline{z}))^{\overline{h}} = 1 + \overline{h}\partial_{\overline{z}} \overline{\epsilon}(\overline{z}) + \mathcal{O}(\overline{\epsilon}^2).$$

Thus, under this infinitesimal transformation, the field $\Phi(z, \overline{z})$ transforms as

$$\Phi(z, \overline{z}) \mapsto \left(1 + h\partial_z \epsilon(z) + \mathcal{O}(\epsilon^2)\right)\left(1 + \overline{h}\partial_{\overline{z}} \overline{\epsilon}(\overline{z}) + \mathcal{O}(\overline{\epsilon}^2)\right)\Phi(f(z), \overline{f}(\overline{z}))$$

$$= \left(1 + h\partial_z \epsilon(z) + \mathcal{O}(\epsilon^2)\right)\left(1 + \overline{h}\partial_{\overline{z}} \overline{\epsilon}(\overline{z}) + \mathcal{O}(\overline{\epsilon}^2)\right)\Phi(z + \epsilon(z), \overline{z} + \overline{\epsilon}(\overline{z}))$$

$$= \left(1 + h\partial_z \epsilon(z) + \mathcal{O}(\epsilon^2)\right)\left(1 + \overline{h}\partial_{\overline{z}} \overline{\epsilon}(\overline{z}) + \mathcal{O}(\overline{\epsilon}^2)\right)\left(\Phi(z, \overline{z}) + \epsilon(z)\partial_z \Phi(z, \overline{z})\right.$$

$$\left. + \overline{\epsilon}(\overline{z})\partial_{\overline{z}}\Phi(z, \overline{z}) + \mathcal{O}(\epsilon^2, \overline{\epsilon}^2, \epsilon\overline{\epsilon})\right)$$

$$= \Phi(z, \overline{z}) + \epsilon(z)\partial_z \Phi(z, \overline{z}) + h\partial_z \epsilon(z)\Phi(z, \overline{z}) + \overline{\epsilon}(\overline{z})\partial_{\overline{z}}\Phi(z, \overline{z}) + \overline{h}\partial_{\overline{z}}\overline{\epsilon}(\overline{z})\Phi(z, \overline{z})$$

$$+ \mathcal{O}(\epsilon^2, \overline{\epsilon}^2, \epsilon\overline{\epsilon})$$

$$= \Phi(z, \overline{z}) + \left[\left(h\partial_z \epsilon(z) + \epsilon(z)\partial_z\right) + \left(\overline{h}\partial_{\overline{z}}\overline{\epsilon}(\overline{z}) + \overline{\epsilon}(\overline{z})\partial_{\overline{z}}\right)\right]\Phi(z, \overline{z}) + \mathcal{O}(\epsilon^2, \overline{\epsilon}^2, \epsilon\overline{\epsilon}).$$

And so, we see that, under an infinitesimal transformation, the variation of the field $\Phi(z, \overline{z})$ is given by, up to first order in ϵ and $\overline{\epsilon}$,

$$\delta_{\epsilon, \overline{\epsilon}}\Phi(z, \overline{z}) = \left[\left(h\partial_z \epsilon(z) + \epsilon(z)\partial_z\right) + \left(\overline{h}\partial_{\overline{z}}\overline{\epsilon}(\overline{z}) + \overline{\epsilon}(\overline{z})\partial_{\overline{z}}\right)\right]\Phi(z, \overline{z}). \tag{6.23}$$

6.5.1 Constraints of Conformal Invariance in 2 Dimensions

Having the two dimensional primary fields transform in this way leads to constraints on the correlation functions of primary fields, just as before for the correlation functions of quasi-primary fields in a d Dimensional spacetime. For example, we know that a 2-point correlation function of primary fields, $G^{(2)}(z_i, \overline{z}_i) \equiv \langle \Phi_1(z_1, \overline{z}_1)\Phi_2(z_2, \overline{z}_2)\rangle$, is invariant under an infinitesimal conformal transformation $\delta_{\epsilon\overline{\epsilon}}$ since the fields of a CFT are invariant under conformal transformations and thus, so is the correlation function.

Now, if we assume that the transformation $\delta_{\epsilon,\bar{\epsilon}}$ is a derivation (linear map and satisfies the Leibniz rule) then we have that

$$0 = \delta_{\epsilon,\bar{\epsilon}} G^{(2)}(z_i, \bar{z}_i) = \delta_{\epsilon,\bar{\epsilon}} \big\langle \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \overline{z_2}) \big\rangle$$

$$= \big\langle (\delta_{\epsilon,\bar{\epsilon}} \Phi_1) \Phi_2 \big\rangle + \big\langle \Phi_1 (\delta_{\epsilon,\bar{\epsilon}} \Phi_2) \big\rangle$$

$$= \Big\langle \Big(\epsilon(z_1) \partial_{z_1} \Phi_1 + h_1 \partial_{z_1} \epsilon(z_1) \Phi_1 \Big) \Phi_2 \Big\rangle + \Big\langle \Big(\bar{\epsilon}(\bar{z}_1) \partial_{\bar{z}_1} \Phi_1 + \bar{h}_1 \partial_{\bar{z}_1} \bar{\epsilon}(\bar{z}_1) \Phi_1 \Big) \Phi_2 \Big\rangle$$

$$+ \Big\langle \Phi_1 \Big(\epsilon(z_2) \partial_{z_2} \Phi_2 + h_2 \partial_{z_2} \epsilon(z_2) \Phi_2 \Big) \Big\rangle + \Big\langle \Phi_1 \Big(\bar{\epsilon}(\bar{z}_2) \partial_{\bar{z}_2} \Phi_2 + \bar{h}_2 \partial_{\bar{z}_2} \bar{\epsilon}(\bar{z}_2) \Phi_2 \Big) \Big\rangle$$

$$= \Big[\Big(\epsilon(z_1) \partial_{z_1} + h_1 \partial_{z_1} \epsilon(z_1) \Big) + \Big(\epsilon(z_2) \partial_{z_2} + h_2 \partial_{z_2} \epsilon(z_2) \Big)$$

$$+ \Big(\bar{\epsilon}(\bar{z}_1) \partial_{\bar{z}_1} + \bar{h}_1 \partial_{\bar{z}_1} \bar{\epsilon}(\bar{z}_1) \Big) + \Big(\bar{\epsilon}(\bar{z}_2) \partial_{\bar{z}_2} + \bar{h}_2 \partial_{\bar{z}_2} \bar{\epsilon}(\bar{z}_2) \Big) \Big] G^{(2)}(z_i, \bar{z}_i) .$$

Thus, we are left with a differential equation for $G^{(2)}(z_i, \bar{z}_i)$ given by

$$0 = \Big[\Big(\epsilon(z_1) \partial_{z_1} + h_1 \partial_{z_1} \epsilon(z_1) \Big) + \Big(\epsilon(z_2) \partial_{z_2} + h_2 \partial_{z_2} \epsilon(z_2) \Big)$$

$$+ \Big(\bar{\epsilon}(\bar{z}_1) \partial_{\bar{z}_1} + \bar{h}_1 \partial_{\bar{z}_1} \bar{\epsilon}(\bar{z}_1) \Big) + \Big(\bar{\epsilon}(\bar{z}_2) \partial_{\bar{z}_2} + \bar{h}_2 \partial_{\bar{z}_2} \bar{\epsilon}(\bar{z}_2) \Big) \Big] G^{(2)}(z_i, \bar{z}_i) .$$

We can solve this equation, see Ginsparg "Applied Conformal Field Theory" pg 13-14, to see that the 2-point correlation function is constrained to take the form

$$G^{(2)}(z_i, \bar{z}_i) = \begin{cases} \frac{C_{12}}{z_{12}^{2h} \bar{z}_{12}^{2\bar{h}}} & \text{if } h_1 = h_2 = h \text{ and } \bar{h}_1 = \bar{h}_2 = \bar{h} , \\ 0 & \text{if } h_1 \neq h_2 \text{ or } \bar{h}_1 \neq \bar{h}_2 . \end{cases} \tag{6.24}$$

Similarly, one can show that the 3-point correlation function of primary fields takes the form

$$G^{(3)}(z_i, \bar{z}_i) = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}, \tag{6.25}$$

where C_{123} is a constant and $z_{ij} \equiv z_i - z_j$. We can in fact show that $G^{(3)}(z_i, \bar{z}_i)$ only depends on the constant C_{123}. This is due to the correlation function being invariant

under the group $SL(2, \mathbb{C})/\mathbb{Z}_2{}^{\ddagger}$. Under the action of $SL(2, \mathbb{C})/\mathbb{Z}_2$, the points z_1, z_2 and z_3 transform as

$$z_1 \mapsto \frac{az_1 + b}{cz_1 + d} = \alpha_1,$$

$$z_2 \mapsto \frac{az_2 + b}{cz_2 + d} = \alpha_2,$$

$$z_3 \mapsto \frac{az_3 + b}{cz_3 + d} = \alpha_3,$$

where α_i are arbitrary constants. So we can send the three points z_1, z_2 and z_3 to any other arbitrary points α_1, α_2 and α_3, and since our correlation function is invariant to this transformation its value will not change under this transformation. Thus, if we take, for e.g., $\alpha_1 = \infty$, $\alpha_2 = 1$ and $\alpha_3 = 0$ we then have that

$$G^{(3)}(z_i, \overline{z}_i) \mapsto G^{(3)}(\infty, 1, 0) = \frac{C_{123}}{\lim_{z_1 \to \infty} z_1^{2h_1} \overline{z}_1^{2\overline{h}_1}},$$

or that $\lim_{z_1 \to \infty} z_1^{2h_1} \overline{z}_1^{2\overline{h}_1} G^{(3)} = C_{123}$. So, we have shown that $G^{(3)}$ only depends on the constant C_{123}.

6.6 Role of Conformal Field Theories in String Theory

A question you should be asking yourself is why are conformal field theories important in string theory? It turns out that two-dimensional conformal field theories are very important in the study of worldsheet dynamics.

A string has internal degrees of freedom determined by its vibrational modes. The different vibrational modes of the string are interpreted as particles in the theory. That is, the different ways that the string vibrates against the background spacetime determine what kind of particle the string is seen to exist as. So in one vibrational mode, the string is an electron, while in another, the string is a quark, for example. Yet a third vibrational mode is a photon.

As we have seen already, the vibrational modes of the string can be studied by examining the worldsheet, which is a two-dimensional surface. It turns out that when

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‡Recall that in two dimensions the (global) conformal group is isomorphic to the quotient group $SL(2, \mathbb{C})/\mathbb{Z}_2$ and so, when one discusses the conformal group in two dimensions they could equally well replace this group with $SL(2, \mathbb{C})/\mathbb{Z}_2$. Thus, since the correlation functions are invariant under the action of the two dimensional conformal group and since this group is isomorphic to $SL(2, \mathbb{C})/\mathbb{Z}_2$ we could equivalently say that the correlation functions are invariant under the action of $SL(2, \mathbb{C})/\mathbb{Z}_2$.

studying the worldsheet, the vibrational modes of the string are described by a two-dimensional conformal field theory.

If the string is closed, we have two vibrational modes (left movers and right movers) moving around the string independently. Each of these can be described by a conformal field theory. Since the modes have direction we call the theories that describe these two independent modes chiral conformal field theories. This will be important for open strings as well.

In the next chapter we will continue with the study of conformal field theories in two dimensions and, in particular, we will look at radial quantization, conserved currents of conformal transformations,[‡] and operator product expansions.

[‡]Recall that for the conformal transformation to be a global transformation, and thus yield a conserved current, it implies that we will be working with a fixed background metric.

6.7 Exercises

Problem 1

In this problem we will prove that the conformal group in d dimensions ($d > 2$) consists of translations, rotations, scale transformations and special conformal transformations. As discussed in this chapter, infinitesimal conformal transformations are given by coordinate transformations

$$\delta x^{\mu} = -\epsilon^{\mu}, \tag{6.26}$$

where ϵ^{μ} satisfies

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}, \tag{6.27}$$

with $\eta_{\mu\nu}$ the Minkowski metric. The set of conformal transformations is obtained by finding the most general solution of (6.27). Useful equations are obtained by acting on (6.27) with $\partial^{\rho}\partial^{\sigma}$.

a) Show that the choice $\rho = \mu, \sigma = \nu$ leads to

$$\Box(\partial \cdot \epsilon) = 0. \tag{6.28}$$

b) Now choose $\sigma = \nu$ and prove

$$\partial_{\mu}\partial_{\rho}(\partial \cdot \epsilon) = 0. \tag{6.29}$$

c) Show that this implies

$$\partial \cdot \epsilon = d(\lambda - 2b_{\alpha}x^{\alpha}), \tag{6.30}$$

where λ and b_{α} are constants and the other numerical constants are chosen for later convenience.

d) Differentiate (6.27) with ∂_{α} and show that the resulting equation can be processed to the following form,

$$\partial_{\mu}(\partial_{\alpha}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\alpha}) = 4(\eta_{\mu\alpha}b_{\nu} - \eta_{\mu\nu}b_{\alpha}). \tag{6.31}$$

e) Show that this implies

$$\partial_{\alpha}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\alpha} = 4(x_{\alpha}b_{\nu} - x_{\nu}b_{\alpha}) + 2\omega_{\alpha\nu}, \tag{6.32}$$

where $\omega_{\alpha\nu}$ is a constant antisymmetric tensor.

f) Use the results obtained so far and (6.27) to prove

$$\epsilon_{\mu} = a_{\mu} + \lambda x_{\mu} + \omega_{\nu\mu}x^{\nu} + b_{\mu}x^2 - 2(b \cdot x)x_{\mu}. \tag{6.33}$$

7. Radial Quantization and Operator Product Expansions

The next topic that will be studied is that of radial quantization, i.e. we will look at defining a quantum field theory on the plane. This will prove to be useful when we want to look more deeply into the consequences of conformal invariance.

7.1 Radial Quantization

We will begin with a flat two dimensional Euclidean surface with coordinates labeled by σ^0 for timelike positions and σ^1 for spacelike positions, i.e. a point on our surface is specified by a time and space position, (σ^0, σ^1). The metric on the surface is given by

$$ds^2 = (d\sigma^0)^2 + (d\sigma^1)^2. \tag{7.1}$$

Note that the left- and right-moving boson fields become Euclidean fields that have purely holomorphic or anti-holomorphic dependence on the coordinates. Also, note that in order to eliminate any infrared divergences we will compactify the space coordinate of our surface, i.e. we will take σ^1 to be periodic, $\sigma^1 = \sigma^1 + 2\pi$, which gives us the topology of an infinitely long (in both directions of σ^0) cylinder for our surface. Thus, we can think of our Euclidean surface as the product space $\mathbb{R} \times S^1$, where S^1 denotes the circle.

We can define light-cone coordinates for our Euclidean surface, $\zeta, \overline{\zeta} = \sigma^0 \pm i\sigma^1$, which are just Wick rotations of the light-cone coordinates previously used for the string worldsheet in the previous sections. In terms of these coordinates, the metric becomes

$$ds^2 = d\zeta d\overline{\zeta}. \tag{7.2}$$

Now, for reasons that will become clear later[‡], we will map the infinitely long cylinder to the complex plane, coordinatized by z, via the map $\exp : \mathbb{R} \times S^1 \to \mathbb{C}$ defined by

$$\zeta \mapsto z = \exp(\zeta) = e^{(\sigma^0 + i\sigma^1)},$$

$$\overline{\zeta} \mapsto \overline{z} = \exp(\overline{\zeta}) = e^{(\sigma^0 - i\sigma^1)}. \tag{7.3}$$
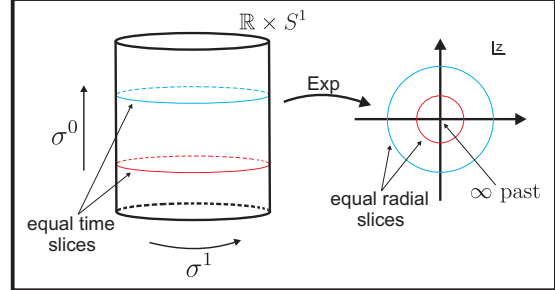
[‡]By defining our CFT on the complex plane we will see that many of the components of the theory will split up into holomorphic and anti-holomorphic parts. Thus, one can use the properties of complex analysis to study the theory.

From the mapping we can see that the infinite past and future of the cylinder, $\sigma^0 = \pm\infty$, are mapped to the points $z = 0$, for infinite past, and $z = \infty$, for infinite future, in the complex plane. Also, equal time slices of the cylinder, i.e. the surface defined by σ^0 =constant and σ^1 taking all values in $[0, 2\pi)$, become circles of constant radius $\exp(\sigma^0)$ in the complex plane (see the previous figure). Time translations, $\sigma^0 + a$ where a is a constant, are mapped to $\exp(a)\exp(\sigma^0 + i\sigma^1)$, i.e. $z \mapsto \exp(a)z$, which are the dilatations in the complex plane. Note that since the Hamiltonian generates time translations we can see that the dilatation generator on the complex plane corresponds to the Hamiltonian on the cylinder. And so, the Hilbert space defined on the cylinder is built up of constant time slices while the Hilbert space defined on the complex plane is built up of circles of constant radius. Finally, a word about nomenclature, this procedure of quantizing a theory on a manifold whose geometry is given by the complex plane is known as radial quantization. Also, it is useful to radially quantize two dimensional CFT's since it allows for one to use complex analysis to analyze short distance operator expansions, conserved charges, etc. as we will see later.

7.2 Conserved Currents and Symmetry Generators

For what follows, we will treat z and $\bar{z}$ as independent coordinates. Thus, we are really mapping the cylinder (i.e. subset of $\mathbb{R}^2$) to $\mathbb{C}^2$. And so, one must remember throughout that we are really sitting on the real slice $\mathbb{R}^2 \subset \mathbb{C}^2$ defined by setting $\bar{z} = z^*$.

In general, symmetry generators can be constructed via the Noether method which states that if your $d + 1$ dimensional quantum theory has an exact symmetry then associated to this symmetry is a conserved current j^μ. For example, if the theory is invariant under an infinitesimal coordinate transformation, $x^\mu \mapsto x^\mu + \delta x^\mu = x^\mu + \epsilon^\mu$, then the corresponding conserved current is given by

$$j_\mu = T_{\mu\nu}\epsilon^\nu, \tag{7.4}$$

where $T_{\mu\nu}$ is the stress-energy tensor. In particular, for translations along x^α by a we have that

$$\epsilon^\mu = a\delta^\mu_\alpha, \tag{7.5}$$

and so our current is given by $j_\mu = aT_{\mu\alpha}$. If our theory is translationally invariant we would have that this current is conserved. While for dilatations, i.e. scaling the coordinates, we have that

$$\epsilon^\mu = \lambda x^\mu, \tag{7.6}$$

and so the current corresponding to this transformation j_μ is proportional to $T_{\mu\nu}x^\nu$. If our theory is invariant under dilatations then we have that $\partial^\mu j_\mu = 0$, but this implies

that

$$\partial^\mu j_\mu = \partial^\mu (bT_{\mu\nu}x^\nu),$$

where b is the constant of proportionality, and so

$$\partial^\mu (bT_{\mu\nu}x^\nu) = b\partial^\mu T_{\mu\nu}x^\nu$$

$$= bT_{\mu\nu}\partial^\mu x^\nu$$

$$= bT_{\mu\nu}\delta^{\nu\mu}$$

$$= bT_\mu{}^\mu.$$

Thus, in a conformally invariant classical field theory we have that the stress-energy tensor is traceless, $T_\mu{}^\mu = 0$. Note that even when the conformal invariance survives in a 2d quantum theory, the vanishing trace of the stress-energy tensor will only turn out to hold in flat space.

Furthermore, there also exists a conserved charge Q, defined by

$$Q = \int_{\partial\mathcal{M}} d^d x j^0, \tag{7.7}$$

where $\partial\mathcal{M}$ is the d dimensional manifold constructed by taking a fixed time slice of the $d+1$ dimensional spacetime manifold in which your theory is defined. The conservation of Q, i.e.

$$\frac{d}{d\tau}Q = 0,$$

follows straight from Stokes' theorem. The conserved charge Q generates the symmetry, i.e. if one has a field A and a symmetry of your theory then under this symmetry the variation of the field A is given by the expression

$$\delta_\epsilon A = \epsilon[Q, A].$$

We will now specialize to the case of two dimensions.

Before we begin to study the conserved charges on the complex z-plane, we need to first introduce some properties of our plane, namely we need to know the components of the metric and stress-energy tensor in complex coordinates z and $\bar{z}$. So, to start we have that on the Euclidean surface (i.e. the cylinder) our metric is given, in light-cone coordinates, by (7.2)

$$ds^2 = d\zeta d\bar{\zeta}.$$

To transform the previous expression for the metric into complex coordinates we note that according to (7.3) one has that $\zeta = \ln(z)$ and $\overline{\zeta} = \ln(\overline{z})$ which gives us that

$$ds^2 = \frac{1}{|z|^2} dz d\overline{z}.$$

Now, the scaling factor of $1/|z|^2$ can be removed via a conformal transformation and since our field theory is a CFT it will be invariant under this transformation. Therefore, we can, without loss of generality, take the metric on our complex plane to be given by

$$ds^2 = dz d\overline{z}. \tag{7.8}$$

In order to find the components of the metric in the $z, \overline{z}$ coordinate system we can simply read off from equation (7.8) to get that $g_{zz} = g_{\overline{z}\overline{z}} = 0$ while $g_{z\overline{z}} = g_{\overline{z}z} = 1/2$. Now that we have the components of the metric in the complex coordinates, we can find the components of the stress-energy tensor in terms of these components as well and we see that

$$T_{zz} = \frac{1}{4}\left(T_{00} - 2iT_{10} - T_{11}\right), \tag{7.9}$$

$$T_{\overline{z}\overline{z}} = \frac{1}{4}\left(T_{00} + 2iT_{10} - T_{11}\right), \tag{7.10}$$

$$T_{\overline{z}z} = T_{z\overline{z}} = \frac{1}{4}\left(T_{00} + T_{11}\right) = \frac{1}{4}T^{\mu}{}_{\mu}. \tag{7.11}$$

Now, by translational invariance, we have that $\partial^{\nu}T_{\mu\nu} = 0$ which implies that

$$\partial_{\overline{z}}T_{zz} + \partial_{z}T_{\overline{z}z} = 0, \tag{7.12}$$

$$\partial_{z}T_{\overline{z}\overline{z}} + \partial_{\overline{z}}T_{z\overline{z}} = 0. \tag{7.13}$$

Also, imposing dilatation invariance gives us that the stress-energy tensor is traceless, $T^{\mu}{}_{\mu} = 0$, and so we see that $T_{00} + T_{11} = 0$ which, from (7.11), implies that $T_{\overline{z}z} = T_{z\overline{z}} = 0$. Combining this result with equations (7.12) - (7.13) gives us that

$$\partial_{z}T_{\overline{z}\overline{z}} = 0, \tag{7.14}$$

and

$$\partial_{\overline{z}}T_{zz} = 0. \tag{7.15}$$

Equation (7.14) tells us that $T_{\overline{z}\overline{z}}$ is a holomorphic function of $\overline{z}$ only, while equation (7.15) tells us that T_{zz} is an anti-holomorphic function of z only. We will, for simplicity,

denote these functions by $T(z) \equiv T_{zz}(z)$ and $\overline{T}(\overline{z}) \equiv T_{\overline{z}\overline{z}}(\overline{z})$. Thus, the only nonvanishing components of the stress-energy tensor, for a two-dimensional CFT, are given by $T(z)$ and $\overline{T}(\overline{z})$, and so we see that the stress-energy tensor factorizes into holomorphic and antiholomorphic pieces. We will see this factorization for other quantities as we proceed further into analyzing symmetries of CFT's living on the plane.

Now that we have the components of the metric and stress-energy tensor in terms of complex coordinates, we are in a position to study symmetries and their corresponding conserved currents for two dimensional CFT's on a plane. So, consider the generator of a general coordinate transformation $\delta x^\mu = \epsilon^\mu$ or, in terms of complex coordinates,

$$\delta z = \epsilon(z),$$

$$\delta \overline{z} = \overline{\epsilon}(\overline{z}),$$

where $\epsilon(z)$ is a holomorphic function and $\overline{\epsilon}(\overline{z})$ is an anti-holomorphic function. The corresponding charge to this transformation is given by

$$Q = \int_{\partial \mathcal{M}} j^0 d\sigma^1, \tag{7.16}$$

where again we integrate over a constant time slice of the cylinder or, in terms of complex coordinates, the corresponding charge is given by

$$Q = \frac{1}{2\pi i} \oint_{\mathcal{C}} \left(dz\, T(z)\epsilon(z) + d\overline{z}\, \overline{T}(\overline{z})\overline{\epsilon}(\overline{z}) \right), \tag{7.17}$$

where the contour $\mathcal{C}$ is over a circle in the complex plane whose radius corresponds with the value of σ^0 for the time slice of the cylinder, i.e. if we pick a time slice at the value $\sigma^0 = s$, for some $s \in [-\infty, \infty]$, then the corresponding contour is a circle of radius $\exp(s)$. Also, we will chose the positive orientation of the circle to be in the counter-clockwise sense.

The variation of a field $\Phi(w, \overline{w})$ with respect to the above transformation is given by the "equal-time" commutator of $\Phi(w, \overline{w})$ with the charge Q, which follows from (7.17),

$$\delta_{\epsilon,\overline{\epsilon}}\Phi(w, \overline{w}) \equiv [Q, \Phi(w, \overline{w})]$$

$$= \frac{1}{2\pi i}\left[\oint dz\, T(z)\epsilon(z), \Phi(w, \overline{w}) \right] + \left[\oint d\overline{z}\, \overline{T}(\overline{z})\overline{\epsilon}(\overline{z}), \Phi(w, \overline{w}) \right]. \tag{7.18}$$

What does this equal-time commutator mean? First, note that the operators are $T(z)$, $\overline{T}(\overline{z})$, and $\Phi(w, \overline{w})$ and we can think of the commutators in (7.18) as, for example just considering the first commutator, letting $\Phi(w, \overline{w})$ act first and then acting with $T(z)$, but this only well-defined when $|w| < |z|$ which is equivalent to extending our contour $\mathcal{C}$ to enclose the point w in the complex plane (see figure 6), plus the contribution when we let $T(z)$ act first then $\Phi(w, \overline{w})$, which is equivalent to shrinking the contour to exclude the point w, i.e. $|z| < |w|$.
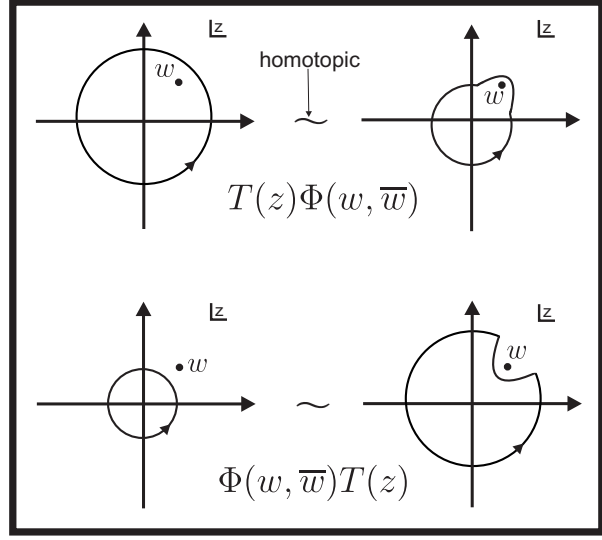
If we define the radial ordering[‡] operator R, of two operators $A(z)$ and $B(w)$, as

Figure 6:

$$R\Big[A(z)B(w)\Big] = \begin{cases} A(z)B(w) & \text{if } |w| < |z| \ , \\ B(w)A(z) & \text{if } |z| < |w| \ , \end{cases} \tag{7.19}$$

then we can rewrite $\delta_{\epsilon,\overline{\epsilon}}\Phi(w, \overline{w})$ as

$$\delta_{\epsilon,\overline{\epsilon}}\Phi(w, \overline{w}) = \frac{1}{2\pi i}\left(\oint_{|w|<|z|} - \oint_{|z|<|w|}\right)\left(dz\ \epsilon(z)R\Big[T(z)\Phi(w, \overline{w})\Big] + d\overline{z}\ \overline{\epsilon}(\overline{z})R\Big[\overline{T}(\overline{z})\Phi(w, \overline{w})\Big]\right). \tag{7.20}$$

[‡]Note that radially ordering operators on the complex plane corresponds to time ordering the operators on the cylinder. This is because we have that constant time slices map to circles of constant radius in the complex plane. So, we have that, for example, a correlation function of the fields $A_1(t_1), \cdots A_n(t_n)$ on the cylinder, given by

$$\langle k|\mathcal{T}\Big[A_1(t_1)\cdots A_n(t_n)\Big]|k'\rangle,$$

becomes, after the conformal mapping, the correlation function of the fields $A_1(z_1, \overline{z}_1) \cdots A_n(z_n, \overline{z}_n)$ on the complex plane given by

$$\langle k|R\Big[A_1(z_1, \overline{z}_1)\cdots A_n(z_n, \overline{z}_n)\Big]|k'\rangle,$$

where $\mathcal{T}$ is the time ordering operator, k and k' are arbitrary in and out states and R is the radial ordering operator.

This expression can be further evaluated by noting that the difference of the two contours, $|w| < |z|$ and $|z| < |w|$, is homotopy equivalent to the one contour centered around the point w in the complex plane, see figure 7.

So, we have that

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\overline{w}) \;=\; \frac{1}{2\pi i}\oint_{\mathcal{C}'}\Big\{\,\times$$

$$\times\; dz\; \epsilon(z)\; R\Big[T(z)\Phi(w,\overline{w})\Big]$$

$$+\; d\bar{z}\; \bar{\epsilon}(\bar{z})\; R\Big[\overline{T}(\bar{z})\Phi(w,\overline{w})\Big]\Big\}\quad (7.21)$$

where the contour $\mathcal{C}'$ is the contour enclosing the point w, see figure 7. But we know what this should be equal to because we know how Φ transforms since it is composed of primary fields. Namely, we know that

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\overline{w}) = h\partial_w\epsilon(w)\Phi(w,\overline{w}) + \epsilon(w)\partial_w\Phi(w,\overline{w})$$

$$+\,\overline{h}\partial_{\overline{w}}\bar{\epsilon}(\overline{w})\Phi(w,\overline{w})$$

$$+\,\bar{\epsilon}(\overline{w})\partial_{\overline{w}}\Phi(w,\overline{w})\,.\qquad (7.22)$$

Thus, setting (7.22) equal to (7.21) we see that in order for the charge Q, given by equation (7.17), to induce the correct infinitesimal conformal transformations, we infer that the short distance, i.e. when $z \mapsto w$, singularities of T and $\overline{T}$ with Φ should be given by[†]

Figure 7: The commutator $\big[T(z),\Phi(w,\overline{w})\big]$ of the two operators, $T(z)$ and $\Phi(w,\overline{w})$, is homotopic to a closed contour enveloping the point w.

$$R\Big[T(z)\Phi(w,\overline{w})\Big] = \frac{h}{(z-w)^2}\Phi(w,\overline{w}) + \frac{1}{z-w}\partial_w\Phi(w,\overline{w}) +\ \text{regular terms}\,,\ (7.23)$$

$$R\Big[\overline{T}(\bar{z})\Phi(w,\overline{w})\Big] = \frac{\overline{h}}{(\bar{z}-\overline{w})^2}\Phi(w,\overline{w}) + \frac{1}{\bar{z}-\overline{w}}\partial_{\overline{w}}\Phi(w,\overline{w}) +\ \text{regular terms}[¶]\,,(7.24)$$

[†]By regular terms, in the following expressions, we mean terms that are non-singular, or equivalently, terms with zeroth order poles.

where $(h, \overline{h})$ are called the conformal weights of the primary field $\Phi(w, \overline{w})$. To see this is indeed correct, note that from (7.21) we have that the variation of $\Phi(w, \overline{w})$, due to a general coordinate transform $\delta z = \epsilon(z)$, is given by

$$\delta_\epsilon \Phi(w, \overline{w}) = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz \ \epsilon(z) R\Big[T(z)\Phi(w, \overline{w})\Big],$$

where $\mathcal{C}$ is a closed contour enclosing the point w. Plugging in for $R\Big[T(z)\Phi(w, \overline{w})\Big]$, from (7.23), the above becomes

$$\delta_\epsilon \Phi(w, \overline{w}) = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz \ \epsilon(z) \left(h\frac{\Phi(w, \overline{w})}{(z - w)^2} + \frac{\partial_w \Phi(w, \overline{w})}{(z - w)} \right).$$

Now, since $\epsilon(z)$ is a holomorphic function, i.e. it has no singularities, its Laurent expansion around the point w is given by

$$\epsilon(z) = \epsilon(w) + \partial_w \epsilon(w)(z - w) + \cdots .$$

Plugging this in gives

$$\delta_\epsilon \Phi(w, \overline{w}) = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz \ \big(\epsilon(w) + \partial_w \epsilon(w)(z - w) + \cdots\big) \left(h\frac{\Phi(w, \overline{w})}{(z - w)^2} + \frac{\partial_w \Phi(w, \overline{w})}{(z - w)} \right)$$

$$= \frac{1}{2\pi i} \oint_{\mathcal{C}} dz \ \Big(\epsilon(w) h\Phi(w, \overline{w})\frac{1}{(z - w)^2} + \epsilon(w)\partial_w \Phi(w, \overline{w})\frac{1}{(z - w)}$$

$$+ \partial_w \epsilon(w) h\Phi(w, \overline{w})\frac{1}{(z - w)} + \partial_w \epsilon(w)\partial_w \Phi(w, \overline{w}) + \cdots\Big)$$

$$= \epsilon(w) h\Phi(w, \overline{w})\frac{1}{2\pi i} \oint_{\mathcal{C}} dz\frac{1}{(z - w)^2} + \epsilon(w)\partial_w \Phi(w, \overline{w})\frac{1}{2\pi i} \oint_{\mathcal{C}} dz\frac{1}{(z - w)}$$

$$+ \partial_w \epsilon(w) h\Phi(w, \overline{w})\frac{1}{2\pi i} \oint_{\mathcal{C}} dz\frac{1}{(z - w)} + \partial_w \epsilon(w)\partial_w \Phi(w, \overline{w})\frac{1}{2\pi i} \oint_{\mathcal{C}} dz(1)$$

$$+ (\cdots)\frac{1}{2\pi i} \oint_{\mathcal{C}} dz(1) .$$

This can be simplified via the residue theorem[‡] to give

$$\delta_\epsilon \Phi(w, \overline{w}) = h\partial_w \epsilon(w)\Phi(w, \overline{w}) + \epsilon(w)\partial_w \Phi(w, \overline{w}),$$

which is what we should get, see (7.22).

[‡]See A.

We have just seen that the transformation property of primary fields leads to a short distance operator product expansion (OPE) for the holomorphic and anti-holomorphic stress-energy tensors, T and $\overline{T}$, with the field Φ. Note that if the OPE of a field, with the stress-energy tensor, is not of this form, then the field called primary. For example, we will see later on that the field $X(z, \overline{z})$ is not a primary field.

7.3 Operator Product Expansion (OPE)

The operator product expansion (OPE) tells one what happens as a collection of local operators approach each other, i.e. if one has two operators $A(x)$ and $B(y)$ and wants to see what happens as they approach each other, $x \mapsto y$, then one needs the OPE of $A(x)$ and $B(y)$. The basic idea behind the OPE is that it should be possible to write the product of local operators, at close points, as a linear combination of a complete set of operators at one of the two points. Thus, if $A_i(z, \overline{z})$ is a local operator at $(z, \overline{z})$ and if $A_j(w, \overline{w})$ is a local operator at $(w, \overline{w})$ then the OPE of the product $A_i(z, \overline{z})A_j(w, \overline{w})$ is given by

$$A_i(z, \overline{z})A_j(w, \overline{w}) = \sum_{k \in \mathfrak{I}} c_{ij}^k(z - w, \overline{z} - \overline{w})\mathcal{O}_k(w, \overline{w}), \qquad (7.25)$$

where the indexing set $\mathfrak{I}$ can either be finite or countable. Also, the functions $c_{ij}^k(z - w, \overline{z} - \overline{w})$ depend only on the differences between the points $(z, \overline{z})$ and $(w, \overline{w})$, due to translational invariance, and the set of operators $\{\mathcal{O}_k\}$ is complete. Note that the above expression is an operator expression and thus only holds inside a general (time/radial-ordered) expectation value,

$$\langle A_i(z, \overline{z})A_j(w, \overline{w}) \cdots \rangle =$$

$$= \sum_{k \in \mathfrak{I}} c_{ij}^k(z - w, \overline{z} - \overline{w})\langle \mathcal{O}_k(w, \overline{w}) \cdots \rangle, \quad (7.26)$$

Figure 8: This is a figure of local operators defined on the complex plane, $\mathbb{C}$. The OPE for $A_i(z)A_j(w)$ has radius of convergence given, pictorially, by the circle.

where the dots represent other local operators defined at points which are much further away from $(w, \overline{w})$ than $|z - w|$, i.e. $A_i(z, \overline{z})$ and $A_j(w, \overline{w})$ are much closer to each other than to any of the other local operators. Also, the above expression for the OPE is, in a two dimensional CFT, a convergent sequence whose radius of convergence is equal to the distance of the next nearest operator, see figure 8. Finally, in the limit $z \mapsto w$ we have that

some of the functions become singular. But, one should not get concerned because this is precisely the part of the OPE that matters to us as can be seen from the relations (7.23) and (7.24) and so, in general, we will write the OPE sum up to singular terms only since we will not care about the regular terms in the expansion, which in fact vanish when inside a contour integral.

So, to recap, we have seen that the transformation law for primary fields leads to a short distance OPE for the holomorphic and anti-holomorphic stress-energy tensors, T and $\overline{T}$, with a primary field Φ. From now on we shall drop the R symbol and consider the OPE as a shorthand notation for radially ordered products. In general, we have that the OPE of a primary field $\Phi(w, \overline{w})$, of conformal weight $(h, \overline{h})$, with $T(z)$ and $\overline{T}(\overline{z})$ is given by

$$T(z)\Phi(w, \overline{w}) \sim \frac{h}{(z - w)^2}\Phi(w, \overline{w}) + \frac{1}{z - w}\partial_w \Phi(w, \overline{w}), \qquad (7.27)$$

$$\overline{T}(\overline{z})\Phi(w, \overline{w}) \sim \frac{\overline{h}}{(\overline{z} - \overline{w})^2}\Phi(w, \overline{w}) + \frac{1}{\overline{z} - \overline{w}}\partial_{\overline{w}} \Phi(w, \overline{w}), \qquad (7.28)$$

where $\sim$ means up to regular terms (i.e. non-singular terms). Note that we could, in fact, take this to be the definition of a primary field and thus any field whose OPE with $T(z)$ and $\overline{T}(\overline{z})$ is not of this form is not a primary field. Also, the weights, h and $\overline{h}$, are not as unfamiliar as they appear. They simply tell us how operators transform under rotations and scalings. But we already have names for these concepts from undergraduate days. The eigenvalue under rotation is usually called the spin, s, and is given in terms of the weights as

$$s = h - \overline{h}. \qquad (7.29)$$

Whereas the scaling dimension, $\triangle$, of an operator is defined by

$$\triangle = h + \overline{h}. \qquad (7.30)$$

In the next chapter we will calculate the OPE's for some specific quantities in the free bosonic field theory. Then we will show that the corresponding charges for the conserved current arising from global conformal transformation are the Virasoro generators, L_m. Afterwords we will look at the link between physical states and highest weight representations of the Virasoro algebra. Finally, we will define the Ward identities for a conformal field theory.

7.4 Exercises

Problem 1

The expansion of the free massless scalar X living on a Lorentzian cylinder is

$$X(\tau, \sigma) = x + 4p\tau + i \sum_{n \neq 0} \frac{1}{n} (\alpha_n e^{2in\sigma} + \tilde{\alpha}_n e^{-2in\sigma}) e^{-2in\tau}, \tag{7.31}$$

where $\sigma \cong \sigma + \pi$. This is obtained by setting $l_s = 2$ (or $\alpha' = 2$) in the expansion formula for the X^μ fields for closed string (BBS (2.40), (2.41)).

(a) Write down the expansion formula on a Euclidean cylinder. First do a Wick rotation $\tau \to -i\tau$ and then express the result in terms of the complex coordinates $\zeta = 2(\tau - i\sigma)$, $\bar{\zeta} = 2(\tau + i\sigma)$.

(b) Derive the following expansion formula for a complex z-plane, by defining $z = e^\zeta$, $\bar{z} = e^{\bar{\zeta}}$:

$$X(z, \bar{z}) = x - ip \log |z|^2 + i \sum_{n \neq 0} \frac{1}{n} (\alpha_n z^{-n} + \tilde{\alpha}_n \bar{z}^{-n}). \tag{7.32}$$

(c) The commutation relations for $x, p, \alpha_n, \tilde{\alpha}_n$ are given by

$$[x, p] = i, \qquad [\alpha_m, \alpha_n] = [\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta_{m+n,0}, \tag{7.33}$$

with all other commutators vanishing. We define the creation-annihilation normal ordering by

$$:xp: \, = \, :px: \, = xp,$$
$$:\alpha_m \alpha_{-n}: \, = \, :\alpha_{-n}\alpha_m: \, = \alpha_{-n}\alpha_m, \tag{7.34}$$
$$:\tilde{\alpha}_m \tilde{\alpha}_{-n}: \, = \, :\tilde{\alpha}_{-n}\tilde{\alpha}_m: \, = \tilde{\alpha}_{-n}\tilde{\alpha}_m,$$

where $m, n > 0$. Namely, it places all lowering operators ($\alpha_n, \tilde{\alpha}_n$ with $n > 0$) to the right of all raising operators ($\alpha_n, \tilde{\alpha}_n$ with $n < 0$). We include p with the lowering operators and x with the raising operators. Show that the following relation holds

$$X(z, \bar{z})X(w, \bar{w}) = \, :X(z, \bar{z})X(w, \bar{w}): - \log |z - w|^2, \tag{7.35}$$

if $|z| > |w|$. Note the identity

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1 - x), \qquad |x| < 1. \tag{7.36}$$

Problem 2

Consider the 2-point function of primary fields in a two-dimensional CFT,

$$G(z_1, \bar{z}_1, z_2, \bar{z}_2) = \langle \Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_2) \rangle. \tag{7.37}$$

Let the conformal dimensions of Φ_1, Φ_2 be $(h_1, \bar{h}_1)$ and $(h_2, \bar{h}_2)$. Namely, under the infinitesimal conformal transformation $z \to z + \epsilon(z)$, the fields $\Phi_{1,2}$ transform as

$$\delta_\epsilon \Phi_i(z) = [\epsilon(z)\partial + h_i \partial\epsilon(z)]\Phi_i(z), \qquad i = 1, 2. \tag{7.38}$$

(Actually there is also an antiholomorphic part depending on $\bar{\epsilon}(\bar{z})$, but in this problem we focus on the holomophic part only.)

(a) Show that the invariance of the 2-point function under a conformal transformation implies the following equation:

$$[\epsilon(z_1)\partial_1 + h_1\partial\epsilon(z_1) + \epsilon(z_2)\partial_2 + h_2\partial\epsilon(z_2)]G(z_1, z_2) = 0. \tag{7.39}$$

(b) By setting $\epsilon(z) = 1$ in (C.193), show that $G(z_1, z_2)$ is a function of $x = z_1 - z_2$ only.

(c) By setting $\epsilon(z) = z$ in (C.193), show that $G(x)$ takes the following form:

$$G(x) = \frac{C}{x^{h_1+h_2}}, \tag{7.40}$$

with C a constant.

(d) By setting $\epsilon(z) = z^2$ in (C.193), show that $G(x)$ vanishes unless $h_1 = h_2$.

So, in this problem, we have shown

$$G(z_1, z_2) = \langle \Phi_1(z_1)\Phi_2(z_2) \rangle = \begin{cases} \dfrac{C}{(z_1 - z_2)^{2h}} & (h_1 = h_2 = h), \\ 0 & (h_1 \neq h_2). \end{cases} \tag{7.41}$$

8. OPE Redux, the Virasoro Algebra and Physical States

8.1 The Free Massless Bosonic Field

We begin this lecture with an example in order to further instill the topics that were mentioned earlier. The example will be of the free massless bosonic field (recall that since a CFT has no length scales it also implies that it cannot have any masses). The action for this theory is given by

$$S = \frac{1}{2\pi} \int dz d\bar{z} \; \partial_z X(z,\bar{z}) \; \partial_{\bar{z}} X(z,\bar{z}). \tag{8.1}$$

It is not hard to show that the equations of motion, resulting from the above action, are given by

$$\partial_z \partial_{\bar{z}} X(z,\bar{z}) = 0. \tag{8.2}$$

Thus, we see that the field decomposes into a holomorphic piece and an anti-holomorphic piece[§],

$$X(z,\bar{z}) = X(z) + \overline{X}(\bar{z}). \tag{8.3}$$

For the calculations that follow we will need to know the propagator $\overbrace{X(z,\bar{z})X(w,\overline{w})}$ for this theory[‡]. To calculate this quantity we will proceed as follows. From problem 7.1 we know that

$$X(z,\bar{z})X(w,\overline{w}) = \;:X(z,\bar{z})X(w,\overline{w})\!: - \log|z-w|^2, \tag{8.4}$$

where $::$ is the normal ordering of the two operators. Also, remember that the LHS of (8.4) really means the radial ordering of the the two operators (recall that we are no longer using the R symbol), i.e (8.4) really is saying that

$$R\Big[X(z,\bar{z})X(w,\overline{w})\Big] = \;:X(z,\bar{z})X(w,\overline{w})\!: - \log|z-w|^2.$$

[§]Since our theory decomposes into these two parts we can consider each one separately. For example, primary fields are holomorphic (anti-holomorphic) with weights h $(\overline{h})$.

[‡]Here we will adopt the contraction notation

$$\overbrace{X(z,\bar{z})X}(w,\overline{w}),$$

rather than the two-point notation given by

$$\langle \; X(z,\bar{z})X(w,\overline{w}) \; \rangle.$$

Now, from Wick's theorem for two bosonic fields $X(z, \overline{z})$ and $X(w, \overline{w})$, which states that the radial ordering of the fields is equal to the normal ordering of the same fields plus their contraction,

$$X(z, \overline{z})X(w, \overline{w}) = \; :X(z, \overline{z})X(w, \overline{w}): + \overbracket{X(z, \overline{z})X}(w, \overline{w}), \tag{8.5}$$

we can see that the contraction of $X(z, \overline{z})$ and $X(w, \overline{w})$ is given by

$$\overbracket{X(z, \overline{z})X}(w, \overline{w}) = -\log\left(|z - w|^2\right). \tag{8.6}$$

Also, if we want to work with just, for example, the holomorphic parts of the fields $X(z, \overline{z})$ and $X(w, \overline{w})$ then we need to know their contraction, $\overbracket{X(z)X}(w)$. So, to begin we have that

$$\overbracket{X(z, \overline{z})X}(w, \overline{w}) = \left(X(z) + \overline{X}(\overline{z})\right)\left(X(w) + \overline{X}(\overline{w})\right)$$

$$= \overbracket{X(z)X}(w) + \overbracket{\overline{X}(\overline{z})X}(w) + \overbracket{X(z)\overline{X}}(\overline{w}) + \overbracket{\overline{X}(\overline{z})\overline{X}}(\overline{w}),$$

and also

$$\overbracket{X(z, \overline{z})X}(w, \overline{w}) = -\log\left(|z - w|^2\right)$$

$$= -\log\left((z - w)(\overline{z} - \overline{w})\right)$$

$$= -\log(z - w) - \log(\overline{z} - \overline{w}).$$

Thus, we have four contractions (or correlation functions) on the LHS and they are equal to a function of $z - w$ plus a function of $\overline{z} - \overline{w}$. Now, we know, see section 6.5.1, that the correlation function of two CFT's can only depend on the difference of the fields arguments, i.e. the contraction $\overbracket{X(z)\overline{X}}(\overline{w})$ is equal to some function of $z - \overline{w}$. Using this we can quickly see that, for the massless bosonic field,

$$\overbracket{X(z)\overline{X}}(\overline{w}) = 0, \qquad \overbracket{\overline{X}(\overline{z})X}(w) = 0, \tag{8.7}$$

while

$$\overbracket{X(z)X}(w) = -\log(z - w), \qquad \overbracket{\overline{X}(\overline{z})\overline{X}}(\overline{w}) = -\log(\overline{z} - \overline{w}). \tag{8.8}$$

With all of the propagator business behind us, we will now define the stress-energy tensor for the theory in question. From the action

$$S = \frac{1}{2\pi} \int dz d\bar{z} \ \partial_z X(z, \bar{z}) \ \partial_{\bar{z}} X(z, \bar{z}),$$

and the definition of T in terms of the variation of the action with respect to a metric, we would think that the holomorphic part of the stress-energy tensor should be simply given by[‡]

$$T(z) = -\frac{1}{2} \partial_z X(z) \partial_z X(z).$$

But this expression runs into the problem of singularities when it is quantized since then we would have the product of two operators, $\partial_z X(z)$, at the same point. So, we can fix this problem by normal ordering the expression (since we are dealing with a free theory, normal ordering is all that is required). Keeping in mind the previous arguments, we define the stress-energy tensor for the free massless bosonic field to be,

$$T(z) = -\frac{1}{2} {:} \partial_z X(z) \partial_z X(z) {:} \equiv -\frac{1}{2} \lim_{z \mapsto w} \left(\partial_z X(z) \partial_w X(w) - \text{singularity} \right).$$

Now, what is the singularity in the expression for $T(z)$? Once again from Wick's theorem, the singularity is given by

$$\text{singularity} = \overbrace{\partial_z X(z) \partial_w X(w)}.$$

[‡]To see this note that the previous action is really the action

$$S = \frac{1}{2\pi} \int d\tau d\sigma \ \partial_\alpha X \partial^\alpha X,$$

with a flat metric. Now, varying this with respect to the metric gives us, also noting the definition of the stress-energy tensor,

$$T_{\alpha\beta} = -\frac{1}{2} \left(\partial_\alpha X \partial_\beta X - \frac{1}{2} \delta_{\alpha\beta} (\partial X)^2 \right).$$

Simplifying this to the flat metric and using our complex coordinates, we get

$$T(z) = -\frac{1}{2} \partial_z X(z) \partial_z X(z), \qquad \overline{T}(\bar{z}) = -\frac{1}{2} \partial_{\bar{z}} \overline{X}(\bar{z}) \partial_{\bar{z}} \overline{X}(\bar{z}).$$

We can calculate this singularity using the propagator as follows,

$$\partial_z \overset{\frown}{X(z) \partial_w} X(w) = \partial_z \partial_w \left(\overset{\frown}{X(z) X}(w) \right)$$

$$= \partial_z \partial_w \left(- \log(z - w) \right)$$

$$= -\frac{1}{(z - w)^2} \, .$$

Thus, finally, we define the holomorphic part of the stress-energy tensor to be

$$T(z) = -\frac{1}{2} : \partial_z X(z) \partial_z X(z) : \equiv -\frac{1}{2} \lim_{z \mapsto w} \left(\partial_z X(z) \partial_w X(w) + \frac{1}{(z - w)^2} \right), \qquad (8.9)$$

while for the anti-holomorphic part we have

$$\overline{T}(\overline{z}) = -\frac{1}{2} : \partial_{\overline{z}} \overline{X}(\overline{z}) \partial_{\overline{z}} \overline{X}(\overline{z}) : \equiv -\frac{1}{2} \lim_{\overline{z} \mapsto \overline{w}} \left(\partial_{\overline{z}} \overline{X}(\overline{z}) \partial_{\overline{w}} \overline{X}(\overline{w}) + \frac{1}{(\overline{z} - \overline{w})^2} \right). \qquad (8.10)$$

With this expression for the stress-energy tensor being given we can now compute the OPE's of certain fields with the stress-energy tensor to see which ones are primary fields in the free bosonic field theory. The first field we want to look at is the field $X(w)$. For this field, the OPE with $T(z)$ is given by[¶]

$$T(z)X(w) = -\frac{1}{2} : \partial_z X(z) \partial_z X(z) : X(w)$$

$$= -\frac{1}{2} : \partial_z X(z) \partial_z X(z) X(w) : - \frac{1}{2} \partial_z X(z) \partial_z \left(\overset{\frown}{X(z) X}(w) \right)$$

$$- \frac{1}{2} \partial_z X(z) \partial_z \left(\overset{\frown}{X(z) X}(w) \right) + \text{regular terms}$$

$$= -\partial_z X(z) \partial_z \left(\overset{\frown}{X(z) X}(w) \right) + \text{reg. terms}$$

$$= -\partial_z X(z) \partial_z \left(- \log(z - w) \right) + \text{reg. terms}$$

$$\sim \partial_z X(z) \frac{1}{z - w} \, ,$$

[¶]For what follows we will use Wick's theorem for several fields and one can consult B for a review of the idea behind the theorem.

where as before $\sim$ implies equivalence up to regular terms. Now, we need for the LHS of this expression to only include functions at w, see the expression for an OPE, and thus we expand $\partial_z X(z)$ around the point w. Doing this we get

$$T(z)X(w) = \partial_z X(z) \frac{1}{z-w}$$

$$= \left(\partial_w X(w) + \partial_w^2 X(w)(z-w) + \cdots \right) \left(\frac{1}{z-w} + \text{reg. terms} \right)$$

$$\sim \frac{\partial_w X(w)}{z-w}.$$

And so, we see that the field $X(w)$ is not a primary field. Recall that a primary field of weight $(h, \overline{h})$ has an OPE with the stress-energy tensor of the form

$$T(z)\Phi(w, \overline{w}) \sim \frac{h}{(z-w)^2}\Phi(w, \overline{w}) + \frac{1}{z-w}\partial_w \Phi(w, \overline{w}),$$

$$\overline{T}(\overline{z})\Phi(w, \overline{w}) \sim \frac{\overline{h}}{(\overline{z}-\overline{w})^2}\Phi(w, \overline{w}) + \frac{1}{\overline{z}-\overline{w}}\partial_{\overline{w}}\Phi(w, \overline{w}),$$

and so, for $X(w)$ to be a primary holomorphic field, of weight h, its OPE should be of the form,

$$T(z)X(w) \sim \frac{h}{(z-w)^2}X(w) + \frac{1}{z-w}\partial_w X(w),$$

which it is not. Similarly, we have that

$$\overline{T}(\overline{z})\overline{X}(\overline{w}) \sim \frac{\partial_{\overline{w}}\overline{X}(w)}{\overline{z}-\overline{w}}, \tag{8.11}$$

and so $\overline{X}(\overline{w})$ is not an antiholomorphic primary field either. Thus, their combination $X(w, \overline{w})$ is not a primary field.

Next, let's consider the field $\partial_w X(w)$. This field's OPE with $T(z)$ is given by

$$T(z)\partial_w X(w) = -\frac{1}{2}{:}\partial_z X(z)\partial_z X(z){:}\partial_w X(w)$$

$$\sim -\partial_z X(z)\partial_z \partial_w \left(\overline{X(z)X(w)} \right)$$

$$\sim \partial_z X(z)\partial_z \partial_w \left(\log(z-w) \right)$$

$$\sim \partial_z X(z) \frac{1}{(z-w)^2}$$

$$\sim \left(\partial_w X(w) + (z-w)\partial_w^2 X(w) + \cdots \right) \frac{1}{(z-w)^2}$$

$$\sim \frac{\partial_w X(w)}{(z-w)^2} + \frac{\partial_w \partial_w X(w)}{(z-w)} .$$

From the above definition for a primary field, we see that $\partial_w X(w)$ is a primary field of conformal weight $h = 1$. Also, we have that

$$\overline{T}(\overline{z})\partial_{\overline{w}}\overline{X}(\overline{w}) \sim \frac{\partial_{\overline{w}}\overline{X}(\overline{w})}{(\overline{z}-\overline{w})^2} + \frac{\partial_{\overline{w}}\partial_{\overline{w}}\overline{X}(\overline{w})}{(\overline{z}-\overline{w})},$$

and so $\partial_{\overline{w}}\overline{X}(\overline{w})$ is a primary antiholomorphic field of weight $\overline{h} = 1$. We can combine these results, noting that $T(z)\partial_{\overline{w}}\overline{X}(\overline{w}) = 0$ and $\overline{T}(\overline{z})\partial_w X(w) = 0$ which follows simply from the above arguments leading to (8.7) and (8.8) (just examine what kind of contractions will arise from these OPE's), to give that

$$T(z)\partial_w X(w,\overline{w}) = \frac{\partial_w X(w)}{(z-w)^2} + \frac{\partial_w \partial_w X(w)}{(z-w)},$$

$$\overline{T}(\overline{z})\partial_w X(w,\overline{w}) = 0, \qquad (8.12)$$

and

$$T(z)\partial_{\overline{w}}\overline{X}(\overline{w}) = 0,$$

$$\overline{T}(\overline{z})\partial_{\overline{w}}\overline{X}(\overline{w}) = \frac{\partial_{\overline{w}}\overline{X}(\overline{w})}{(\overline{z}-\overline{w})^2} + \frac{\partial_{\overline{w}}\partial_{\overline{w}}\overline{X}(\overline{w})}{(\overline{z}-\overline{w})}, \qquad (8.13)$$

which implies that $\partial_w X(w,\overline{w})$ is a primary field of weight $(1,0)$ and $\partial_{\overline{w}}X(w,\overline{w})$ is a primary field of weight $(0,1)$.

The next OPE we want to construct is the OPE of $T(w)$ with $T(z)$. This is given by

$$T(z)T(w) = \frac{1}{4}:\partial_z X(z)\partial_z X(z)::\partial_w X(w)\partial_w X(w):$$

$$\sim \partial_z\partial_w\left(\overbrace{X(z)X}(w)\right):\partial_z X(z)\partial_w X(w): + \frac{1}{2}\left(\partial_z\partial_w(\overbrace{X(z)X}(w))\right)^2$$

$$\sim \frac{1}{(z-w)^2}:\partial_z X(z)\partial_w X(w): + \frac{1/2}{(z-w)^4}$$

$$\sim \frac{:\partial_w X(w)\partial_w X(w):}{(z-w)^2} + \frac{:\partial_w\partial_w X(w)\partial_w X(w):}{(z-w)} + \frac{1/2}{(z-w)^4}$$

$$\sim \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \frac{1/2}{(z-w)^4} .$$

We see that $T(w)$ is almost primary except for the fourth order pole. Note that the value $1/2$ appearing in the fourth order pole above depends on the particular theory in question. For example, suppose we have d scalar fields, $X^\mu(z)$ with $\mu = 0, ..., d-1$, then for $T(z)$ we get

$$T(z) = -\frac{1}{2}:\partial_z X^\mu(z)\partial^z X_\mu(z): , \tag{8.14}$$

and the only difference in calculating the OPE of $T(z)$ with $T(w)$, with the previous calculation, comes in the contractions

$$: \partial_z X^\mu \partial^z X_\mu :: \partial_z X^\nu \partial^z X_\nu :\sim -\frac{1}{(z-w)^2}\delta^\mu_\nu \delta^\nu_\mu = -\frac{1}{(z-w)^2}d.$$

So, in general, if we have c fields, $X^\mu(z)$, and $\overline{c}$ fields, $\overline{X}^\mu(\overline{z})$, then

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{reg. terms}, \tag{8.15}$$

and also

$$\overline{T}(\overline{z})\overline{T}(\overline{w}) = \frac{\overline{c}/2}{(\overline{z}-\overline{w})^4} + \frac{2\overline{T}(\overline{w})}{(\overline{z}-\overline{w})^2} + \frac{\partial_{\overline{w}}\overline{T}(\overline{w})}{\overline{z}-\overline{w}} + \text{reg. terms}, \tag{8.16}$$

where c and $\overline{c}$ are called the central charges of our CFT. Some things should be mentioned here. First, it is clear that c and $\overline{c}$ somehow measure the number of degrees of freedom of our CFT[‡]. And also, there is nothing that says, in general, that c has to be related to $\overline{c}$ or that they have to be integer valued. However, due to modular invariance we have that $c - \overline{c} = 0 \ Mod \ 24$. Also, a theory with a Lorentz invariant, conserved two-point function $\langle T_{\mu\nu}(p)T_{\alpha\beta}(-p)\rangle$ requires that $c = \overline{c}$. This is equivalent to requiring cancellation of local gravitational anomalies, allowing the system to be consistently coupled to two dimensional gravity. Finally, note that for the example we are doing we do in fact agree with the general formula, (8.15), since our theory splits into two independent sectors with one field in each, $X(z)$ in the holomorphic sector and $\overline{X}(\overline{z})$ in the antiholomorphic sector, and we get that $c = 1$.

The final example of an OPE we will mention is that of the field $:e^{i\alpha X(w)}:$ with the stress-energy tensor, where we write the $::$ to remind ourselves that there are no

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[‡]Note that the bosonic string theory, i.e. the Polyakvo action, is constructed out of 26 free bosonic fields X^μ, and thus we see, as before, that the central charge of this theory is 26.

singularities within the operator $e^{i\alpha X(w)}$. This OPE is given by

$$T(z)\,:e^{i\alpha X(w)}:\,=-\frac{1}{2}:\partial_z X(z)\partial_z X(z)::e^{i\alpha X(w)}:\,=-\frac{1}{2}:\partial_z X(z)\partial_z X(z)::\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\big(X(w)\big)^n:$$

$$=\quad\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\Big[-\frac{1}{2}:\partial_z X(z)\partial_z X(z)\underbrace{X(w)\cdots X(w)}_{n\text{ times}}:\Big]$$

$$\sim\quad-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\Big[\partial_z\big(\overline{X(z)X}(w)\big):\partial_z X(z)\underbrace{X(w)\cdots X(w)}_{n-1\text{ times}}:+\cdots$$

$$+\quad\partial_z\big(\overline{X(z)X}(w)\big):\partial_z X(z)\underbrace{X(w)\cdots X(w)}_{n-1\text{ times}}:)+\partial_z\big(\overline{X(z)X}(w)\big):\partial_z X(z)\underbrace{X(w)\cdots X(w)}_{n-1\text{ times}}:$$

$$+\quad\cdots+\partial_z\big(\overline{X(z)X}(w)\big):\partial_z X(z)\underbrace{X(w)\cdots X(w)}_{n-1\text{ times}}:\Big]$$

$$-\quad\frac{1}{2}\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\Big[\partial_z\big(\overline{X(z)X}(w)\big):\partial_z X(z)\underbrace{X(w)\cdots X(w)}_{n-1\text{ times}}:+\cdots$$

$$+\quad\partial_z\big(\overline{X(z)X}(w)\big):\partial_z X(z)\underbrace{X(w)\cdots X(w)}_{n-1\text{ times}}:)+\partial_z\big(\overline{X(z)X}(w)\big):\partial_z X(z)\underbrace{X(w)\cdots X(w)}_{n-1\text{ times}}:$$

$$+\quad\cdots+\partial_z\big(\overline{X(z)X}(w)\big):\partial_z X(z)\underbrace{X(w)\cdots X(w)}_{n-1\text{ times}}:\Big]$$

$$-\quad\frac{1}{2}\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\Big[\partial_z\big(\overline{X(z)X}(w)\big)\partial_z\big(\overline{X(z)X}(w)\big):\underbrace{X(w)\cdots X(w)}_{n-2\text{ times}}:$$

$$+\quad\cdots+\partial_z\big(\overline{X(z)X}(w)\big)\partial_z\big(\overline{X(z)X}(w)\big):\underbrace{X(w)\cdots X(w)}_{n-2\text{ times}}:\Big]$$

$$\sim\quad-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\Big[2n\partial_z\big(\overline{X(z)X}(w)\big):\partial_z X(z)\underbrace{X(w)\cdots X(w)}_{n-1\text{ times}}:\Big]$$

$$-\quad\frac{1}{2}\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\Big[n(n-1)\partial_z\big(\overline{X(z)X}(w)\big)\partial_z\big(\overline{X(z)X}(w)\big):\underbrace{X(w)\cdots X(w)}_{n-2\text{ times}}:\Big]$$

$$\sim\quad-i\alpha\sum_{n=1}^{\infty}\frac{(i\alpha)^{n-1}}{(n-1)!}\Big[\partial_z\big(\overline{X(z)X}(w)\big):\partial_z X(z)\underbrace{X(w)\cdots X(w)}_{n-1\text{ times}}:\Big]$$

$$-\quad\frac{(i\alpha)^2}{2}\sum_{n=2}^{\infty}\frac{(i\alpha)^{n-2}}{(n-2)!}\Big[\partial_z\big(\overline{X(z)X}(w)\big)\partial_z\big(\overline{X(z)X}(w)\big):\underbrace{X(w)\cdots X(w)}_{n-2\text{ times}}:\Big]$$

$$\sim -i\alpha \partial_z \big(\overbrace{X(z)X}(w)\big){:}\partial_z X(z) \sum_{n=1}^{\infty} \frac{(i\alpha)^{n-1}}{(n-1)!}(X(w))^{n-1}{:}$$

$$-\frac{(i\alpha)^2}{2}\partial_z\big(\overbrace{X(z)X}(w)\big)\partial_z\big(\overbrace{X(z)X}(w)\big){:}\sum_{n=2}^{\infty}\frac{(i\alpha)^{n-2}}{(n-2)!}(X(w))^{n-2}{:}\bigg]$$

$$\sim -i\alpha\partial_z\big(\overbrace{X(z)X}(w)\big){:}\partial_z X(z)e^{i\alpha X(w)}{:} - \frac{(i\alpha)^2}{2}\partial_z\big(\overbrace{X(z)X}(w)\big)\partial_z\big(\overbrace{X(z)X}(w)\big){:}e^{i\alpha X(w)}{:}$$

$$\sim \frac{i\alpha}{(z-w)}{:}\partial_z X(z)e^{i\alpha X(w)}{:} - \frac{(i\alpha)^2/2}{(z-w)^2}{:}e^{i\alpha X(w)}{:}$$

$$\sim \frac{i\alpha}{(z-w)}{:}(\partial_w X(w)+\cdots)e^{i\alpha X(w)}{:} - \frac{(i\alpha)^2/2}{(z-w)^2}{:}e^{i\alpha X(w)}{:}$$

$$\sim \frac{i\alpha}{(z-w)}{:}\partial_w X(w)e^{i\alpha X(w)}{:} - \frac{(i\alpha)^2/2}{(z-w)^2}{:}e^{i\alpha X(w)}{:}$$

$$\sim \frac{1}{(z-w)}{:}\partial_w(i\alpha e^{i\alpha X(w)}){:} - \frac{i^2\alpha^2/2}{(z-w)^2}{:}e^{i\alpha X(w)}{:}$$

$$\sim \frac{1}{(z-w)}\partial_w{:}e^{i\alpha X(w)}{:} + \frac{\alpha^2/2}{(z-w)^2}{:}e^{i\alpha X(w)}{:}$$

$$\sim \frac{\alpha^2/2}{(z-w)^2}{:}e^{i\alpha X(w)}{:} + \frac{1}{(z-w)}\partial_w{:}e^{i\alpha X(w)}{:}\,.$$

Thus, we see that ${:}e^{i\alpha X(w)}{:}$ is a primary holomorphic field with conformal weight $h = \alpha^2/2$. In an analogous manner, one can show that ${:}e^{i\bar\alpha \overline{X}(\overline{w})}{:}$ is a primary antiholomorphic field of weight $\bar\alpha^2/2$. We can combine the holomorphic and anti-holomorphic pieces as before to show that the composition ${:}e^{i\alpha X(w)+i\bar\alpha\overline{X}(\overline{w})}{:}$ is a primary field with weight $(\alpha^2/2, \bar\alpha^2/2)$. After this painful calculation we decide to leave OPE's for now; for more examples see the exercises.

8.2 Charges of the Conformal Symmetry Current

Recall that we had previously mentioned that the current corresponding to conformal symmetry was given by

$$J(z) = T(z)\epsilon(z), \tag{8.17}$$

were $\epsilon(z)$ is some holomorphic function. Similarly, we also have that the antiholomorphic part of the current $\overline{J}(\overline{z}) = \overline{T}(\overline{z})\bar\epsilon(\overline{z})$ where $\bar\epsilon(\overline{z})$ is some anti-holomorphic function, but we will not concern ourselves with this since all the derivations that follow are completely similarly for $J(z)$ and $\overline{J}(\overline{z})$.

Now, since $\epsilon(z)$ is holomorphic it is natural to expand it in terms of modes. Note that the particular mode expansion depends on the surface in which we are working. For example, in our case, the Riemann sphere, we expect $\epsilon(z)$ to take the form z^{n+1}. And thus, we get an infinite set of currents $J^n(z) = T(z)z^{n+1}$ corresponding to each value of $n \in \mathbb{Z}$. With this expression for the current we can write its corresponding charges, which we denote by L_n, as

$$L_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz \, T(z)z^{n+1}, \tag{8.18}$$

where $\mathcal{C}$ is a closed contour which encloses the origin, $z = 0$. We can formally invert this relation, via Cauchy's theorem, to give the mode expansion for $T(z)$,

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2}L_n. \tag{8.19}$$

As was already stated, in radial quantization the charge L_n is the conserved charge associated to the conformal transformation $\delta z = z^{n+1}$. To see this, recall that the corresponding Noether current, is $J(z) = z^{n+1}T(z)$. Moreover, the contour integral $\oint dz$ maps to the integral around spatial slices on the cylinder. This tells us that L_n is the conserved charge, where conserved means that it is constant under time evolution on the cylinder, or under radial evolution on the plane. Also note that, in general, when we quantize a theory the conserved charges of some symmetry become the generators of the transformations. And so, we see that operators L_n generate the conformal transformations $\delta z = z^{n+1}$.
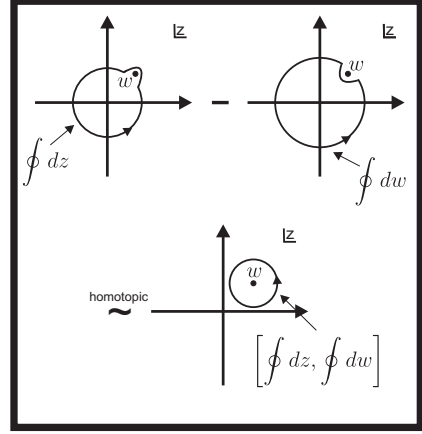
Figure 9:

To compute the algebra of commutators satisfied by the operators L_n, we proceed as follows. Recall that the commutator of two contour integrations $[\oint dz, \oint dw]$ is evaluated by first fixing w and deforming the difference between the two z integrations into a single z contour around the point w. Then, in evaluating the z contour integral, we may perform OPEs to identify the leading behavior as z approaches w. Finally, the w integration is then performed, see figure 9. Using this procedure we can compute $[L_m, L_n]$. This is given by

$$[L_m, L_n] = \left(\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \right) z^{m+1}T(z)w^{n+1}T(w)$$

$$= \oint_{\text{origin}} \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} T(z)T(w)z^{m+1}w^{n+1}$$

– 116 –

$$= \oint_{\text{origin}} \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} z^{m+1} w^{n+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{regular terms} \right),$$

where $\oint_{\text{origin}}$ means the closed contour around the origin while $\oint_w$ is the closed contour around w. Now, to evaluate the z integral we do what was just outlined above and we expand z^{m+1} around w to reduce all higher order poles to be of order one. The expansion of z^{m+1} is given by

$$z^{m+1} = w^{m+1} + (m+1)w^m(z-w) + \frac{1}{2}m(m+1)w^{m-1}(z-w)^2 + \frac{1}{6}m(m^2-1)w^{m-2}(z-w)^3 + \cdots$$

Plugging this in and performing the z integration gives us (only worrying about terms which could be singular since the rest vanish by Cauchy's theorem)

$$[L_m, L_n] = \oint_{\text{origin}} \frac{dw}{2\pi i} w^{n+1} \left[\partial_w T(w) w^{m+1} + 2(m+1)T(w)w^m + \frac{c}{12}m(m^2-1)w^{m-2} \right].$$

To proceed, we integrate the first term by parts and combine it with the second term, which gives us

$$\oint \frac{dw}{2\pi i} w^{n+1} \left(\partial_w T(w) w^{m+1} + 2(m+1)T(w)w^m \right)$$

$$= \oint \frac{dw}{2\pi i} \partial_w T(w) w^{m+1} w^{n+1} + \oint \frac{dw}{2\pi i} 2(m+1)T(w)w^m w^{n+1}$$

$$= -\oint \frac{dw}{2\pi i}(n+1)T(w)w^{m+1}w^n - \oint \frac{dw}{2\pi i}(m+1)T(w)w^m w^{n+1}$$

$$+ \oint \frac{dw}{2\pi i} 2(m+1)T(w)w^m w^{n+1}$$

$$= \oint \frac{dw}{2\pi i} \left((m+1)w^{m+n+1}T(w) - (n+1)w^{m+n+1}T(w) \right)$$

$$= \oint \frac{dw}{2\pi i}(m-n)w^{m+n+1}T(w)$$

$$= (m-n)L_{m+n}.$$

For the third term we have

$$\oint_{\text{origin}} \frac{dw}{2\pi i} w^{n+1} w^{m-2} \frac{c}{12}m(m^2-1),$$

which is nonzero only when $w^{n+1}w^{m-2} = w^{-1}$, i.e. when $m = -n$. So, integrating the third term gives us

$$\oint_{\text{origin}} \frac{dw}{2\pi i} w^{n+1} w^{m-2} \frac{c}{12} m(m^2 - 1) = \frac{c}{12} m(m^2 - 1)\delta_{m,-n}.$$

Combining all of these results we see that the algebra of commutators for the operators L_n is given by

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{m,-n}, \tag{8.20}$$

which is the same algebraic structure obeyed by the Virasoro algebra. Thus, we see that the set of generators of conformal transformations, corresponding to the holomorphic part of the theory, is isomorphic with the Virasoro algebra[‡].

Note that, not surprisingly, a term proportional to the central charge c appears. Strictly speaking, such a constant term is not allowed in the algebra since an algebra must be closed with respect to its multiplication and c is not an operator while our algebra is an algebra of operators. Thus, we are forced to change our view from c being some number to treating c as an operator which commutes with every element of the algebra, i.e. due to Schur's lemma we can treat c as some constant, which we will also call c, times the identity operator. From this it follows that on any representation of the algebra this c operator has a constant value, which we denote by c. It is common practice to call operators which only appear on the RHS of a commutation relation central charges. This is in close analogy with the central extension of an algebra. As an aside, note that the Virasoro algebra can be constructed by adding a central extension, $\mathbb{C}c$, to the Witt algebra and adding to the usual Witt commutators the commutator $[L_n, c] = 0$ for all elements of the Witt algebra L_n. For a more detailed review of this approach see Kac "Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras".

Also, note that because of the central charge the classical symmetry is not preserved when we quantize the theory in question. In particular, the central charge prohibits the vacuum from having the full symmetry since we cannot have that all of the generators L_n annihilate the vacuum without introducing a contradiction in the algebra. This is closely analogous to the situation in quantum mechanics were the position and momentum operators cannot annihilate the vacuum simultaneously.

We next move to the representation theory of the Virasoro algebra and its relation to physical states.

[‡]This implies that the algebra of the generators for the total theory, i.e. the holomorphic and antiholomorphic sectors, is isomorphic to the direct sum of two copies of Virasoro algebras.

8.3 Representation Theory of the Virasoro Algebra

When one is given an algebra the first thing they usually do is try to find its representations. For example, in quantum mechanics one tries to find the representations of the angular momentum algebra. This algebra consists of the three generators J^-, J^+, J_3 along with their usual commutation relations, $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$. Among the generators one looks for a maximal commuting set of generators (in the usual case consisting of the operator J_3 and the Casimir element $J^{2\ddagger}$). Casimir elements commute with all the generators, which follows from the fact that the Casimir element lives in the center of the algebra's universal enveloping algebra, and the eigenvalue of J^2 is used to label the representation (j). Within the $(2j + 1)$ dimensional representation space $V^{(j)}$ one can label the eigenstates $|j, m\rangle$ with the eigenvalues of J_3, denoted by m. The other generators, J^- and J^+, then transform between the states in the representation $V^{(j)}$. Thus, we can find all of the states in a certain representation, say j, by starting with the state in this representation with maximal J_3, which we denote by $|j, m_{max}\rangle$, which is therefore annihilated by J^+, $J^+|j, m_{max}\rangle = 0^\P$. The state obeying this property is called a highest weight state. The other states in this representation are obtained by acting on this highest weight state with J^-. It turns out that there are only a finite number of these states since it can be shown that the norm of the state given by $(J^-)^{2j+1}|j, m_{max}\rangle$ is zero. These states are called null states and are usually set to zero.

Now, we want to mimic this construction with the Virasoro algebra. We will only be interested in unitary representations, where a representation of the Virasoro algebra is called unitary if all the generators L_n are realized as operators acting in a Hilbert space, along with the condition that $L_n^\dagger = L_{-n}$. So, to start, we need to find a maximal set of generators which commute with all of the generators of the algebra. This set is given by the central charge, c, and L_0. Now that we have our maximal set of commuting operators we can see that our representations will be labeled by the eigenvalues of c, denoted by c, and each state inside this representation will be labeled by the eigenvalues of L_0, denoted by h, i.e. the states will be denoted by $|h, c\rangle$.

The next thing that we need to define is the highest weight representation for the Virasoro algebra. So, by definition, a highest weight representation is a representation containing a state with a smallest eigenvalue of L_0, i.e. there exists a state $|h, c\rangle$ such

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‡Note that the Casimir element does not live in the algebra itself, but rather in the universal enveloping algebra of the angular momentum algebra. However, there is a 1-1 relation between the angular momentum algebra and its universal enveloping algebra.

¶This follows from the fact that J^+ raises the value of m. So, if we are at the maximum value of m then we must impose that J^+ annihilates this state.

that

$$L_0|h, c\rangle = h|h, c\rangle, \tag{8.21}$$

where h is bounded from below. It follows from the commutation relations that $L_{n>0}$ decreases the eigenvalue of L_0 by n, since

$$L_0 L_n|\psi, c\rangle = \left(L_n L_0 - n L_n\right)|\psi, c\rangle = L_n(\psi - n)|\psi, c\rangle = (\psi - n)L_n|\psi, c\rangle, \tag{8.22}$$

where ψ is the eigenvalue of L_0. So, if $|h, c\rangle$ is a highest weight state, then $|h, c\rangle$ is annihilated by all generators $L_{n>0}$ because if it was not then we could obtain a state with a lower eigenvalue than h, which is not allowed since we assumed that for a highest weight state the eigenvalue of L_0 was bounded below by h.

The generators $L_{n<0}$ do not annihilate the highest weight state and they can be used to generate other states, just like the J^- from before. These states are called descendant states. Since there is an infinite number of generators $L_{n<0}$ we see that there is an infinite number of descendant states corresponding to each highest weight state. So, we can think of a representation, corresponding to some value for c, as a pyramid of states with the highest weight state at the top and below it is the (level 1) state created by acting on the highest weight state by L_{-1}, then below this are the two (level 2) states created by acting on the highest weight state with L_{-2} and then the state created by acting with $L_{-1}L_{-1}$,

$$|h, c\rangle$$

$$L_{-1}|h, c\rangle$$

$$L_{-2}|h, c\rangle, L_{-1}^2|h, c\rangle$$

$$L_{-3}|h, c\rangle, L_{-1}L_{-2}|h, c\rangle, L_{-1}^3|h, c\rangle$$

$$.$$
$$.$$
$$. \tag{8.23}$$

The whole set of states is called a Verma module. They are the irreducible representations of the Virasoro algebra. This means that if we know the spectrum of primary states, then we know the spectrum of the whole theory.

Vacuum State

The vacuum of the theory can be defined by the condition that it respects the maximum number of symmetries, i.e. it should be annihilated by the maximum number of conserved charges. In this present context it means that the vacuum should satisfy $L_n|0\rangle = 0$ for all n. But this is not possible due to the central charge. For example, consider the generator L_{-n} (with $n \geq 0$) acting on the vacuum. If we assume that L_{-n} annihilates the vacuum then

$$0 = L_{-n}|0\rangle = [L_n, L_{-n}]|0\rangle = \left(2nL_0 + \frac{c}{12}n(n^2 - 1)\right)|0\rangle,$$

where the first equality holds because if we assume that $L_{-n}|0\rangle = 0$, and we know that $L_n|0\rangle = 0$, then their commutator also annihilates the vacuum, while the second equality holds from the commutation relations for the L's. Ok, so for how many n's, where $n \geq 0$, can $L_n|0\rangle = 0$ hold?

- $n = 0$: If $L_0|0\rangle = 0$ then we must have that, for the vacuum, $h = 0$, since $L_0|0\rangle = h|0\rangle$.

- $n = -1$: If $L_{-1}|0\rangle = 0$ then we must have $h = 0$ since

$$L_{-1}|0\rangle = \left(2nL_0 + \frac{c}{12}n(n^2 - 1)\right)|0\rangle = \left(2(-1)L_0 + \frac{c}{12}(-1)1(1^2 - 1)\right)|0\rangle = -2L_0|0\rangle.$$

- $n \leq -2$: These generators can never annihilate the vacuum since even if $h = 0$ they still can never get rid of the term $n(n^2 - 1)$.

Thus, the maximal symmetry we can impose on the vacuum is

$$L_n|0\rangle = 0, \qquad \forall\, n \geq -1. \tag{8.24}$$

Conformal Fields and States

There is a relation between highest weight states and primary conformal fields. Consider a primary conformal field $\phi(z, \overline{z})$ with weights h and $\overline{h}$. Now, define the state $|h, \overline{h}, c\rangle$ by[§]

$$|h, \overline{h}, c\rangle = \phi(0, 0)|0\rangle, \tag{8.25}$$

i.e. we create this state by acting on the vacuum with the primary conformal field $\phi(z, \overline{z})$ (evaluated at $z = \overline{z} = 0$), in much the same way we create states in QFT by

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[§]Note that since the algebra of the complete set of generators, both the holomorphic and antiholomorphic sectors, is isomorphic to the direct sum of two Virasoro algebras, we really should label the states as $|h, \overline{h}, c, \overline{c}\rangle$. But this is redundent since we have that $c = \overline{c}$ due to our theory being Lorentz invariant.

acting on the vacuum with the modes of the fields. We will now show that this state is a highest weight state. To show this we need to show that $L_n|h, \overline{h}, c\rangle = 0$ for all $n > 0$ and also that $L_0|h, \overline{h}, c\rangle = h|h, \overline{h}, c\rangle$ along with $\overline{L}_0|h, \overline{h}, c\rangle = \overline{h}|h, \overline{h}, c\rangle$. The first thing that we need to show, i.e. that $L_{n>0}$ annihilates the state, holds by previous considerations. Thus, we can write $L_n|h, \overline{h}, c\rangle = L_n\phi(0,0)|0\rangle$ as $[L_n, \phi(0,0)]|0\rangle$. Now consider the following

$$[L_n, \phi(0,0)] = \lim_{w, \overline{w} \to 0} \left(\frac{1}{2\pi i} \oint dz \ z^{n+1} T(z)\phi(w, \overline{w}) \right)$$

$$= \lim_{w, \overline{w} \to 0} \left(h(n+1)w^n \phi(w, \overline{w}) + w^{n+1}\partial_w \phi(w, \overline{w}) \right).$$

So, if $n > 0$ then, when $w, \overline{w} \mapsto 0$, we have that $[L_n, \phi(0,0)] = 0$, which implies that $L_n|h, \overline{h}, c\rangle = [L_n, \phi(0,0)]|h, \overline{h}, c\rangle = 0$. Also, when $n = 0$ and $w \mapsto 0$ we get that $[L_0, \phi(0,0)] = h\phi(0,0)$, which implies that $L_0|h, \overline{h}, c\rangle = [L_0, \phi(0,0)]|h, \overline{h}\rangle = h|h, \overline{h}, c\rangle$. Similarly for the anti-holomorphic sectors. Thus, the state created from the vacuum by a primary conformal field of weight $(h, \overline{h})$ is a highest weight state, with weights (or eigenvalues of $L_0, \overline{L}_0$) give by h and $\overline{h}$.

As was mentioned at the beginning, representations of the Virasoro algebra start with a single highest weight state, which we have just seen corresponds to a primary conformal field acting on the vacuum, and then fans out with the descendant states below it. It turns out that we can find fields which generate descendant fields from the vacuum. These fields are called descendant fields, which we denote by $\hat{L}_{-n}\phi(w, \overline{w})$, and they can be extracted from the less-singular parts of the OPE of a primary field $\phi(w, \overline{w})$ with the energy momentum tensor,

$$T(z)\phi(w, \overline{w}) \equiv \sum_{n \geq 0} (z - w)^{n-2}\hat{L}_{-n}\phi(w, \overline{w})$$

$$= \frac{1}{(z-w)^2}\hat{L}_0\phi(w, \overline{w}) + \frac{1}{(z-w)}\hat{L}_{-1}\phi(w, \overline{w}) + \hat{L}_{-2}\phi(w, \overline{w})$$

$$+ (z - w)\hat{L}_{-3}\phi(w, \overline{w}) + (z - w)^2\hat{L}_{-4}\phi(w, \overline{w}) + \cdots.$$

We can now project out one of the descendant fields $\hat{L}_{-n}\phi(w, \overline{w})$ from the sum by

$$\hat{L}_{-n}\phi(w, \overline{w}) = \frac{1}{2\pi i} \oint dz \frac{1}{(z-w)^{n-1}} T(z)\phi(w, \overline{w}). \tag{8.26}$$

Using this we can see that

$$\hat{L}_{-n}\phi(0,0)|0\rangle = \frac{1}{2\pi i} \oint dz \frac{1}{(z)^{n-1}} T(z)\phi(0,0)|0\rangle = L_{-n}\phi(0,0)|0\rangle. \tag{8.27}$$

Thus, we see that $\hat{L}_{-n}\phi(w, \overline{w})$ generates the L_{-n} descendant of the highest weight state $|h, \overline{h}, c\rangle$. Descendant states are not primary conformal fields. However, when they are commuted with the generators L_0 and L_{-1} (scaling, rotations, and translations) they behave like primary conformal fields, but not when commuted with L_1 (special conformal transformations). Thus, we can assign a conformal weight to them, which is equal to the L_0 eigenvalue of the the state $\hat{L}_{-n}\phi(0, 0)|0\rangle$ they create from the vacuum. This weight is given by $(h + n, \overline{h})$ for a field $\hat{L}_{-n}\phi(w, \overline{w})$, where $(h, \overline{h})$ is the weight of the primary field $\phi(w, \overline{w})$. An example of a level two descendant field is given by

$$\left(\hat{L}_{-2}I\right)(w) = \frac{1}{2\pi i} \oint dz \frac{1}{(z - w)} T(z)I(w) = T(w), \qquad (8.28)$$

where I is the identity operator. Thus, $I^{(-2)}(w) = \left(\hat{L}_{-2}I\right)(w) = T(w)$ and we see that the stress-energy tensor is always a level two descendant of the identity operator. Also, for $n > 0$, primary fields $\phi(w, \overline{w})$ satisfy $\hat{L}_n\phi(w, \overline{w}) = 0$. Below is a table of the first few descendant fields, ordered according to their conformal weights,

level	dimension	field
0	h	ϕ
1	h+1	$\hat{L}_{-1}\phi$
2	h+2	$\hat{L}_{-2}\phi,\ \hat{L}_{-1}^2\phi$
3	h+3	$\hat{L}_{-3}\phi,\ \hat{L}_{-1}\hat{L}_{-2}\phi,\ \hat{L}_{-1}^3\phi$
.	.	.
.	.	.
.	.	.
N	h + N	$P(N)$ fields ,

where $P(N)$ is the number of partitions of N into positive integers, i.e. in terms of the generating function

$$\left(\prod_{n=1}^{\infty}(1 - q^n)\right)^{-1} = \sum_{n=0}^{\infty} P(N)q^N,$$

with $P(0) \equiv 1$.

So, to recap, we have seen that when we act on the vacuum of the Virasoro algebra with primary conformal fields we get highest weight states and when we act on the vacuum with descendant fields we get descendant states.

We now move to the discussion of the conformal Ward identities.

8.4 Conformal Ward Identities

Ward identities are generally identities satisfied by correlation functions as a reflection of symmetries possessed by a theory, i.e. the spirit of Noethers theorem in quantum field theories is captured by these Ward identities. These identities, or operator equations, are easily derived via functional integral formulation of correlation functions, for example by requiring that they be independent of a change of dummy integration variables. The Ward identities for conformal symmetry can thus be derived by considering the behavior of n-point functions under a conformal transformation. This should be considered to take place in some localized region containing all the operators in question, and can then be related to a surface integral about the boundary of the region.

For the two dimensional conformal theories of interest here, we shall instead implement this procedure in the operator form of the correlation functions. In what follows, unless otherwise noted we will be considering correlation functions of primary conformal fields. Due to the fields being conformal we know that these correlation functions must have global conformal invariance and thus should satisfy (see 6.14)

$$\langle \phi_1(z_1, \overline{z}_1) \cdots \phi_n(z_n, \overline{z}_n) \rangle = \prod_j \left(\partial_{z_j} f(z_j) \right)^{h_j} \left(\partial_{\overline{z}_j} \overline{f}(\overline{z}_j) \right)^{\overline{h}_j} \langle \phi_1(w_1, \overline{w}_1) \cdots \phi_n(w_n, \overline{w}_n) \rangle,$$

$$(8.29)$$

with $w = f(z)$ and $\overline{w} = \overline{f}(\overline{z})$ of the form of (6.11) and (6.12). We now consider, to gain additional information from the *global* conformal algebra, an assemblage of operators at the points w_i (see figure 10) and then perform a conformal transformation on the interior of the region bounded by the z contour by line integrating $\epsilon(z)T(z)$ around it. To perform this integral we can deform the contour enclosing

Figure 10:

all the poles to a sum of contours surrounding only one pole each. This gives us,
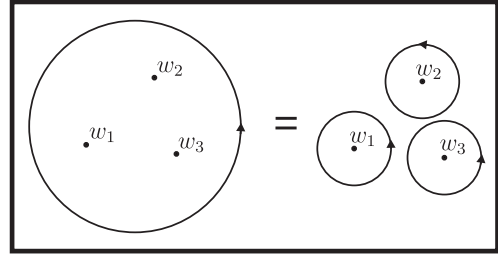
$$\left\langle \frac{1}{2\pi i} \oint dz \epsilon(z) T(z) \phi_1(w_1, \overline{w}_1) \cdots \phi_n(w_n, \overline{w}_n) \right\rangle$$

$$= \sum_{j=1}^{n} \left\langle \phi_1(w_1, \overline{w}_1 \cdots \left(\frac{1}{2\pi i} \oint dz \epsilon(z) T(z) \phi_j(w_j, \overline{w}_j) \right) \cdots \phi_n(w_n, \overline{w}_n) \right\rangle$$

$$= \sum_{j=1}^{n} \langle \phi_1(w_1, \overline{w}_1 \cdots \delta_\epsilon \phi_j(w_j, \overline{w}_j) \cdots \phi_n(w_n, \overline{w}_n) \rangle , \qquad (8.30)$$

where the last line comes from

$$\delta_\epsilon \phi(w, \overline{w}) = \frac{1}{2\pi i} \oint dz \epsilon(z) T(z) \phi(w, \overline{w}) = \Big(\epsilon(w) \partial_w + h \partial_w \epsilon(w) \Big) \phi(w, \overline{w}).$$

Now, since the above equation (8.30) is true for any $\epsilon(z)$ and since $\oint d\overline{z}\, T(z) = 0$, we can ignore the integrals to get

$$\big\langle T(z) \phi_1(w_1, \overline{w}_1) \cdots \phi_n(w_n, \overline{w}_n) \big\rangle = \sum_{j=1}^{n} \left(\frac{h_j}{(z - w_j)^2} + \frac{1}{z - w_j} \frac{\partial}{\partial w_j} \right) \big\langle \phi_1(w_1, \overline{w}_1) \cdots \phi_n(w_n, \overline{w}_n) \big\rangle.$$

(8.31)

This expression states that the correlation functions are meromorphic functions of z with singularities at the positions of the inserted operators. Also, one can show that all of the correlation functions of descendant fields can be obtained from the correlation functions of primary fields by acting on them with differential operators. For example if we let $z \mapsto w_n$ in (8.31) and then expand it in powers of $z - w_n$, while noting the definition of descendant fields, then we get

$$\big\langle \phi_1(w_1, \overline{w}_1) \cdots \phi_{n-1}(w_{n-1}, \overline{w}_{n-1}) (\hat{L}_{-k} \phi)(z, \overline{z}) \big\rangle$$

$$= \mathcal{L}_{-k} \big\langle \phi_1(w_1, \overline{w}_1) \cdots \phi_{n-1}(w_{n-1}, \overline{w}_{n-1}) \phi(z, \overline{z}) \big\rangle,$$

where $\mathcal{L}_{-k}$ is defined, for $k \geq 2$, by

$$\mathcal{L}_{-k} = -\sum_{j=1}^{n-1} \left(\frac{(1-k)h_j}{(w_j - z)^k} + \frac{1}{(w_j - z)^{k-1}} \frac{\partial}{\partial w_j} \right).$$

In general we can write down expressions for correlation functions of arbitrary secondary fields in terms of those for primaries, but there is no convenient closed form expression in the most general case. For more information on this see Ginsparg "Applied Conformal Field Theory" page 41.

In the next chapter we will look at another way to quantize the bosonic string theory called BRST quantization. This will involve adding ghost fields to our theory and additional terms to the action. We will first tackle this approach from a general point of view and then restrict to the case of the string.

8.5 Exercises

Problem 1

We will simplify notation by writing $ABC(z)$ for $A(z)B(z)C(z)$, i.e by $:\partial X \partial X(z):$ we really mean $:\partial_z X(z)\partial_z X(z):$. Consider a free boson X CFT with the XX OPE

$$X(z,\bar{z})X(w,\bar{w}) \sim -\ln|z-w|^2, \tag{8.32}$$

and the stress-energy tensor

$$T(z) = -\frac{1}{2}:\partial X \partial X(z): , \qquad \bar{T}(\bar{z}) = -\frac{1}{2}:\bar{\partial}X\bar{\partial}X(\bar{z}): . \tag{8.33}$$

(i) Derive the OPE of $T(z)$ and $\bar{T}(\bar{z})$ with $X(w,\bar{w})$, $\partial X(w,\bar{w})$, $\bar{\partial}X(w,\bar{w})$, $\partial^2 X(w,\bar{w})$, and $:\exp(i\sqrt{2}X)(w,\bar{w}): $.

(ii) What do these results imply for the conformal dimension $(h,\bar{h})$ in each case?

Note: if you want to draw hooks for contractions using LaTeX, try googling for `simplewick.sty` or `wick.sty`.

Problem 2

Let A and B be two free fields whose contractions with themselves and each other are c numbers. We denote by $\overparen{\mathcal{F}\mathcal{G}}$ the contraction between two operators $\mathcal{F},\mathcal{G}$ which are functions of A,B. Recall that

$$\mathcal{F}\mathcal{G} = :\mathcal{F}\mathcal{G}: + \overparen{\mathcal{F}\mathcal{G}}. \tag{8.34}$$

(i) Show by recursion that

$$\overparen{A(z) :B^n(w):} = n\overparen{A(z)B(w)}:B^{n-1}(w): . \tag{8.35}$$

(ii) Use this result to prove

$$\overparen{A(z) :\exp B(w):} = \overparen{A(z)B(w)}:\exp B(w): . \tag{8.36}$$

(iii) By counting multiple contractions, show that

$$:\exp \overparen{A(z): :\exp B(w):}$$

$$= \sum_{m,n=0}^{\infty} \sum_{1 \le k \le m,n} \frac{k!}{m!\,n!}\binom{m}{k}\binom{n}{k}\left(\overparen{A(z)B(w)}\right)^k :A^{m-k}(z)B^{n-k}(w):$$

$$= \left[\exp\left(\overparen{A(z)B(w)}\right) - 1\right] :\exp A(z)\exp B(w): . \tag{8.37}$$

(iv) Consider the free boson X as in Problem 1 and compute

$$\langle :\exp(iaX)(z)::\exp(-iaX)(w): \rangle. \tag{8.38}$$

Use the answer to determine the conformal weight of $:\exp(iaX)(z):$.

Problem 3

Show that the correlation function containing one secondary field can be obtained from a correlation function of only primaries fields by acting with a differential operator (we ignore the anti-holomorphic part in what follows):

$$\langle \phi_1(w_1) \cdots \phi_n(w_n)(\hat{L}_{-k}\phi)(z) \rangle$$
$$= \mathcal{L}_{-k}\langle \phi_1(w_1) \cdots \phi_n(w_n)\phi(z) \rangle, \tag{8.39}$$

where

$$\mathcal{L}_{-k} = -\sum_{j=1}^{n} \left(\frac{(1-k)h_j}{(w_j - z)^k} + \frac{1}{(w_j - z)^{k-1}} \frac{\partial}{\partial w_j} \right). \tag{8.40}$$

Here ϕ_i are chiral primaries of weight h_i and

$$(\hat{L}_{-k}\phi)(w) = \oint \frac{dz}{2\pi i} \frac{1}{(z-w)^{k-1}} T(z)\phi(w), \tag{8.41}$$

is a descendant of the chiral primary ϕ.

9. BRST Quantization of the Bosonic String

9.1 BRST Quantization in General

So far we have seen two methods that can be utilized to quantize the string: the covariant approach and light-cone quantization. Each offers its advantages. Covariant quantization makes Lorentz invariance manifest but allows for the existence of ghost states (states with negative norm) in the theory. In contrast, light-cone quantization is ghost free. However, Lorentz invariance is no longer obvious. Another trade-off is that the proof of the number of space-time dimensions ($c = D = 26$ for the bosonic theory) is rather difficult in covariant quantization, but its rather straightforward in light-cone quantization. Finally, identifying the physical states is easier in the light-cone approach. Another method of quantization, that in some ways is a more advanced approach, is called BRST quantization. This approach takes a middle ground between the two methods outlined above. BRST quantization is manifestly Lorentz invariant, but includes ghost states in the theory. Despite this, BRST quantization makes it easier to identify the physical states of the theory and to extract the number of space-time dimensions relatively easily.

We have previously seen that the Polyakov action,

$$S_\sigma = -\frac{T}{2} \int d\tau d\sigma \sqrt{-h} \, h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu, \tag{9.1}$$

is invariant under local (gauge) symmetries. In particular, we know that S_σ is invariant under:

- Reparameterization Invariance:

$$\delta h_{\alpha\beta} = D_\alpha \xi_\beta + D_\beta \xi_\alpha,$$

$$\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu,$$

  where by D_α we mean the covariant derivative which, in the case of a reparameterization, is given by

$$D_\alpha \xi_\beta = \partial_\alpha \xi_\beta + \Gamma^\lambda_{\alpha\beta} \xi_\lambda. \tag{9.2}$$

  Note that, in general, if a theory has gauge transformations, it means that some physical properties of certain equations are preserved under those transformations. Likewise, the covariant derivative is the ordinary derivative, ∂_α, modified in such a way as to make it behave like a true vector operator, so that equations written using the covariant derivative preserve their physical properties under gauge transformations.

- Weyl Invariance: The action, S_σ, is also invariant under the transformations given by

$$\delta h_{\alpha\beta} = e^\phi h_{\alpha\beta},$$

$$\delta X^\mu = 0\,.$$

We then used these gauge symmetries to simplify the metric; we were able to fix a gauge which set the metric $h_{\alpha\beta}$ equal to the flat metric with Minkowski signature $\eta_{\alpha\beta}$, and then we proceeded to quantize the action. However, in general, to quantize a theory which has gauge symmetry properly we should introduce ghost fields, anti-ghost fields, auxiliary fields, gauge fixing terms, etc. and also add two additional pieces to the action, the so-called gauge fixing action and the ghost action. All of the techniques for doing this are given by the BRST approach to quantization, which we now discuss in the general case; only later do we restrict to the bosonic string theory.

9.1.1 BRST Quantization: A Primer

The BRST approach to quantizing a theory with gauge symmetry proceeds as follows.

Suppose that we have an action $S(\Phi^I)$ which depends on the different fields Φ^I; here we take the index I to be generalized in such a way that Φ^I could be a collection of the same type of field and also other fields in general, for example we could have that $\{\Phi^I\} = \{X^0, X^1, \cdots, X^\mu, h_{\alpha\beta}, \cdots\}$. We will assume that indices can either be discrete or continuous. For example, we will write e^τ when we mean the field $e(\tau)$, which is a function of τ, and repeated indices means that you sum over them if they are discrete, or integrate over them if they are continuous, i.e.

$$A^x B_x \equiv \int dx A(x) B(x).$$

Also, suppose that the action is invariant under the gauge transformation given by

$$\delta\Phi^I = \epsilon^\alpha \delta_\alpha \Phi^I, \tag{9.3}$$

where ϵ^α is a local parameter. Note that by the action being invariant under this transformation we get that

$$S\big[\Phi^I + \delta\Phi^I\big] = S\big[\Phi^I\big] \;\Rightarrow \delta S = 0.$$

Next, consider the important fact that symmetry transformations form an algebra. This can be thought of as the following. Let the action S be invariant under a certain

transformation, call it T_1, and also another transformation, call it T_2. Then, since S is invariant under both of these transformations, it will also be invariant under their composition. Thus, the set of symmetry transformations is closed under composition. Now, to define the algebra we need to specify the multiplication of the symmetry transformations in the set. We do this by defining the multiplication, in general, of two elements, δ_α and δ_β, in the set of symmetry transformations, S, as

$$[\delta_\alpha, \delta_\beta] = f^\gamma_{\alpha\beta} \, \delta_\gamma \; + \; \text{some kind of field equations}, \tag{9.4}$$

where $f^\gamma_{\alpha\beta}$ are known as the structure constants. Furthermore, we take the extra field equations to vanish in order to have the multiplication close, and also we assume that the bracket multiplication, $[\cdot, \cdot] : S \times S \to S$, obeys the following properties:

1. The bracket operation is bilinear, i.e. we have that

$$[cA + dB, c'A' + d'B'] = cc'[A, A'] + cd'[A, B'] + dc'[B, A'] + dd'[B, B'],$$

   where c, d, c' and d' are elements of the field over which the algebra is constructed, and A, B, A' and B' are elements of the algebra itself.

2. $[\delta_\alpha, \delta_\alpha] = 0$ for all δ_α in the set of transformations S.

3. $[\delta_\alpha, [\delta_\beta, \delta_\gamma]] + [\delta_\beta, [\delta_\gamma, \delta_\alpha]] + [\delta_\gamma, [\delta_\alpha, \delta_\beta]] = 0$ for δ_α, δ_β, and δ_γ in S. This is known as the Jacobi identity.

Thus, our algebra of symmetry transformations has the structure of a Lie algebra[‡]. This is because the set of transformations forms a vector space along with the fact that the multiplication, $[\cdot, \cdot]$, obeys the above properties. Note that, in general, the structure constants are not so constant and will, in fact, depend on the particular fields, i.e. $f^\gamma_{\alpha\beta} = f^\gamma_{\alpha\beta}(\Phi^I)$. However, in this section we will assume that they are constant.

Finally, to gauge fix this gauge symmetry, we need to give a gauge fixing condition,

$$F^A(\Phi^I) = 0. \tag{9.5}$$

For example, earlier we fixed the gauge symmetry by setting the metric $h_{\alpha\beta}$ equal to the flat metric $\eta_{\alpha\beta}$. In terms of this choice of gauge (or gauge slice) the F^A would be given by $F_{\alpha\beta} = h_{\alpha\beta} - \eta_{\alpha\beta} = 0$.

Next we give the rules for quantizing this gauge theory properly.

[‡]For a detailed account of Lie algebras see Humphreys "Lie Algebras and Representation Theory" and Carter "Lie Algebras of Finite and Affine Type".

BRST Rules For The Quantization of Gauge Theories

1. The first thing we do is to introduce new fields, called ghost fields, one for every gauge parameter; i.e. for each ϵ^α we introduce a field c^α, where the ghost field c^α has opposite statistics as the parameter ϵ^α (if ϵ^α is bosonic then c^α will be fermionic and vice versa). So, if we have two parameters ϵ^1 and ϵ^2, then we have to introduce two ghost fields, c^1 and c^2. We will take ϵ^α to have bosonic statistics unless otherwise specified.

2. Next we introduce anti-ghost fields b_A and auxiliary fields B_A, one of each for each gauge fixing condition F^A. Where b_A has the same statistics as the ghost fields c^α and B_A has the same statistics as the gauge fixing parameter F^A.

3. Then we add to the action S the two terms

$$S_2 = -iB_A F^A(\Phi^I) \qquad \text{gauge fixing action,} \tag{9.6}$$

$$S_3 = b_A c^\alpha \delta_\alpha F^A(\Phi^I) \qquad \text{ghost action} . \tag{9.7}$$

Thus, the new action is given by $S + S_2 + S_3$, where S was the original action that was invariant under the gauge algebra given in (9.4).

4. Finally, the quantization of the theory is given by the partition function

$$Z = \int \mathcal{D}\Phi^I \, \mathcal{D}B_A \, \mathcal{D}c^\alpha \, \mathcal{D}b_A \, e^{-(S+S_2+S_3)}, \tag{9.8}$$

where $\mathcal{D}\Phi^I$ means to integrate over all fields Φ^I, $\mathcal{D}B_A$ means to integrate over all auxiliary fields, etc.

(COULD PUT THE FADEEV-POPOV STUFF HERE FROM TONG AND PAGES 86-90 OF POLCHINSKI AND 3.7 OF KRISTSIS)

BRST Symmetry

Now, we claim that the quantum action $S + S_2 + S_3$ has a global symmetry. This new symmetry is called the BRST symmetry. The transformation which generates this symmetry is given by defining its action on all the fields present in the theory, and this is given by[‡]

$$\delta_B \Phi^I = -i\kappa c^\alpha \delta_\alpha \Phi^I, \tag{9.9}$$

[‡]Note that κ is some arbitrary global parameter, which we take to have the same statistics as the ghost fields c^α so that both sides of the following equations have matching statistics.

which is basically the same as the gauge transformation from earlier, (9.3), with ϵ^α replaced by c^α, and also

$$\delta_B c^\alpha = -\frac{i}{2}\kappa f^\alpha_{\beta\gamma} c^\beta c^\gamma, \tag{9.10}$$

$$\delta_B b_A = \kappa B_A, \tag{9.11}$$

$$\delta_B B_A = 0. \tag{9.12}$$

As an example, let's workout the BRST transformation of the combination $b_A F^A(\Phi^I)$. We have

$$\delta_B(b_A F^A(\Phi^I)) = \left(\kappa B_A\right) F^A(\Phi^I) + b_A\left(-i\kappa c^\alpha \delta_\alpha F^A(\Phi^I)\right)$$

$$= i\kappa\left(-iB_A F^A(\Phi^I) \underbrace{+ b_A}_{\{\kappa,b_A\}=0} c^\alpha \delta_\alpha F^A(\Phi^I)\right)$$

$$= i\kappa(S_2 + S_3), \tag{9.13}$$

where in the second line we get the plus sign from the fact that b_A anti-commutes with κ, $\{\kappa, b_A\} \equiv \kappa b_A + b_A \kappa = 0$, since κ is a fermionic parameter and if ϵ^α is bosonic, which we have been assuming, then b_A has fermionic statistics.

Proposition 9.1 *The BRST transformation is nilpotent, of order 2, i.e. we have that*

$$\delta_B \delta_B = 0. \tag{9.14}$$

Proof To prove the claim we need to show that (9.14) holds for each field in the BRST quantization scheme, i.e. we need to check that $\delta_B^2(b_A) = \delta_B^2(B_A) = \delta_B^2(\Phi^I) = \delta_B^2(c^\alpha) = 0$. So, to begin we have that

$$\delta_B^2(b_A) = \delta_B(\delta_B b_A)$$

$$= \delta_B(\kappa B_A)$$

$$= \kappa(\underbrace{\delta_B B_A}_{=0}) = 0.$$

Also, we have $\delta_B^2(B_A) = 0$ by definition of the BRST transformation acting on B_A. Next, we have that

$$\delta_B^2(\Phi^I) = \delta_B(-i\kappa_1 c^\alpha \delta_\alpha \Phi^I)$$

$$= -i\kappa_1 \left(\left(-\frac{i}{2}\kappa_2 f^\alpha_{\beta\gamma} c^\beta c^\gamma \right) \delta_\alpha \Phi^I + c^\alpha \left(-i\kappa_2 c^\beta \delta_\alpha \delta_\beta \Phi^I \right) \right)$$

$$= \kappa_1 \kappa_2 \left(-\frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \delta_\alpha \Phi^I + c^\alpha c^\beta \delta_\alpha \delta_\beta \Phi^I \right).$$

Now, we can do some index switching in the first term and use that $[\delta_\alpha, \delta_\beta] = f^\gamma_{\alpha\beta} \delta_\gamma$ in the second term to show that $\delta_B^2(\Phi^I)$ indeed vanishes (HOW?). Finally, we have

$$\delta_B^2(c^\alpha) = \delta_B \left(-\frac{i}{2}\kappa f^\alpha_{\beta\gamma} c^\beta c^\gamma \right),$$

which, after doing the second transformation, looks like combinations of $ffccc$ with indices contracted in various ways. It can be shown that this vanishes by writing the Jacobi identity for the gauge algebra in terms of the structure constants. Thus, we have proved the claim, namely that the BRST transformation is nilpotent of order 2. $\mathcal{Q.E.D.}$

We will now show that nilpotency of the BRST transformation implies that the quantum action, $S + S_2 + S_3$, is BRST invariant.

Proposition 9.2 *Since δ_B is nilpotent, of order 2, the action, $S + S_2 + S_3$, is BRST invariant.*

Proof We need to show that

$$\delta_B(S + S_2 + S_3) = 0,$$

in order for $S + S_2 + S_3$ to be invariant under the BRST transformation, δ_B, since this implies that $S_{tot}[\Phi^I + \delta_B \Phi^I, c^\alpha + \delta_B c^\alpha, \cdots] = S_{tot}[\Phi^I, c^\alpha, \cdots]$, where $S_{tot} = S + S_2 + S_3$. So, we have already seen that

$$S_2 + S_3 = \delta_B \left(b_A F^A(\Phi^I) \right),$$

up to some, in this case meaningless, constants. Thus, we need to check that

$$\delta_B \left(S + \delta_B \left(b_A F^A(\Phi^I) \right) \right) = 0.$$

But this is just

$$\delta_B\big(S + \delta_B\left(b_A F^A(\Phi^I)\right)\big) = \delta_B(S) + \delta_B(\delta_B\left(b_A F^A(\Phi^I)\right))$$

$$= \delta_B(S) + \delta_B^2\left(b_A F^A(\Phi^I)\right)$$

$$= \delta_B^2\left(b_A F^A(\Phi^I)\right)$$

$$= 0\,.$$

Where the term $\delta_B(S)$ vanishes by gauge invariance of the original action S, and where $\delta_B^2\left(b_A F^A(\Phi^I)\right)$ vanishes by nilpotency.
$\mathcal{Q.E.D.}$

Now, we will derive the Ward identities for our BRST theory.

9.1.2 BRST Ward Identities

Suppose we want to compute the correlation function given by

$$\langle f(\varphi^I)\rangle, \tag{9.15}$$

where f is some arbitrary function of the fields $\varphi^I = \{\Phi^I, b_A, c^\alpha, B_A\}$. In general, the above correlation function is given by

$$\int \mathcal{D}\Phi^I\ \mathcal{D}B_A\ \mathcal{D}c^\alpha\ \mathcal{D}b_A\ f(\varphi^I)e^{-(S+S_2+S_3)},$$

which we will rewrite as

$$\int \mathcal{D}\Phi^I\ \mathcal{D}B_A\ \mathcal{D}c^\alpha\ \mathcal{D}b_A\ f(\varphi^I)e^{-S[\varphi^I]}. \tag{9.16}$$

Now, we can make a change of variables to all the fields in the path integral, $\varphi^I \mapsto \varphi^{I'} = \varphi^I + \delta_B\varphi^I$, which gives

$$\langle f(\varphi^I)\rangle = \int \mathcal{D}\Phi^{I'}\ \mathcal{D}B'_A\ \mathcal{D}c^{\alpha'}\ \mathcal{D}b'_A\ f(\varphi^{I'})e^{-S[\varphi^{I'}]}$$

$$= \int \mathcal{D}\Phi^I\ \mathcal{D}B_A\ \mathcal{D}c^\alpha\ \mathcal{D}b_A\ f(\varphi^{I'})e^{-S[\varphi^{I'}]}\,,$$

where the last line follows by the definition of the measure of the path integral (that is, if it exists). Also, since the action is invariant under the BRST transformation, $S[\varphi^{I'}] = S[\varphi^I]$, gives gives us

$$\langle f(\varphi^I) \rangle = \int \mathcal{D}\Phi^I \, \mathcal{D}B_A \, \mathcal{D}c^\alpha \, \mathcal{D}b_A \, f(\varphi^{I'})e^{-S[\varphi^I]}$$

$$= \int \mathcal{D}\Phi^I \, \mathcal{D}B_A \, \mathcal{D}c^\alpha \, \mathcal{D}b_A \, f(\varphi^I + \delta_B \varphi^I)e^{-S[\varphi^I]}$$

$$= \int \mathcal{D}\Phi^I \, \mathcal{D}B_A \, \mathcal{D}c^\alpha \, \mathcal{D}b_A \, \left(f(\varphi^I)e^{-S[\varphi^I]} + \delta_B f(\varphi^I) \right)e^{-S[\varphi^I]}$$

$$= \langle f(\varphi^I) \rangle + \langle \delta_B f(\varphi^I) \rangle .$$

And so, we have shown that

$$\langle f(\varphi^I) \rangle = \langle f(\varphi^I) \rangle + \langle \delta_B f(\varphi) \rangle,$$

or that $\langle \delta_B f(\varphi) \rangle = 0$. This implies that BRST "exact" terms, $\langle \delta_B f(\varphi) \rangle$, have no physical significance.

9.1.3 BRST Cohomology and Physical States

From Noether's theorem we know that corresponding to the global BRST symmetry, there is the BRST charge operator, denoted by Q_B. Similarly to the BRST transformation, δ_B, the BRST charge is nilpotent of order two, $Q_B^2 = 0$. This has important consequences in the defining of BRST physical states.

First, by varying the gauge fixing condition and observing the induced change in the gauge fixed and ghost actions it can be shown, see Polchinksi page 127, that any physical state $|\psi\rangle$ of our BRST theory must obey,

$$Q_B|\psi\rangle = 0.^\ddagger \tag{9.17}$$

We call a state which is annihilated by Q_B a BRST closed state. Now, since Q_B is nilpotent, any state, $|\psi\rangle$, of the form

$$|\psi\rangle = Q_B|\chi\rangle,$$

will be annihilated by the BRST charge. Thus, any state which can be written in the form $Q_B|\chi\rangle$, called a BRST exact state, is a candidate to be physical since it obeys

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‡Note that this is NOT the definition of a physical state, as we are about to see, it is merely a minimum requirement that our definition of physical states must include.

(9.17). However, note that, as before for physical spurious states, BRST exact states are orthogonal to all other physical states, including itself, since if $|\psi'\rangle$ is BRST exact and $|\psi\rangle$ is physical then

$$\langle\psi|\psi'\rangle = \langle\psi|Q_B|\chi\rangle \qquad \text{(for some } \chi \text{ such that } |\psi'\rangle = Q_B|\chi\rangle)$$

$$= \Big(\langle\psi|Q_B\Big)|\chi\rangle$$

$$= 0\,,$$

where the last line follows due to if $|\psi\rangle$ is physical then $\langle\psi|Q_B = 0^\ddagger$. Thus, as was mentioned before, BRST exact states have no physical significance since their physical amplitudes vanish and so cannot be physical states. Also, this implies that once we do define a state to be BRST physical this definition will not be unique, but only up to some BRST exact state. For instance, assume that $|\psi\rangle$ is a BRST physical state and also that we have calculated the values for its inner product with all other physical states. Now, if we add to $|\psi\rangle$ a BRST exact state then this new state, $|\psi\rangle + Q_B|\chi\rangle$, will have the same values for the inner products with the various physical states as did $|\psi\rangle$ (since the inner product is bilinear). Thus, one cannot differentiate, physically, between the two states $|\psi\rangle$ and $|\psi\rangle + Q_B|\chi\rangle$, and so we must treat these two states as being physically equivalent. Thus, each BRST physical state will generates an equivalence class, i.e. we say that two BRST physical states, $|\psi_1\rangle$ and $|\psi_2\rangle$, are equivalent if and only if the difference between these two states is given by some BRST exact state and then, given a BRST physical state all states equivalent to it form the equivalence class of the state in question.

Now, how should we define a physical state? Well, first of all a candidate for a physical state should be BRST closed which are not exact. Furthermore, we should treat all physical states which differ at most by a BRST exact state as being equivalent. Thus, we define a BRST physical state to be the equivalence class formed by a BRST closed state. With this definition, the BRST Hilbert space, $\mathcal{H}_{BRST}$ is given by taking the quotient of the Hilbert space formed from BRST closed states, $\mathcal{H}_{closed}$, with the Hilbert space formed from BRST exact states, $\mathcal{H}_{exact}$, i.e. our physical Hilbert space is given by

$$\mathcal{H}_{BRST} = \frac{\mathcal{H}_{closed}}{\mathcal{H}_{exact}}. \tag{9.18}$$

Finally, we would like to mention another useful quantity in the BRST theory, that of the ghost number. The ghost number is a constant that is allowed to take values in $\mathbb{R}$.

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‡Here we are assuming that $Q_B^\dagger = Q_B$.

One assigns ghost number $+1$ to the ghost fields c^α, ghost number -1 to the anti-ghost fields b_A as well as the parameter κ and ghost number 0 to the other fields. Thus, if one starts with a Fock-space state of a certain ghost number and acts on it with various oscillators, the ghost number of the resulting state is the initial ghost number plus the number of c^α-oscillator excitations minus the number of b_A-oscillator excitations. So, physical states are also classified by their ghost number. This is an additive global symmetry of the quantum action $(S + S_2 + S_3)$, so there is a corresponding conserved ghost-number current and ghost-number charge.

It turns out that the BRST charge operator Q_B raises the ghost number of a state by 1. Thus, if we let A_n be the subspace of all states of ghost number n, then Q_B, restricted to A_n, maps A_n to the space A_{n+1} and so, since $Q_B^2 = 0$, we get a cochain sequence[‡],

$$\cdots \xrightarrow{Q_{B,-3}} A_{-2} \xrightarrow{Q_{B,-2}} A_{-1} \xrightarrow{Q_{B,-1}} A_0 \xrightarrow{Q_{B,0}} A_1 \xrightarrow{Q_{B,1}} A_2 \xrightarrow{Q_{B,2}} \cdots , \qquad (9.20)$$

where $Q_{B,n}$ symbolizes the operator Q_B restricted to A_n. This in turn allows for us to define the cohomology groups of Q_B, where the $n-th$ cohomology group of Q_B, which we denote by $H^n(Q_B)$, is defined to be the quotient group given by

$$H^n(Q_B) = \frac{\ker Q_{B,n+1}}{\operatorname{Im} Q_{B,n}}, \qquad (9.21)$$

But wait a minute. If φ is an element of $\ker Q_{B,n+1}$ then it is annihilated by $Q_{B,n+1}$, which is the same property of a BRST closed state. Similarly, if φ is an element of $\operatorname{Im} Q_{B,n}$ then it can be written as $Q_{B,n}\varphi'$ for some φ', which is the same property of a BRST exact state. Thus, we have a 1-1 correspondence between BRST physical states, of ghost number n, and the $n-th$ cohomology group of Q_B, $H^n(Q_B)$. It turns out, as we will see, that in the case of the bosonic string we will take our physical states to have ghost number $-1/2$.

[‡]A cochain complex $(B^\bullet, d^\bullet)$ is a sequence of modules (or abelian groups),

$$\cdots, B_{-2}, B_{-1}, B_0, B_1, B_2, \cdots,$$

which are connected by homomorphisms $d_n : B_n \to B_{n+1}$, where the composition of any two homomorphisms is zero ($d_n \circ d_{n+1} = 0$ for all n). One usually writes all of this in the compact notation given by

$$\cdots \xrightarrow{d_{-3}} B_{-2} \xrightarrow{d_{-2}} B_{-1} \xrightarrow{d_{-1}} B_0 \xrightarrow{d_0} B_1 \xrightarrow{d_1} B_2 \xrightarrow{d_2} \cdots . \qquad (9.19)$$

For an excellent introduction to these mathematical concepts see Massey "A Basic Course in Algebraic Topology (Graduate Texts in Mathematics)" or also Bott "Differential Forms in Algebraic Topology (Graduate Texts in Mathematics)".

For an example of calculating the physical states let's consider the BRST quantization of a point particle (see Polchinski p 129 - 131 and problem 9.1). For what follows, we will only list results in order to get a feel for how to define BRST physical states. First, the ghosts b and c generate a two-state system and so a complete set of states is given by $|k, \uparrow\rangle$ and $|k, \downarrow\rangle$, where

$$p^\mu|k^\mu; \uparrow\rangle = k^\mu|k^\mu; \uparrow\rangle, \qquad p^\mu|k^\mu; \downarrow\rangle = k^\mu|k^\mu; \downarrow\rangle,$$

$$b|k^\mu; \downarrow\rangle = 0, \qquad b|k^\mu; \uparrow\rangle = |k^\mu; \downarrow\rangle,$$

$$c|k^\mu; \downarrow\rangle = |k^\mu; \uparrow\rangle, \qquad c|k^\mu; \uparrow\rangle = 0.$$

Also, in this theory the BRST charge operator Q_B is given by the ghost field c times the Hamiltonian,

$$Q_B = cH = c(k^2 + m^2).$$

Thus, the action of Q_B on these states is given by

$$Q_B|k^\mu; \downarrow\rangle = \left(k^2 + m^2\right)|k^\mu; \uparrow\rangle, \qquad Q_B|k^\mu; \uparrow\rangle = 0. \tag{9.22}$$

From this we can immediately see that the BRST closed states are given by

$$|k^\mu; \downarrow\rangle, \qquad \text{for } k^2 + m^2 = 0,$$

$$|k^\mu; \uparrow\rangle \qquad, \qquad \text{for all } k^\mu,$$

and the BRST exact states are given by

$$|k^\mu; \uparrow\rangle, \qquad \text{for } k^2 + m^2 \neq 0.$$

Thus, the closed BRST which are not exact are the states of the form

$$|k^\mu; \downarrow\rangle, \qquad \text{for } k^2 + m^2 = 0,$$

$$|k^\mu; \uparrow\rangle, \qquad \text{for } k^2 + m^2 = 0,$$

So, we have just shown that BRST physical states, modulo exact states, are states which satisfy the mass-shell condition as expected. Note, however, that we have two copies of the expected spectrum. But, only states $|k^\mu; \downarrow\rangle$ satisfying the additional condition

$$b|k^\mu; \downarrow\rangle = 0,$$

appear in physical amplitudes. This is because for $k^2 + m^2 \neq 0$ the states $|k^\mu; \uparrow\rangle$ are orthogonal to all other physical states. Which implies that these amplitudes can only be proportional to $\delta(k^2 + m^2)$. But in field theory, for $D \neq 2$, amplitudes can never be proportional to delta functions, so these amplitudes must vanish.

We will now apply the general BRST quantization proceedure to the bosonic string.

9.2 BRST Quantization of the Bosonic String

We have seen that the string action, given by

$$S_\sigma = -\frac{T}{2} \int d\tau d\sigma \sqrt{-h} \, h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu, \tag{9.23}$$

is invariant under the local symmetry given by, here we are combining the reparameterization with the Weyl transformations (i.e. a conformal symmetry),

$$\delta h_{\alpha\beta} = D_\alpha \xi_\beta + D_\beta \xi_\alpha + 2w h_{\alpha\beta}, \tag{9.24}$$

$$\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu, \tag{9.25}$$

where in the transformation of the metric $h_{\alpha\beta}$ we are assuming $e^\phi = 2w$. Now, in order to proceed with the BRST quantization we simply follow the steps outlined in the primer, see 9.1.1. So, first, we add the fermionic ghost fields c_α, corresponding to the bosonic parameters ξ_α, and the fermionic ghost field c_w, corresponding to the bosonic parameter w. Note that for the ghost field c_w the subscript w is not an index, it is just there to notify us which ghost field it is.

Now, we need to introduce three gauge fixing terms $F^A(\Phi^I)$, two for ξ_α and one for w. These are given by (using a Euclidean signature for the metric $h_{\alpha\beta}$)

$$F_{\alpha\beta} = h_{\alpha\beta} - \delta_{\alpha\beta},$$

where $\delta_{\alpha\beta}$ is the two dimensional Euclidean metric, i.e. the Kronecker delta. Note that this is indeed three independent expressions since $\alpha, \beta = 0, 1$ gives 4 but then $h_{01} = h_{10}$ as well as $\delta_{01} = \delta_{10}$, which implies that $F_{\alpha\beta} = F_{\beta\alpha}$ which reduces the 4 to 3 independent terms. Also, we need to add the anti-ghost fields $b^{\alpha\beta}$ and auxiliary fields $B^{\alpha\beta}$, three of each, for the gauge fixing terms $F_{\alpha\beta}$.

The next ingredient of the BRST recipe is to add to the Polyakov action the two terms given by

$$S_2 = \int d\tau d\sigma \sqrt{h} \Big(-iB^{\alpha\beta}(h_{\alpha\beta} - \delta_{\alpha\beta}) \Big), \tag{9.26}$$

which is the gauge fixing term, and (for this derivation see below)

$$S_3 = \int d\tau d\sigma \sqrt{h} \, b^{\alpha\beta} c^\gamma \delta_\gamma F_{\alpha\beta}$$

$$= \int d\tau d\sigma \sqrt{h} \, b^{\alpha\beta} \Big(D_\alpha c_\beta + D_\beta c_\alpha + 2 c_w h_{\alpha\beta} \Big), \qquad (9.27)$$

which is the ghost action[‡]. As an example, let us explicitly derive the above expression for the ghost action. From (9.7), and expanding the DeWitt notation, we have

$$b_A c^\lambda \delta_\lambda F^A = \int \int d^2\sigma d^2\sigma' b^{\delta\gamma}(\sigma) c^\lambda(\sigma') \delta_\lambda(\sigma') \Big(h_{\delta\gamma}(\sigma) - \delta_{\delta\gamma}(\sigma) \Big),$$

where $d^2\sigma$ is shorthand for $d\tau d\sigma$ and c^λ is shorthand for both c^α and c_w, i.e. the two ghost fields corresponding to the diffeomorphims, and the ghost field corresponding to the Weyl transformation, respectively. Also, keep in mind that the term $\delta_\lambda(\sigma')$ is a transformation while $\delta_{\delta\gamma}(\sigma)$ is the flat metric. Now, since the flat metric's variation is zero, we have that the above becomes

$$\int \int d^2\sigma d^2\sigma' b^{\delta\gamma}(\sigma) \Big(c^\alpha(\sigma') \delta_\alpha(\sigma') h_{\delta\gamma}(\sigma) + c_w(\sigma') \delta_w(\sigma') h_{\delta\gamma}(\sigma) \Big),$$

and so we need to see how $h_{\delta\gamma}$ varies under a diffeomorphism, $\delta_\alpha h_{\delta\gamma}$, and also how it varies under a Weyl transformation, $\delta_w h_{\delta\gamma}$. These are given by[§]

$$\delta_\alpha(\sigma') h_{\delta\gamma}(\sigma) = -h_{\alpha\delta}(\sigma) D_\gamma \delta^2(\sigma - \sigma') - h_{\alpha\gamma}(\sigma) D_\delta \delta^2(\sigma - \sigma'),$$

$$\delta_w(\sigma') h_{\delta\gamma}(\sigma) = 2\delta^2(\sigma - \sigma') h_{\delta\gamma}(\sigma).$$

To check this one needs to show that with these definitions of $\delta_\alpha h_{\delta\gamma}$ and $\delta_w h_{\delta\gamma}$ then $\xi^\lambda \delta_\lambda h_{\delta\gamma} = \xi^\alpha \delta_\alpha h_{\delta\gamma} + w \delta_w h_{\delta\gamma} = \delta h_{\delta\gamma}$ where $\delta h_{\delta\gamma}$ is given by (9.24). This is easily

---

[‡]One should note the missing minus sign in the term $\sqrt{h}$ as compared with the measures we used before. This is because we are taking our worldsheet to have a Euclidean signature and thus the determinant of $h_{\alpha\beta}$ is now positive definite, instead of negative definite as before. Also, on a worldsheet of Euclidean signature we rewrite the quantity $\sqrt{-h}$ as $\sqrt{h}$ and thus, we can write the Polyakov action as

$$S_\sigma = -\frac{T}{2} \int d\tau d\sigma \sqrt{h} \, h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu. \qquad (9.28)$$

[§]See problem 9.1 to see how to do this for the case of a point particle.

computed as follows,

$$\xi^\lambda \delta_\lambda h_{\delta\gamma} = \int d\sigma' \xi^\alpha(\sigma') \Big(-h_{\alpha\delta}(\sigma) D_\gamma \delta^2(\sigma - \sigma') - h_{\alpha\gamma}(\sigma) D_\delta \delta^2(\sigma - \sigma') \Big) + w(\sigma') \Big(2\delta^2(\sigma - \sigma') h_{\delta\gamma}(\sigma) \Big)$$

$$= \int d\sigma' \Big(-\xi_\delta(\sigma') D_\gamma \delta^2(\sigma - \sigma') - \xi_\gamma(\sigma') D_\delta \delta^2(\sigma - \sigma') + 2w(\sigma') \delta^2(\sigma - \sigma') h_{\delta\gamma}(\sigma) \Big)$$

$$= \int d\sigma' \Big(D_\gamma \xi_\delta(\sigma') \delta^2(\sigma - \sigma') + D_\delta \xi_\gamma(\sigma') \delta^2(\sigma - \sigma') + 2w(\sigma') \delta^2(\sigma - \sigma') h_{\delta\gamma}(\sigma) \Big)$$

$$= D_\gamma \xi_\delta(\sigma) + D_\delta \xi_\gamma(\sigma) + 2w(\sigma) h_{\delta\gamma}(\sigma)$$

$$= \delta h_{\delta\gamma} \,,$$

where in the third line we integrated by parts while in the fourth line we performed the integration over σ' using the Dirac delta function $\delta^2(\sigma - \sigma')$.

Plugging the expressions for the transformations into the ghost action gives

$$b_A c^\lambda \delta_\lambda F^A = \int\int d^2\sigma d^2\sigma' \, b^{\delta\gamma}(\sigma) \Big(-c^\alpha(\sigma') h_{\alpha\delta}(\sigma) D_\gamma \delta^2(\sigma - \sigma') - c^\alpha(\sigma') h_{\alpha\gamma}(\sigma) D_\delta \delta^2(\sigma - \sigma')$$

$$+ 2c_w(\sigma') \delta^2(\sigma - \sigma') h_{\delta\gamma}(\sigma) \Big)$$

$$= \int\int d^2\sigma d^2\sigma' \, b^{\delta\gamma}(\sigma) \Big(-c_\delta(\sigma') D_\gamma \delta^2(\sigma - \sigma') - c_\gamma(\sigma') D_\delta \delta^2(\sigma - \sigma')$$

$$+ 2c_w(\sigma') \delta^2(\sigma - \sigma') h_{\delta\gamma}(\sigma) \Big)$$

$$= \int\int d^2\sigma d^2\sigma' \, b^{\delta\gamma}(\sigma) \Big(D_\gamma c_\delta(\sigma') \delta^2(\sigma - \sigma') + D_\delta c_\gamma(\sigma') \delta^2(\sigma - \sigma')$$

$$+ 2c_w(\sigma') \delta^2(\sigma - \sigma') h_{\delta\gamma}(\sigma) \Big)$$

$$= \int d^2\sigma \, b^{\delta\gamma}(\sigma) \Big(D_\gamma c_\delta(\sigma) + D_\delta c_\gamma(\sigma) + 2c_w(\sigma) h_{\delta\gamma}(\sigma) \Big) \,.$$

Thus, we have just shown that the correct expression for the ghost action is given by

$$\int d^2\sigma \, b^{\delta\gamma}(\sigma) \Big(D_\gamma c_\delta(\sigma) + D_\delta c_\gamma(\sigma) + 2c_w(\sigma) h_{\delta\gamma}(\sigma) \Big), \tag{9.29}$$

which is what was stated before, see (9.27).

Now that we have the new action, $S_\sigma + S_2 + S_3$, we quantize the theory, which is the final ingredient of the BRST recipe, by defining the partition function to be

$$Z = \int \mathcal{D}X^\mu \, \mathcal{D}h_{\alpha\beta} \, \mathcal{D}B^{\alpha\beta} \, \mathcal{D}b^{\alpha\beta} \, \mathcal{D}c^\alpha \, \mathcal{D}c_w \, e^{-(S_\sigma + S_2 + S_3)} \tag{9.30}$$

$$= \int \mathcal{D}X^\mu \, \mathcal{D}h_{\alpha\beta} \, \mathcal{D}B^{\alpha\beta} \, \mathcal{D}b^{\alpha\beta} \, \mathcal{D}c^\alpha \, \mathcal{D}c_w \, \times$$

$$\times \exp\left(\int d\tau d\sigma \sqrt{h} \left\{ -\frac{T}{2} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + iB^{\alpha\beta}(h_{\alpha\beta} - \delta_{\alpha\beta}) - b^{\alpha\beta}\left(D_\alpha c_\beta + D_\beta c_\alpha + 2c_w h_{\alpha\beta}\right) \right\}\right).$$

Thus, we have now officially quantized our Polyakov action, which describes a bosonic string propagating throughout a 26 dimensional spacetime, in the BRST fashion. And as before, this total action $S_\sigma + S_2 + S_3$ has a global symmetry, the BRST symmetry. From (9.9), we see that the BRST transformation is given by

$$\delta_B h_{\alpha\beta} = -i\kappa\left(D_\alpha c_\beta + D_\beta c_\alpha + 2c_w h_{\alpha\beta}\right), \tag{9.31}$$

$$\delta_B X^\mu = -i\kappa\left(c^\alpha \partial_\alpha X^\mu\right). \tag{9.32}$$

Next, we need to find the structure constants

$$[\delta_\alpha, \delta_\beta] = f^\gamma_{\alpha\beta}\delta_\gamma.$$

When we do this[‡] we then get, from (9.10), that

$$\delta_B c^\alpha = -i\kappa c^\beta \partial_\beta c^\alpha, \tag{9.33}$$

and also that

$$\delta_B c_w = -i\kappa c^\alpha \partial_\alpha c_w, \tag{9.34}$$

along with, coming from (9.11) and (9.12),

$$\delta_B b^{\alpha\beta} = \kappa B^{\alpha\beta}, \tag{9.35}$$

$$\delta_B B^{\alpha\beta} = 0. \tag{9.36}$$

The BRST transformation is given by (9.31) - (9.36).

Now, we would like to simplify the above and to further investigate this BRST quantized theory.

[‡]See problem 9.1 for an example of how to do this for a point particle.

To proceed further, we can integrate out the c_w ghost field since it only appears linearly in the partition function. Using the fact that[§]

$$\int \mathcal{D}c_w e^{\int d\tau d\sigma \sqrt{h}\, c_w b^{\alpha\beta} h_{\alpha\beta}} = \delta(b^{\alpha\beta} h_{\alpha\beta} - 0), \tag{9.37}$$

we see that integrating out the c_w adds a Dirac delta function to the partition function which removes the trace of $b^{\alpha\beta}$. After this integration our partition function takes the form

$$= \int \mathcal{D}X^\mu \, \mathcal{D}h_{\alpha\beta} \, \mathcal{D}B^{\alpha\beta} \, \mathcal{D}b^{\alpha\beta} \, \mathcal{D}c^\alpha \; \delta(b^{\alpha\beta} h_{\alpha\beta} - 0) \times$$

$$\times \exp\left(\int d\tau d\sigma \sqrt{h} \left\{ -\frac{T}{2} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + iB^{\alpha\beta}(h_{\alpha\beta} - \delta_{\alpha\beta}) - b^{\alpha\beta}\left(D_\alpha c_\beta + D_\beta c_\alpha\right) \right\}\right).$$

Next, since $B^{\alpha\beta}$ also appears only linearly in the partition function, we can integrate it out as well. Doing this adds another Dirac delta function to the partition function, namely $\delta(h_{\alpha\beta} - \delta_{\alpha\beta})$ which fixes the metric $h_{\alpha\beta}$ to be flat, i.e. integrating out the $B^{\alpha\beta}$ field introduces our gauge fixing term into the partition function.

Now, we can perform the integration over the $h_{\alpha\beta}$ field using the delta function $\delta(h_{\alpha\beta} - \delta_{\alpha\beta})$, which in essence is done by simply replacing $h_{\alpha\beta}$ with the flat metric, in Euclidean spacetime, $\delta_{\alpha\beta}$. Doing this gives us

$$Z = \int \mathcal{D}X^\mu \, \mathcal{D}b^{\alpha\beta} \, \mathcal{D}c^\alpha \; \exp\left(\int d\tau d\sigma \left(-\frac{T}{2} \partial_\alpha X^\mu \partial^\alpha X_\mu - b^{\alpha\beta}\left(D_\alpha c_\beta + D_\beta c_\alpha\right)\right)\right),$$

which can be further simplified by noting that since we have been working on a Euclidean worldsheet $D_\alpha \mapsto \partial_\alpha$ the partition function becomes

$$Z = \int \mathcal{D}X^\mu \, \mathcal{D}b^{\alpha\beta} \, \mathcal{D}c^\alpha \; \exp\left(\int d\tau d\sigma \left(-\frac{T}{2} \partial_\alpha X^\mu \partial^\alpha X_\mu - b^{\alpha\beta}\left(\partial_\alpha c_\beta + \partial_\beta c_\alpha\right)\right)\right).$$

[§]This follows from the expression

$$\delta(x - 0) = \int \frac{dp}{2\pi} \, e^{ipx},$$

by analytic continuation of $ix \mapsto y$ and dropping constants since,

$$\int \frac{dp}{2\pi} \, e^{py} \sim \delta(y - 0).$$

Thus, we have that

$$\int \mathcal{D}c_w e^{\int d\tau d\sigma \sqrt{h}\, c_w b^{\alpha\beta} h_{\alpha\beta}} \sim \delta(b^{\alpha\beta} h_{\alpha\beta} - 0),$$

and we don't care about the constant of proportionality.

Also, since we are working on a Euclidean worldsheet we can map the partition function to the complex plane by mapping the fields $X^\mu(\sigma,\tau) \mapsto X^\mu(z,\overline{z})$ etc., as we did previously, to give

$$Z = \int \mathcal{D}X \, \mathcal{D}b \, \mathcal{D}c \, \exp\left(\frac{1}{2\pi\alpha'} \int dz d\overline{z} \left(\partial X^\mu \overline{\partial} X_\mu \right) + \frac{1}{2\pi} \int dz d\overline{z} \left(b \overline{\partial} c + \overline{b} \partial \overline{c} \right) \right), \quad (9.38)$$

where, as before, $\partial \equiv \partial_z$, $\overline{\partial} \equiv \partial_{\overline{z}}$, $c \equiv c^z$, $\overline{c} \equiv c^{\overline{z}}$, $b \equiv b_{zz}$, $\overline{b} \equiv b_{\overline{z}\overline{z}}$ and $\alpha' = 1/2\pi T$.

Now, it can be shown (see problem 9.1 for doing this for a point particle) that the above action, given in (9.38), is further invariant under the (gauge fixed) BRST transformation given by

$$\delta_B X^\mu = -i\kappa \left(c \partial X^\mu + \overline{c} \, \overline{\partial} X^\mu \right), \quad (9.39)$$

$$\delta_B c = -i\kappa c \partial c, \quad (9.40)$$

$$\delta_B \overline{c} = -i\kappa \overline{c} \, \overline{\partial} \overline{c}, \quad (9.41)$$

$$\delta_B b = -i\kappa T, \quad (9.42)$$

$$\delta_B \overline{b} = -i\kappa \overline{T}, \quad (9.43)$$

where T is the total stress-energy tensor, i.e. it is the sum of the stress-energy tensor coming from the matter part of the action (the usual one from before), T^M, and the stress-energy tensor coming from the ghost part of the action, T^{gh}. The expressions for these quantities, which are calculated from the gauge fixed action, are given by[‡]

$$T^M = -\frac{1}{\alpha'} (\partial X)^2, \quad (9.44)$$

$$T^{gh} = -2 : b\partial c : + : c\partial b : . \quad (9.45)$$

This gauge fixed BRST transformation acts exactly the same as the non-gauge fixed BRST transformation on the fields X and c but its action on the b field has to be modified due to integrating out the B field. We will now see how to get this transformation for b.

[‡]Here we are only concerning ourselves with the holomorphic parts, as we will do for the remainder of this chapter unless otherwise noted.

The non-gauge fixed BRST transformation for b is given by $\delta_B b = \kappa B$. However, since we have now integrated out the B field, it no longer makes since to define anything in terms of it. What we need to do is to find an expression for the B field in terms of the fields that are still present in our theory. This can be done as follows. When we integrated out the B field we were left with a Dirac delta function which fixed the metric $h_{\alpha\beta}$ to the flat metric. This delta function was then used to integrate out $h_{\alpha\beta}$ and so it is the field equation for $h_{\alpha\beta}$ which will tell us the expression for B in terms of the other fields. To begin, we have that

$$S = S_\sigma - iB^{\alpha\beta}(h_{\alpha\beta} - \delta_{\alpha\beta}) + S_3.$$

Now, when we vary the action, S, with respect to $h_{\alpha\beta}$ we get

$$\delta_h S = \sqrt{h}\delta h_{\alpha\beta}(-iB^{\alpha\beta} + T^M + T^G),$$

and so the field equations imply that

$$B^{\alpha\beta} = -iT^{\alpha\beta},$$

which, in turn, implies that the gauge fixed BRST transformation for b is given by

$$\delta_B b = -i\kappa T.$$

Next we will discuss the OPE's of our BRST quantized theory.

9.2.1 The Ghost CFT

Previously we saw that by defining the quantities

$$b = b_{zz}, \qquad \bar{b} = b_{\bar{z}\bar{z}},$$

$$c = c^z, \qquad \bar{c} = c^{\bar{z}},$$

we could then write the ghost action, after integrating out the c_w and B fields, as

$$S_{gh} = \frac{1}{2\pi} \int dz d\bar{z} \big(b\bar{\partial}c + \bar{b}\partial\bar{c} \big). \tag{9.46}$$

This ghost action yields the following equations of motion for the ghost and anti-ghost fields,

$$\partial\bar{b} = \bar{\partial}b = \partial\bar{c} = \bar{\partial}c = 0, \tag{9.47}$$

which tells us that the fields b and c are holomorphic, while the fields $\bar{b}$ and $\bar{c}$ are anti-holomorphic. Thus, we can only focus on one part, either holomorphic or anti-holomorphic, since the other follows by replacing/removing c.c. bars. We will focus on the holomorphic part.

In order to derive the OPE of c and b we begin with the ghost partition function and the assumption that the path integral of a total derivative vanishes, just like we do for ordinary integrals. Thus, we can write

$$
\begin{aligned}
0 &= \int \mathcal{D}b \; \mathcal{D}c \; \frac{\delta}{\delta b(z)}\left(e^{-S_{gh}}b(w)\right), \\
&= \int \mathcal{D}b \; \mathcal{D}c \left(\left(\frac{\delta}{\delta b(z)}e^{-S_{gh}}\right)b(w) + e^{-S_{gh}}\left(\frac{\delta}{\delta b(z)}b(w)\right)\right) \\
&= \int \mathcal{D}b \; \mathcal{D}c \left(\left(-e^{-S_{gh}}\frac{1}{2\pi}\bar{\partial}c(z)\right)b(w) + e^{-S_{gh}}\delta(z-w)\right) \\
&= \int \mathcal{D}b \; \mathcal{D}c \; e^{-S_{gh}} \left(-\frac{1}{2\pi}\bar{\partial}c(z)b(w) + \delta(z-w)\right),
\end{aligned}
$$

which implies that

$$
\bar{\partial}c(z)b(w) = 2\pi\delta(z-w). \tag{9.48}
$$

This expression can be integrated on both sides to obtain[§] the OPE for $c(z)b(w)$, namely

$$
c(z)b(w) \sim \frac{1}{z-w}. \tag{9.49}
$$

Looking at $\frac{\delta}{\delta c(z)}$ instead of $\frac{\delta}{\delta b(z)}$ gives us the following OPE for $b(z)c(w)$

$$
b(z)c(w) \sim \frac{1}{z-w}, \tag{9.50}
$$

which also follows from the fact that, in our case, the b and c fields are fermionic since

---

[§]One also uses the identity

$$
2\pi\delta(z-w) = \bar{\partial}(\frac{1}{z-w}).
$$

we have that

$$c(z)b(w) \sim \frac{1}{z-w}$$

$$\Rightarrow -b(w)c(z) \sim \frac{1}{z-w}$$

$$\Rightarrow b(w)c(z) \sim -\frac{1}{z-w}$$

$$\Rightarrow b(z)c(w) \sim -\frac{1}{w-z}$$

$$\Rightarrow b(z)c(w) \sim \frac{1}{z-w}.$$

While $c(z)c(w)$ and $b(z)b(w)$ are both regular, i.e. non-singular.

Now that we know the OPE for c and b we can prove the following claim.

Proposition 9.3 *The (holomorphic) fields $c(z)$ and $b(z)$ are primary ghost fields with conformal weights $h = -1$ and $h = 2$, respectively.*

Proof First, recall that for $\Phi(w)$ to be a primary field of conformal weight h it means that its OPE with the stress-energy tensor $T(z)$ is of the form

$$T(z)\Phi(w) \sim \frac{h}{(z-w)^2}\Phi(w) + \frac{1}{z-w}\partial_w\Phi(w), \tag{9.51}$$

where T is the stress-energy tensor. Also, although it doesn't affect the calculations, recall that $T(z)c(w)$ is really shorthand notation for $R\left[T(z)c(w)\right]$, where R is the radial ordering operator. Now, since fields of different types have non-singular short distance behavoir, when we calculate the OPE of a matter field with the total stress-energy tensor $(T = T^M + T^{gh})$ we only have to concern ourselves with the matter part of the stress-energy tensor since the OPE of T^{gh} with the matter field is regular. Similary for the ghost fields. Thus, in order to show that $c(z)$ is a primary ghost field of conformal weight $h = -1$ we need to show that

$$T^{gh}(z)c(w) \sim \frac{-c(w)}{(z-w)^2} + \frac{\partial_w c(w)}{z-w}.$$

We have

$$T^{gh}(z)c(w) = \left(-2 : \partial_z c(z)b(z) : \; + \; : c(z)\partial_z b(z) : \right) c(w)$$

$$= -2 : \partial_z c(z)b(z) : c(w) \; + \; : c(z)\partial_z b(z) : c(w)$$

$$= -2 : \partial_z c(z) : \overbracket{b(z)\rangle\rangle c(w)} - 2 : b(z) : \underbracket{\overbracket{\partial_z c(z)\rangle\rangle c(w)}}_{\text{reg. term}} + : c(z) : \overbracket{\partial_z b(z)\rangle\rangle c(w)}$$

$$- : \partial_z b(z) : \underbracket{\overbracket{c(z)\rangle\rangle c(w)}}_{\text{reg. term}}$$

$$\sim -2\partial_z c(z)\left(\frac{-1}{z-w}\right) + c(z)\partial_z\left(\frac{-1}{(z-w)}\right)$$

$$\sim 2\partial_z c(z)\left(\frac{1}{z-w}\right) - \frac{c(z)}{(z-w)^2}$$

$$\sim \frac{2\partial_w c(w)}{z-w} - \frac{c(w)}{(z-w)^2} - \frac{\partial_w c(w)}{z-w}$$

$$\sim \frac{-c(w)}{(z-w)^2} + \frac{\partial_w c(w)}{z-w},$$

where in the third line we get the last minus sign due to the fact that the fields anti-commute and we have to move ∂b around c, while in the second to last line we expanded the functions $c(z)$ and $\partial_z c(z)$ around w, see the previous chapter for more examples of calculating OPE's.

For the b field we have that

$$T^{gh}(z)b(w) = \left(-2 : \partial_z c(z)b(z) : \; + \; : c(z)\partial_z b(z) : \right) b(w)$$

$$= -2 : \partial_z c(z)b(z) : b(w) \; + \; : c(z)\partial_z b(z) : b(w)$$

$$= -2 : \partial_z c(z) : \underbracket{\overbracket{b(z)\rangle\rangle b(w)}}_{\text{reg. term}} + 2 : b(z) : \overbracket{\partial_z c(z)\rangle\rangle b(w)} + : c(z) : \underbracket{\overbracket{\partial_z b(z)\rangle\rangle b(w)}}_{\text{reg. term}}$$

$$- : \partial_z b(z) : \overbracket{c(z)\rangle\rangle b(w)}$$

$$\sim 2b(z)\partial_z\left(\frac{1}{z-w}\right) + \frac{\partial_z b(z)}{z-w}$$

– 148 –

$$\sim \frac{2b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{z-w},$$

where in the third line we pick up the minus signs since the fields are fermionic (i.e. when we move $: \partial_z b(z) :$ around the $c(z)$ in the Wick expansion we have to take into account their statistics and add a minus sign), in the fourth line we get the minus signs due to the OPE of c with b and in the fifth line we expand the functions of z around w as before. Thus, we have shown that $c(z)$ is a primary field of weight $h = -1$ while $b(z)$ is a primary field of weight $h = 2$.

$\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

With the expression for the ghost stress-energy tensor T^{gh} we can compute the central charge[¶] of the (b,c) ghost system and it is given by

$$T^{gh}(z)T^{gh}(w) = \frac{-13}{(z-w)^4} + \frac{2T^{gh}(w)}{(z-w)^2} + \frac{\partial T^{gh}(w)}{z-w}, \tag{9.52}$$

i.e. the central charge is $c^{gh} = 2(-13) = -26$. Also, the total central charge, c, of the matter + ghost theory is given by $c^M + c^{gh}$, as can be seen from computing the OPE of the total stress-energy tensor, $T = T^M + T^{gh}$, with itself. One should recall here that the OPE of two different types of fields, i.e. a matter field and a ghost field, vanishes since the short distance behavoir of the pair is regular. Now, in order to not have the Weyl symmetry become anomalus[‡] it can be shown that the total central charge of the theory must equal 0[§]. This implies that the central charge of the matter theory, c^M,

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[¶]Recall that the central charge of a theory with stress-energy tensor T is given by computing the OPE of T with itself and then multiplying the constant sitting above the fourth order pole by 2 in this expansion, i.e. if T has an OPE with itself of the form

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{reg. terms},$$

then the central charge is given by c.

[‡]Symmetries of a classical theory which do not survive the transition to the quantum theory are called anomalous.

[§]As an aside: We have seen that in a CFT the trace of the stress-energy tensor must vanish, i.e. $T^\alpha{}_\alpha = 0$, classically. Now, when one quantizes the theory they find that the expectation value of the trace of the stress-energy tensor is given by

$$\langle T^\alpha{}_\alpha \rangle = -\frac{c}{12}R, \tag{9.53}$$

where R is the Ricci scalar. Thus, only if $R = 0$ or if $c = 0$ do we have that expectation value of the trace vanish. But, if we set $R = 0$ then we are only allowed to work in flat space and so we must, if we want to include varying spacetimes, set the central charge equal to zero, $c = 0$. Since the trace of T vanishes, classically, due to the CFT being Weyl invariant and since when we quantize the CFT this vanishing is no longer guaranteed, we refer to (9.53) as the Weyl anomaly.

must be equal to 26, i.e. the critical dimension of our bosonic theory is 26 as we showed in the previous chapters.

Note that the simplest way to achieve this value for the c^M is to add 26 bosonic free scalar fields to the theory since each one will add a value of 1 to c^M. However, this is not the only way to get this value. We only need to add a CFT that has central charge $c = 26$. Each such CFT describes a different background in which a string can propagate. If you like, the space of CFTs with $c = 26$ can be thought of as the space of classical solutions of string theory. Thus, we learn that the critical dimension of string theory is something of a misnomer: it is really a critical central charge. Only for rather special CFTs can this central charge be thought of as a spacetime dimension. For example, if we wish to describe strings moving in 4d Minkowski space, we can take 4 free scalars (one of which will be timelike) together with some other $c = 22$ CFT. This CFT may have a geometrical interpretation, or it may be something more abstract. The CFT with $c = 22$ is sometimes called the internal sector of the theory. This is what one really means when they talk about the extra hidden dimensions of string theory[¶].

9.2.2 BRST Current and Charge

Since the gauge fixed BRST transformations, (9.39) - (9.43), form a global symmetry for the gauge fixed action, (9.38), they have a corresponding Noether current. By the Noether method for calculating currents, we have that the holomorphic part of the current, j_B, is given by (HOW?)

$$j_B(z) = c(z)T^M(z) + \frac{1}{2} : c(z)T^{gh}(z) : + \frac{3}{2}\partial^2 c(z), \tag{9.54}$$

$$= c(z)T^M(z) + : b(z)c(z)\partial c(z) : + \frac{3}{2}\partial^2 c(z).$$

and similarly for $\bar{j}_B$. Corresponding to the Noether current there exists the Noether charge, Q_B, which is given by

$$Q_B = \frac{1}{2\pi} \oint dz \Big(c(z)T^M(z) + : b(z)c(z)\partial c(z) : \Big). \tag{9.55}$$

Note that the final term in the BRST current j_B, the total derivative term, does not contribute to the BRST charge Q_B. This term is put into the expression by hand in order to have j_B transform as a conformal tensor.

[¶]See David Tong's lectures on string theory.

Now, we can mode expand the ghost and anti-ghost fields,

$$b(z) = \sum_{m=-\infty}^{\infty} \frac{b_m}{z^{m+2}}, \tag{9.56}$$

$$c(z) = \sum_{m=-\infty}^{\infty} \frac{c_m}{z^{m-1}}, \tag{9.57}$$

and insert them into the BRST charge to give

$$Q_B = \sum_{m=-\infty}^{\infty} \left(L_{-m}^{X} - \delta_{m,0} \right) c_m - \sum_{m,n=-\infty}^{\infty} (m-n) : c_{-m}c_{-n}b_{m+n} : \;, \tag{9.58}$$

where L_{-m}^{X} are the Virasoro generators from the previous lectures, they just have an index X to remind us that they are arising from the matter part of the theory (i.e. the X fields). Also, note the appearance of the combination $L_0^X - 1$, the same combination that gives the mass-shell condition, in the coefficient of c_0. Finally, note that the above mode expansions for the b and c fields are of the form

$$\varphi(z) = \sum_{m=-\infty}^{\infty} \frac{\varphi_m}{z^{m+h}}, \tag{9.59}$$

where h is the conformal weight of φ.

9.2.3 Vacuum of the BRST Quantized String Theory

Since the BRST theory splits into a part containing the matter fields and a part containing the ghost fields, the vacuum of our theory, $|0\rangle_T$¶, will be given by the tensor product of the matter vacuum and the ghost vacuum, $|0\rangle_T = |0\rangle \otimes |0\rangle_{gh}$. Also, since we have already defined the matter vacuum, see previous lectures, we are only left to define the ghost vacuum. We do this by defining $|0\rangle_{gh}$ to be the state such that

$$b_n|0\rangle_{gh} = c_n|0\rangle_{gh} = 0, \tag{9.60}$$

for all $n \geq 1$, where b_n and c_n are the modes defined earlier in (9.56) and (9.57), respectively. But, now an interesting question arises, what about for the $n = 0$ modes? Due to these zero modes, the ground state (vacuum) is doubly degenerate (WHY?) and so we see that our state $|0\rangle_{gh}$ splits into two states, which we will denote by $|\uparrow\rangle$ and $|\downarrow\rangle$. We further define these states to obey

$$b_0|\uparrow\rangle = |\downarrow\rangle, \qquad b_0|\downarrow\rangle = 0,$$

$$c_0|\uparrow\rangle = 0, \qquad b_0|\downarrow\rangle = |\uparrow\rangle.$$

¶Here, and usually in what follows, we are supressing the momentum label, k^μ.

along with the conditions from earlier, namely

$$b_n | \uparrow \rangle = 0,$$

$$b_n | \downarrow \rangle = 0,$$

$$c_n | \uparrow \rangle = 0,$$

$$c_n | \downarrow \rangle = 0 \,,$$

for all $n \geq 1$. As an aside: since our physical space has a doubly degenerate ground state and also since our states are characterized by ghost numbers we see that our physical Hilbert space $\mathcal{H}_{BRST}$ is $\mathbb{Z}_2 \times \mathbb{R}$ graded, where $\mathbb{Z}_2$ corresponds to the two choices for a ground state and $\mathbb{R}$ corresponds to the ghost number. And thus, ever operator defined on the space must also be $\mathbb{Z}_2 \times \mathbb{R}$ graded.

Now, which of the two ground states do we take, i.e. which of the two states, $| \uparrow \rangle$ and $| \downarrow \rangle$, corresponds to the tachyon? It turns out, see Becker, Becker and Schwarz, to be the state $| \downarrow \rangle$. Thus our BRST vacuum is given by

$$|0\rangle_T = |0\rangle \otimes | \downarrow \rangle. \tag{9.61}$$

The physical states of the BRST theory are given by acting on the vacuum with the BRST operator Q_B. For instance we have

$$Q_B |0\rangle_T = Q_B \Big(|0\rangle \otimes | \downarrow \rangle \Big)$$

$$= \Big((L_0^X - 1)|0\rangle \Big) \Big(c_0 | \downarrow \rangle \Big) + \sum_{m>0} \Big(L_m^X |0\rangle \Big) \Big(c_{-m} | \downarrow \rangle \Big) \,,$$

where in the last line we only keep the terms for $m > 0$ since the $m < 0$ terms for c_{-m} annihilate $| \downarrow \rangle$. Now, when Q_B annihilates the ground state we see that $(L_0^X - 1)|\rangle) = 0$ and $L_M^X |0\rangle = 0$ for all $m > 0$. Which is none other than the physical state condition from earlier, leading to the tachyon. Also, note that imposing the extra condition on physical states given by $b_n |\psi\rangle = 0$ implies that a physical state can contain no c oscillator modes. While fixing the ghost number, which we will see to be given by $-1/2$, prohibits the physical states from having any b oscillator modes. Thus, we are in complete agreement with the physical states defined in the previous chapters. To see the complete spectrum of the BRST physical states see Polchinski p 134 - 137. Also note that one can show that the BRST cohomology is isomorphic to the canonical

quantization and light cone quantization spectra as well as having a positive definite inner product. By having a positive definite inner product, the BRST theory has no ghost states, i.e. physical states with negative norm, this is called the no-ghost or Goddard-Thorn theorem. And since there is an isomorphism between the BRST physical states and the BRST cohomology, which in turn is isomorphic to the physical states we derived earlier via the canonical quantization and light-cone quantization procedures, we see that we do in fact recover all the physical states with the BRST approach to quantization.

To end this chapter we will discuss the ghost current and its charge.

9.2.4 Ghost Current and Charge

We define the ghost transformation to be

$$\delta_{gh}(c) = c, \tag{9.62}$$

$$\delta_{gh}(b) = -b, \tag{9.63}$$

$$\delta_{gh}(\text{all other fields}) = 0. \tag{9.64}$$

Note that we have already seen that this transformation leaves the action invariant. Thus there exists a current and charge corresponding to this symmetry.

The current which generates the ghost transformations is given by

$$j_{gh} = - : bc : , \tag{9.65}$$

and its charge is given by

$$Q_{gh} = \frac{1}{2\pi} \oint dz \; j_{gh}. \tag{9.66}$$

To see this, note that computing the OPEs of j_{gh} with $b(w)$ and $c(w)$ gives,

$$j_{gh}(z)b(w) = - : b(z)c(z) : b(w) = -\frac{b(w)}{z - w},$$

$$j_{gh}(z)c(w) = - : b(z)c(z) : c(w) = \frac{c(w)}{z - w},$$

and so we see that

$$[Q_{gh}, b(w)] = -b(w),$$

$$[Q_{gh}, c(w)] = c(w),$$

$$[Q_{gh}, X^\mu(w)] = 0,$$

which are the ghost transformations.

We can mode expand the ghost charge, also called the ghost number operator, to give

$$Q_{gh} = \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{n=1}^{\infty}(c_{-n} b_n - b_{-n} c_n), \tag{9.67}$$

and we can then use this operator to compute the ghost number of a state. For our case we have that

$$Q_{gh}\Big(|0\rangle \otimes |\downarrow\rangle\Big) = \left(\frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{n=1}^{\infty}(c_{-n} b_n - b_{-n} c_n)\right)\Big(|0\rangle \otimes |\downarrow\rangle\Big)$$

$$= \frac{1}{2}c_0 b_0\Big(|0\rangle \otimes |\downarrow\rangle\Big) - \frac{1}{2}b_0 c_0\Big(|0\rangle \otimes |\downarrow\rangle\Big) + \sum_{n=1}^{\infty} c_{-n} b_n \Big(|0\rangle \otimes |\downarrow\rangle\Big)$$

$$- \sum_{n=1}^{\infty} b_{-n} c_n \Big(|0\rangle \otimes |\downarrow\rangle\Big)$$

$$= -\frac{1}{2}b_0\Big(|0\rangle \otimes |\uparrow\rangle\Big)$$

$$= -\frac{1}{2}\Big(|0\rangle \otimes |\downarrow\rangle\Big),$$

where we get the second to last line since only this term doesn't annihilate the ghost vacuum. Also note that we have been sloppy with notation and we should technically write the above as $\mathbb{I} \otimes Q_{gh}$ where $\mathbb{I}$ is the identity operator which acts on the matter vacuum. So, from the previous calculation we see, as was stated earlier, that our BRST physical states have ghost number $-1/2$. Thus, we indeed have an isomorphism between our BRST physical states and the BRST cohomology group $H^{-1/2}(Q_B)$. Finally, note that in the case of open strings, this is the whole story. However, in the case of closed strings, this construction has to be carried out for the holomorphic (right-moving) and antiholomorphic (left-moving) sectors separately. The two sectors are then tensored with one another in the usual manner.

In the next chapter we will look at scattering theory in the bosonic string theory.

9.3 Exercises

Problem 1

In this chapter, we discussed BRST quantization in the abstract. We used de Witt's condensed notation in which the symmetry transformations of the fields were denoted by $\epsilon^\alpha \delta_\alpha \phi$. A summation over α, or, in case α is a continuous parameter, an integration, is understood. We express the commutator of two symmetry transformations in terms of structure constants $f^\gamma_{\alpha\beta}$ as $[\delta_\alpha, \delta_\beta] = f^\gamma_{\alpha\beta} \delta_\gamma$.

In this problem we consider diffeomorphism invariance in 1 dimension,

$$\tau \to \tau' = \tau - \xi(\tau),$$

as the gauge symmetry. The indices α, I, A will all get identified with the continuous parameter τ. A scalar field transforms as

$$X'(\tau') = X(\tau) \quad \Rightarrow \quad \delta X(\tau) \equiv X'(\tau) - X(\tau) = \xi(\tau) \dot{X}(\tau), \tag{9.68}$$

where $\dot{X} = \frac{dX}{d\tau}$ and in the last equality we only kept terms linear in ξ. Now in de Witt's notation,

$$\delta X(\tau) \equiv \xi^\tau \delta_\tau X(\tau), \tag{9.69}$$

with ξ^τ a different notation for $\xi(\tau)$. Equality between (9.68) and (9.69) implies

$$\delta_{\tau_1} X(\tau) = \delta(\tau - \tau_1) \dot{X}(\tau), \tag{9.70}$$

because, as can be seen by direct computation,

$$\xi^\tau \delta_\tau X(\tau) = \int d\tau_1 \, \xi(\tau_1) \delta(\tau - \tau_1) \dot{X}(\tau) = \xi(\tau) \dot{X}(\tau).$$

a) Calculate the structure constant $f^{\tau_3}_{\tau_1 \tau_2}$ by computing

$$[\delta_{\tau_1}, \delta_{\tau_2}] X(\tau) = f^{\tau_3}_{\tau_1 \tau_2} \delta_{\tau_3} X(\tau), \tag{9.71}$$

where

$$\delta_{\tau_1}(\delta_{\tau_2} X)(\tau) = \delta(\tau - \tau_1) \, \partial_\tau \left(\delta(\tau - \tau_2) \dot{X}(\tau) \right). \tag{9.72}$$

b) Consider now the action for a massless point particle moving in a flat one dimensional target space (a line):

$$S_1 = \frac{1}{2} \int d\tau \, e^{-1} (\dot{X})^2. \tag{9.73}$$

We have seen earlier in the course that this action is invariant under reparametrizations that act as follows,

$$\delta X = \xi(\tau)\dot{X}(\tau), \qquad \delta e = \frac{d}{d\tau}(\xi(\tau)e(\tau)). \qquad (9.74)$$

We now want to BRST quantize this theory. We introduce a ghost field $c(\tau)$ and an antighost field $b(\tau)$ and a corresponding auxiliary field $B(\tau)$. As we discussed in this chapter, the general form of the BRST transformations acting on fields ϕ^I and ghosts, c^α, antighosts b_A and auxiliary fields B_A is

$$\delta_B\phi^I = -i\kappa c^\alpha \delta_\alpha \phi^I, \qquad \delta_B c^\alpha = -\frac{i}{2}\kappa f^\alpha_{\beta\gamma}c^\beta c^\gamma, \qquad \delta_B b_A = \kappa B_A, \qquad \delta_B B_A = 0, \qquad (9.75)$$

where κ is an anticommuting variable and $f^\alpha_{\beta\gamma}$ are the structure constants.

(i) Write down the BRST transformations for X, e, c, b, B and show that they are nilpotent, i.e., $\delta_B^2 = 0$ on all fields. (Use different anticommuting parameters, say κ and κ', for the two BRST transformations, namely, $\delta_B^2 = \delta_B(\kappa)\delta_B(\kappa')$. Otherwise $\delta_B^2 = 0$ is trivial because $\kappa^2 = 0$!)

(ii) Use as a gauge fixing condition $F(e) = e - 1$ and write down the gauge fixed action.

(iii) Show that, after integrating out the auxiliary field B, the gauge fixed action becomes

$$S = \int d\tau \left(\frac{1}{2}(\dot{X})^2 + b\dot{c}\right). \qquad (9.76)$$

(iv) Show that the action (9.76) is invariant under the BRST transformations,

$$\delta_B X = -i\kappa c\dot{X}, \qquad \delta_B c = -i\kappa c\dot{c}, \qquad \delta_B b = i\kappa\left(\frac{1}{2}(\dot{X})^2 + \dot{b}c\right). \qquad (9.77)$$

You can ignore total derivative terms when checking the invariance of the action. Explain the relation between these transformations and the ones in (i)?

Problem 2

The holomorphic part of the BRST current for the Polyakov action of the bosonic string is given by

$$j_B = cT^m + \frac{1}{2} : cT^{gh} : + \frac{3}{2}\partial^2 c$$

$$= cT^m + : bc\partial c : + \frac{3}{2}\partial^2 c, \qquad (9.78)$$

where T^m is the energy momentum tensor of the matter sector with central charge c_m and T^{gh} is the energy momentum tensor of the bc system. The BRST charge is given by

$$Q_B = \frac{1}{2\pi i} \oint dz \, j_B. \tag{9.79}$$

where we suppress the antiholomorphic contribution.

a) Compute the BRST transformation rule for the ghost field c.

b) Compute the OPE of j_B with the antighost b

c) Compute the OPE of j_B with a matter conformal primary ϕ^h of weight h (i.e. ϕ^h does not depend on the ghost or antighost fields).

d) Show that the OPE between the total energy momentum tensor $T = T^m + T^{gh}$ and j_B is given by

$$T(z)j_B(w) \sim \frac{c_m - 26}{2(z-w)^4}c(w) + \frac{1}{(z-w)^2}j_B(w) + \frac{1}{z-w}\partial j_B(w). \tag{9.80}$$

What does the result imply for j_B?

Problem 3

a) Show that the OPE between two BRST currents (defined in Problem 2) is given by

$$j_B(z)j_B(w) \sim -\frac{c_m - 18}{2(z-w)^3}c\partial c(w) - \frac{c_m - 18}{4(z-w)^2}c\partial^2 c(w) - \frac{c_m - 26}{12(z-w)}c\partial^3 c(w), \tag{9.81}$$

b) Use this OPE to determine the anticommutator of the BRST charge with itself. For what value of c_m does this vanish? What is the significance of this result?

10. Scattering in String Theory

10.1 Vertex Operators

Vertex operators V_ϕ are world-sheet operators that represent the emission or absorption of a physical on-shell string mode $|\phi\rangle$ from a specific point on the string world sheet. There is a one-to-one mapping between physical states and vertex operators. Since physical states are highest-weight representations of the Virasoro algebra (see 8.3), the corresponding vertex operators are primary fields, and the problem of constructing them is thus the inverse of the problem discussed earlier in connection with the state-operator correspondence (WHAT PROBLEM? see BBS page 85). In the case of an open string, the vertex operator must act on a boundary of the world sheet, whereas for a closed string it acts on the interior. Thus, summing over all possible insertion points gives an expression of the form

$$g_0 \oint V_\phi(s)ds$$

in the open-string case. The idea here is that the integral is over a boundary that is parametrized by a real parameter s. In the closed-string case one has

$$g_s \int V_\phi(z, \overline{z})d^2z,$$

which is integrated over the entire world sheet. In each case, the index ϕ is meant to label the specific state that is being emitted or absorbed (including its 26-momentum). There is a string coupling constant g_s that accompanies each closed-string vertex operator. Note that the open-string coupling constant g_0 is related to it by $g_0^2 = g_s$. Also, to compensate for the integration measure, and give a coordinate-independent result, a vertex operator must have conformal dimension 1 in the open-string case and $(1, 1)$ in the closed-string case.

(SEE BBS 3.4, Tong, Kristsis 5.1 and Polchinski 2.8 and Ch 6)

In the next chapter we will finally begin to include fermions in our theory. This will be done by imposing that our worldsheet has a new symmetry, that of supersymmetry. We will define this new string theory, or rather supersting theory, called the RNS supersting theory and then we will show that this theory has fermions, is no longer plagued by the tachyon ground state and has a critical dimension given by $D = 10$.

10.2 Exercises

Problem 1

To compute the closed string tachyon amplitude one can use the following expression for the 3-point function of c ghosts on the sphere:

$$\langle c(z_1)c(z_2)c(z_3)\rangle_{S_2} = C^g_{S_2}(z_1 - z_2)(z_1 - z_3)(z_2 - z_3), \tag{10.1}$$

where $C^g_{S_2}$ is a normalization constant. Following section 6.3 of Polchinski, derive the expression (10.1) using the fact that $c(z)$ is a holomorphic field and that it is anticommuting.

Problem 2

Find the Möbius transformation:

$$z' = \frac{az + b}{cz + d} \qquad (ad - bc = 1), \tag{10.2}$$

that takes three given points, z_1, z_2 and z_3, into 0, 1, ∞, respectively.

Problem 3

Exercise 6.10 of Polchinski.

Problem 4

In this problem we want to study the high energy behavior of closed string scattering amplitudes. Consider the four tachyon scattering amplitude, see (PUT REF HERE) (also see Polchinski 6.6.4-6.6.5). Consider the scattering process:

$$1 + 2 \to 3 + 4, \tag{10.3}$$

so that k_1^0, $k_2^0 > 0$ and $k_3^0, k_4^0 < 0$. We will be working in the 1-2 center-of-mass frame. We call E the center-of-mass energy and θ the scattering angle between particle 1 and particle 3.

1. Show that the s, t, u variables for this process are:

$$s = E^2, \qquad t = (4m^2 - E^2)\sin^2\frac{\theta}{2}, \qquad u = (4m^2 - E^2)\cos^2\frac{\theta}{2}, \tag{10.4}$$

    where m^2 is the mass-squared of the incoming tachyons.

2. We want to study the amplitude in the limit $E \to \infty$, $\theta = $ constant, which in terms of the s, t, u variables is equivalent to $s \to \infty$, $s/t = $ fixed. We need to compute the asymptotic expansion of $J(s, t, u)$ (equation 6.6.5) in this limit. We

will do it using the saddle point approximation. The function J can be written as:

$$J(s, t, u) = \int d^2 z_4 exp\left[(-\alpha' u/2 - 4)\log |z_4| + (-\alpha t/2 - 4)\log |1 - z_4|\right]. \quad (10.5)$$

This is of the form:

$$\int d^2 z_4 exp\left[cf(z_4)\right]. \quad (10.6)$$

In the limit $c \to \infty$ such an integral can be approximated by:

$$\int d^2 z_4 exp\left[cf(z_4)\right] \approx e^{cf(\widetilde{z_4})}, \quad (10.7)$$

where $\widetilde{z_4}$ is the saddle point of f, that is a point where:

$$\frac{df(z_4)}{dz_4} = 0. \quad (10.8)$$

Show that in the limit we are considering the saddle point is at:

$$z_4 = -u/s, \quad (10.9)$$

and that using (10.7) we find that the amplitude behaves like:

$$S \approx \exp\left[-\frac{\alpha'}{2}(s \ln s + t \ln t + u \ln u)\right], \quad (10.10)$$

which can also be written as:

$$S \approx \exp\left[-\frac{\alpha'}{2}E^2 f(\theta)\right], \quad (10.11)$$

where:

$$f(\theta) = -\sin^2\frac{\theta}{2}\ln\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\ln\cos^2\frac{\theta}{2}. \quad (10.12)$$

Notice that the amplitude vanishes exponentially as we increase the scattering energy. This is very different from quantum field theory where the scattering amplitude goes to zero as a power of $\frac{1}{E}$. Power law falloff is characteristic of scattering of pointlike objects. The much softer exponential falloff (10.11) suggests that we are scattering extended objects of typical size $\sqrt{\alpha'}$. Of course this is what we expected due to the finite size of the string.

11. Supersymmetric String Theories (Superstrings)

The bosonic string theory that was discussed in the previous chapters is unsatisfactory in two aspects. First, the closed-string spectrum contains a tachyon. If one chooses to include open strings, then additional open-string tachyons appear. Tachyons are unphysical because they imply an instability of the vacuum. The elimination of open-string tachyons from the physical spectrum has been understood in terms of the decay of D-branes into closed-string radiation. However, the fate of the closed-string tachyon has not been determined yet. The second unsatisfactory feature of the bosonic string theory is that the spectrum (of both open and closed strings) does not contain fermions. Fermions play a crucial role in nature, of course. They include the quarks and leptons in the standard model. As a result, if we would like to use string theory to describe nature, fermions have to be incorporated.

In string theory the inclusion of fermions turns out to require supersymmetry, a symmetry that relates bosons, $X^\mu(\tau, \sigma)$, to fermions, $\Psi^\mu(\tau, \sigma)$, where $\Psi^\mu(\tau, \sigma)$ are two-component spinors, i.e. $\Psi^\mu(\tau, \sigma)$ is really given by

$$\Psi^\mu(\tau, \sigma) = \begin{pmatrix} \psi_-^\mu(\tau, \sigma) \\ \psi_+^\mu(\tau, \sigma) \end{pmatrix},$$

and we call $\psi_A^\mu(\tau, \sigma)$ the chiral components of the spinor, so, for example, $\psi_+^\mu(\tau, \sigma)$ is the $+$ chiral component of the spinor $\Psi^\mu(\tau, \sigma)$. These resulting string theories, i.e. string theories which have supersymmetry, are called superstring theories.

In order to incorporate supersymmetry into string theory, two basic approaches have been developed:

- The first is the Ramond-Neveu-Schwarz (RNS) formalism which is supersymmetric on the string world sheet.

- The other is the Green-Schwarz (GS) formalism which is supersymmetric in ten-dimensional Minkowski background spacetime. This formalism can be generalized to other background spacetime[¶] geometries.

In ten dimensional Minkowski space, these two formalism are equivalent, maybe in other spacetimes as well. We will focus on the RNS formalism in this chapter.

[¶]Recall that the terminology background spacetime and target space are equivalent and will be used interchangeably throughout the whole book.

11.1 Ramond-Neveu-Schwarz Strings

In the RNS formalism we add to our D dimensional bosonic string theory D free fermionic fields $\Psi^\mu(\tau, \sigma)$. The fields $\Psi^\mu(\tau, \sigma)$ are two-component spinors which describe fermions living on the worldsheet which also transform as vectors under a Lorentz transformation on the D dimensional background spacetime. Note that a nessecary condition for a supersymmetric theory is that the number of bosonic degress of freedom be equal to the number of fermionic and this is why we add D fermionic fields to pair up with the D bosonic fields $X^\mu(\tau, \sigma)$.

We incorporate these fermionic fields into our theory by modifying the bosonic action. The new action is now given by the addition of the bosonic action, S_B, (i.e. the Polyakov action) along with the Dirac action for D free massless fermions, S_F, i.e.

$$S = S_B + S_F$$

$$= -\frac{1}{2\pi} \int d\tau d\sigma\, \partial_\alpha X^\mu \partial^a X_\mu - \frac{1}{2\pi} \int d\tau d\sigma\, \overline{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi_\mu, \tag{11.1}$$

where ρ^α, with $\alpha = 0, 1$, is the two dimensional representation of the Dirac matrices,[‡] and $\overline{\Psi}^\mu$ is the Dirac conjugate to Ψ_μ, which is defined by

$$\overline{\Psi}^\mu = (\Psi^\mu)^\dagger i\rho^0, \tag{11.3}$$

with $A^\dagger$ the Hermitian conjugate of A. Also, we will pick a choose such that in this basis the Dirac matrices assume the form

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{11.4}$$

Note that when the Dirac matrices have real components, as above, we call this a Majorana representation of the Dirac matrices. Also, in the Majorana representation we can impose a reality condition on the spinors,

$$(\Psi^\mu)^T C = (\Psi^\mu)^\dagger i\rho^0, \tag{11.5}$$

where C is, in our case, a 2×2 matrix called the charge conjugation matrix. To see that this implies that the spinor has real components, note that in two dimensions the

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[‡]The collection of Dirac matrices, with matrix multiplication, forms a Clifford algebra, i.e. the matrices satisfy

$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}, \tag{11.2}$$

where $\{\cdot, \cdot\}$ is the anti-commutator and $\eta^{\alpha\beta}$ is the flat metric with Minkowskian signature.

charge conjugation matrix is given by $C = i\rho^0$, which implies that

$$(\Psi^\mu)^T C = (\Psi^\mu)^\dagger i\rho^0$$

$$\Rightarrow (\Psi^\mu)^T i\rho^0 = (\Psi^\mu)^\dagger i\rho^0$$

$$\Rightarrow (\Psi^\mu)^T = (\Psi^\mu)^\dagger$$

$$\Rightarrow (\Psi^\mu)^T = (\Psi^\mu)^{T*}$$

$$\Rightarrow \Psi^\mu = (\Psi^\mu)^*$$

$$\Rightarrow \psi_\pm^\mu = (\psi_\pm^\mu)^*,$$

where $(A)^*$ is the complex conjugate of A. Thus, in the Majorana representation our spinor has real components,

$$(\psi_\pm^\mu)^* = \psi_\pm^\mu, \tag{11.6}$$

and we will call spinors of this type, i.e. with real components, Majorana spinors. So, to recap, in our RNS supersting theory we start by adding a Majorana spinor field, one for each bosonic field present, and then we alter the action by adding to it the action describing the Majorana spinors.

Classically, the Majorana spinors (or Majorana fields), since they are fermionic, are functions from the worldsheet to the set of Grassman numbers. This implies that the fields obey the following anti-commutation relations[‡]

$$\{\Psi^\mu, \Psi^\nu\} = 0. \tag{11.7}$$

Now, plugging into the action, (11.1), the forms of the Dirac matrices, (11.4), and writing the explicit components of the Majorana spinor we get, in worldsheet light-cone coordinates,

$$S = \frac{1}{\pi} \int d\sigma^+ d\sigma^- \, \partial_+ X^\mu(\sigma^-, \sigma^+) \, \partial_- X_\mu(\sigma^-, \sigma^+) \tag{11.8}$$

$$+ \frac{i}{2\pi} \int d\sigma^+ d\sigma^- \left(\psi_-^\mu(\sigma^-, \sigma^+)\partial_+ \psi_{-\mu}(\sigma^-, \sigma^+) + \psi_+^\mu(\sigma^-, \sigma^+)\partial_- \psi_{+\mu}(\sigma^-, \sigma^+) \right),$$

[‡]This relation only holds classically and so when we quantize the RNS superstring theory it must be corrected.

where $\partial_\pm$ refers to the worldsheet light-cone coordinates $\sigma^\pm$ introduced earlier. To see that this expression is indeed correct we only need to show it for the fermionic part since we have already shown it for the bosonic part (see problem 4.1). We proceed as follows. Since $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$ we have that, from (11.4),

$$\rho^\alpha \partial_\alpha = \begin{pmatrix} 0 & \partial_1 - \partial_0 \\ \partial_1 + \partial_0 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & -\partial_- \\ \partial_+ & 0 \end{pmatrix}.$$

Next,

$$\Psi^\dagger i \rho^0 = (\psi_-^*, \psi_+^*) i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= i(\psi_+^*, -\psi_-^*)$$

$$= i(\psi_+, -\psi_-),$$

where the last line follows from the fact that $\psi_A^* = \psi_A$ since Ψ is a Majorana spinor. Now we need to calculate the Jacobian for the change of coordinates $(\tau, \sigma) \mapsto (\sigma^-, \sigma^+)$ since under a change of variables the measure transforms as $d\tau d\sigma = J(\sigma^+, \sigma^-)d\sigma^+ d\sigma^-$. This is given by

$$J(\sigma^+, \sigma^-) = \det \begin{pmatrix} \frac{\partial \tau}{\partial \sigma^-} & \frac{\partial \tau}{\partial \sigma^-} \\ \frac{\partial \sigma}{\partial \sigma^-} & \frac{\partial \sigma}{\partial \sigma^-} \end{pmatrix}$$

$$= \det \begin{pmatrix} \frac{\partial}{\partial \sigma^-}(\frac{1}{2}(\sigma^+ + \sigma^-)) & \frac{\partial}{\partial \sigma^-}(\frac{1}{2}(\sigma^+ + \sigma^-)) \\ \frac{\partial}{\partial \sigma^-}(\frac{1}{2}(\sigma^+ - \sigma^-)) & \frac{\partial}{\partial \sigma^-}(\frac{1}{2}(\sigma^+ - \sigma^-)) \end{pmatrix}$$

$$= \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Thus, we have that

$$d\tau d\sigma = J(\sigma^+, \sigma^-)d\sigma^+ d\sigma^- = \frac{1}{2} d\sigma^+ d\sigma^-.$$

Finally, plugging all of this into the fermionic action gives

$$S_F = -\frac{1}{2\pi} \int d\tau d\sigma \ \overline{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi_\mu$$

$$= -\frac{1}{2\pi} \int \left(\frac{1}{2} \right) d\sigma^+ d\sigma^- \ \overline{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi_\mu$$

$$= -\frac{1}{4\pi} \int d\sigma^+ d\sigma^- \ i(\psi_+^\mu, -\psi_-^\mu) 2 \begin{pmatrix} 0 & -\partial_- \\ \partial_+ & 0 \end{pmatrix} \begin{pmatrix} \psi_{-\mu} \\ \psi_{+\mu} \end{pmatrix}$$

$$= -\frac{i}{2\pi} \int d\sigma^+ d\sigma^- \ (\psi_+^\mu, -\psi_-^\mu) \begin{pmatrix} -\partial_- \psi_{+\mu} \\ \partial_+ \psi_{-\mu} \end{pmatrix}$$

$$= -\frac{i}{2\pi} \int d\sigma^+ d\sigma^- \Big(-\psi_+^\mu \partial_- \psi_{+\mu} - \psi_-^\mu \partial_+ \psi_{-\mu} \Big)$$

$$= \frac{i}{2\pi} \int d\sigma^+ d\sigma^- \Big(\psi_+^\mu \partial_- \psi_{+\mu} + \psi_-^\mu \partial_+ \psi_{-\mu} \Big) ,$$

which is indeed equal to (11.8).

From the fermionic action we see that the equation of motion for the two spinor components is given by the Dirac equation, which in the worldsheet light-cone coordinates is given by

$$\partial_+ \psi_-^\mu = 0 \qquad \text{and} \qquad \partial_- \psi_+^\mu = 0. \tag{11.9}$$

Note that the first equation describes a left-moving wave while the second equation describes a right-moving wave.

11.2 Global Worldsheet Supersymmetry

The total action, $S_B + S_F$ (11.8), has a global symmetry, whose action on the fields is given by

$$\delta X^\mu = \bar{\epsilon} \Psi^\mu, \tag{11.10}$$

$$\delta \Psi^\mu = \rho^\alpha \partial_a X^\mu \epsilon, \tag{11.11}$$

where ϵ is a constant infinitesmal Majorana spinor, i.e. we have that

$$\epsilon = \begin{pmatrix} \epsilon_- \\ \epsilon_+ \end{pmatrix}, \tag{11.12}$$

with the components ϵ_- and ϵ_+ being infinitesmal, constant, real and Grassman. Thus, $\bar{\epsilon}$ is defined as $\bar{\epsilon} = \epsilon^\dagger i \rho^0$. Note that since we have a symmetry, this will be shown in a minute, which mixes the bosonic fields and fermionic fields this is in fact a supersymmetery. Thus, our action given by (11.8) really describes a superstring theory. This superstring theory is called a RNS superstring theory since our theory has supersymmetry on the worldsheet, as opposed to the Green-Schwarz (GS) superstring theory which has supersymmetry on the background spacetime in which the theory is defined.

Also note that globality of the supersymmetry (susy) transformations follows directly from the fact that ϵ has no worldsheet coordinate dependence.

Now, we can expand the above transformations in terms of the spinor components to get

$$\delta X^\mu = i\left(\epsilon_+ \psi_-^\mu - \epsilon_- \psi_+^\mu\right), \tag{11.13}$$

$$\delta\psi_-^\mu = -2\partial_- X^\mu \epsilon_+, \tag{11.14}$$

$$\delta\psi_+^\mu = 2\partial_+ X^\mu \epsilon_- . \tag{11.15}$$

To see that the action, written here without Lorentz indices,

$$S = \frac{1}{\pi} \int d\sigma^+ d\sigma^- \left(2\partial_+ X\,\partial_- X + i\psi_- \partial_+ \psi_- + i\psi_+ \partial_- \psi_+\right)$$

is invariant under the above supersymmetry transformation let us see how the action varies under the transformation. The general expression for the varied action is given by

$$\delta S = \frac{1}{\pi} \int d\sigma^+ d\sigma^- \left(2\partial_+(\delta X)\partial_- X + 2\partial_+ X\partial_-(\delta X) + i(\delta\psi_-)\partial_+\psi_-\right.$$

$$\left. + i\psi_- \partial_+(\delta\psi_-) + i(\delta\psi_+)\partial_-\psi_+ + i\psi_+ \partial_-(\delta\psi_+)\right).$$

Plugging in for the susy transformations, (11.13) - (11.15), gives

$$\delta S = \frac{1}{\pi} \int d\sigma^+ d\sigma^- \left(2\partial_+(i\epsilon_+\psi_- - i\epsilon_-\psi_+)\partial_- X + 2\partial_+ X\partial_-(i\epsilon_+\psi_- - i\epsilon_-\psi_+)\right.$$

$$\left. + i(-2\partial_- X\epsilon_+)\partial_+\psi_- + i\psi_-\partial_+(-2\partial_- X\epsilon_+) + i(2\partial_+ X\epsilon_-)\partial_-\psi_+ + i\psi_+\partial_-(2\partial_+ X\epsilon_-)\right),$$

which gives, after grouping the terms (don't forget that ϵ anti-commutes with ψ),

$$\delta S = \frac{2i}{\pi} \int d\sigma^+ d\sigma^- \left(\epsilon_+\partial_+\psi_-\partial_- X - \epsilon_-\partial_+\psi_+\partial_- X + \epsilon_+\partial_+ X\partial_-\psi_- - \epsilon_-\partial_- X\partial_+\psi_+\right.$$

$$\left. - \epsilon_+\partial_- X\partial_+\psi_- + \epsilon_+\psi_-\partial_+\partial_- X + \epsilon_-\partial_+ X\partial_-\psi_+ - \epsilon_-\psi_+\partial_-\partial_+ X\right)$$

$$= \frac{2i}{\pi} \int d\sigma^+ d\sigma^- \left\{\epsilon_+\left(\partial_+\psi_-\partial_- X + \partial_+ X\partial_-\psi_- - \partial_- X\partial_+\psi_- + \psi_-\partial_+\partial_- X\right)\right.$$

$$\left. + \epsilon_-\left(-\partial_+\psi_+\partial_- X - \partial_- X\partial_+\psi_+ + \partial_+ X\partial_-\psi_+ - \psi_+\partial_-\partial_+ X\right)\right\} .$$

Now, integrating the third term in the first sum, $\partial_- X \partial_+ \psi_-$, and third term in the second sum, $\partial_+ X \partial_- \psi_+$, by parts gives

$$\delta S = \frac{2i}{\pi} \int d\sigma^+ d\sigma^- \left\{ \epsilon_+ \left(\partial_+ \psi_- \partial_- X + \partial_+ X \partial_- \psi_- + X \partial_- \partial_+ \psi_- + \psi_- \partial_+ \partial_- X \right) \right.$$

$$\left. + \epsilon_- \left(-\partial_+ \psi_+ \partial_- X - \partial_+ \partial_- \psi_+ - X \partial_+ \partial_- \psi_+ - \psi_+ \partial_- \partial_+ X \right) \right\}$$

$$= \frac{2i}{\pi} \int d\sigma^+ d\sigma^- \epsilon_+ \left(\partial_+ \partial_- (\psi_- X) \right) - \frac{2i}{\pi} \int d\sigma^+ d\sigma^- \epsilon_- \left(\partial_+ \partial_- (\psi_+ X) \right).$$

And so, if we assume that the boundary terms vanish then we finally get that under a susy transformation the variation of the RNS action vanishes,

$$\delta S = 0,$$

which implies that our action is indeed invariant under the susy transformation, thus giving us that there exists a supersymmetry in our theory.

11.3 Supercurrent and the Super-Virasoro Constraints

We now want to proceed, as before in the previous chapters, and canonically quantize the RNS superstring theory. Recall that previously we used the equations of motion to derive the mode expansion for the fields. Then we proceeded to quantize the theory by promoting the modes to operators, which act in the physical Hilbert space of the theory, and replacing Poisson brackets with commutators. We then saw that, in our quantized bosonic string theory, there existed ghost states, states of negative norm. These ghost states were removed from the theory by enforcing that $a = 1$ in the mass-shell relation, $(L_0 - a)|\phi\rangle$, along with having the central charge of the Virasoro algebra equal to 26, $c = 26$. We will see shortly that ghost states will also plague the RNS theory. However, we will be able to eliminate the ghost states in the RNS theory by using the super-Virasoro constraints which, in turn, follow from the superconformal symmetry of the RNS theory, if we have the critical dimension $D = 10$. Also, note that we could follow the light-cone quantization approach by using the superconformal symmetry to fix a light-cone gauge which, as we already saw, gives us a theory which is free of ghost states but is no longer manifestly Lorentz invariant. But, as before, if we work in the critical dimension, $D = 10$, then we regain Lorentz invariance.

In this section we will derive the constraint equations which impose a superconformal invariance on our theory. Latter on we will quantize our theory and derive the result that the critical dimension is 10. In order to derive the constraint equations we begin by looking at the two conserved currents associated to the two global symmetries

of the RNS action, (11.1). These two currents are the supercurrent, which arises from the supersymmetry of the action, and the stress-energy tensor, which arises from the translational symmetry of the action. We will begin with the supercurrent and then proceed to the stress-energy tensor.

Since the supersymmetry is a global worldsheet symmetry we get, by Noether's theorem, an associated conserved current, called the worldsheet supercurrent. The explicit form of the supercurrent is constructed via the Noether method, see 4.1, as follows. By taking the supersymmetry spinor parameter ϵ to be worldsheet coordinate dependent, one finds that up to a total derivative (HOW?) the total action (11.1) varies, under this now local supersymmetry, as

$$\delta S \sim \int d\tau d\sigma (\partial_\alpha \bar{\epsilon}) \left(-\frac{1}{2} \rho^\beta \rho^\alpha \Psi_\mu \partial_\beta X^\mu \right).$$

Thus, the supercurrent is given by (written here with the spinor index A)

$$J_A^\alpha = -\frac{1}{2} \left(\rho^\beta \rho^\alpha \Psi_\mu \right)_A \partial_\beta X^\mu. \tag{11.16}$$

It can be shown that the supercurrent satisfies the following equation

$$(\rho_\alpha)_{AB} J_B^\alpha = 0, \tag{11.17}$$

where A and B are spinor components. This implies that the supercurrent really only has two indepedent components, which we label by j_- and j_+. Although the above is the correct expression for the supercurrent, we really would like to have an expression for the supercurrent in terms of the worldsheet light-cone coordinates. To obtain this expression we could perform a coordinate transformation or, which we will now do, we could use the Noether method on the total action, written in terms of the light-cone coordinates. The variation of the action, in light-cone coordinates, is in general given by, here omitting the Lorentz indicies,

$$\delta S = \frac{1}{\pi} \int d\sigma^+ d\sigma^- \left(2\partial_+(\delta X)\partial_- X + 2\partial_+ X \partial_-(\delta X) + i(\delta\psi_-)\partial_+\psi_- \right.$$

$$\left. + i\psi_-\partial_+(\delta\psi_-) + i(\delta\psi_+)\partial_-\psi_+ + i\psi_+\partial_-(\delta\psi_+) \right).$$

Now, plugging in for the ϵ_- susy transformation[‡] we get for the intergrand

$$2\partial_+(-i\epsilon_-\psi_+)\partial_-X + 2\partial_+X\partial_-(-i\epsilon_-\psi_+) + i(0)\partial_+\psi_- + i\psi_-\partial_+(0)$$

$$+\ i(2\partial_+X\epsilon_-)\partial_-\psi_+ + i\psi_+\partial_-(2\partial_+X\epsilon_-)\,,$$

which is equal to, modulo a total derivative,

$$4i\epsilon_-\partial_-(\psi_+\partial_+X).$$

Plugging this back into the expression for the varied action gives

$$\delta S = \frac{4i}{\pi}\int d\sigma^+ d\sigma^-\ \epsilon_-\partial_-(\psi_+\partial_+X),$$

which after integrating by parts yields

$$\delta S = -\frac{4i}{\pi}\int d\sigma^+ d\sigma^-\ (\partial_-\epsilon_-)(\psi_+\partial_+X).$$

And so, by choosing an appropriate normalization, the supercurrent associated with the ϵ_- transformation is given by

$$j_+ \equiv \psi_+^\mu\partial_+X_\mu. \qquad (11.18)$$

Similarly, doing the same for ϵ_+ gives

$$j_- \equiv \psi_-^\mu\partial_-X_\mu. \qquad (11.19)$$

To see that the supercurrent is conserved consider the following

$$\partial_+j_- = \partial_+(\psi_-^\mu\partial_-X_\mu)$$

$$= \partial_+\psi_-^\mu\partial_-X_\mu + \psi_-^\mu\partial_+\partial_-X^\mu \qquad (11.20)$$

$$= 0\,,$$

[‡]This transformation is given by

$$\delta_-X^\mu = -i\epsilon_-\psi_+^\mu,$$

$$\delta_-\psi_+^\mu = 2\partial_+X^\mu\epsilon_-,$$

$$\delta_-\psi_-^\mu = 0\,,$$

as can be read off from (11.13)-(11.15).

$$- 169 -$$

where the last line follows by the field equations for ψ_- and X. Similiarly, one can show that

$$\partial_- j_+ = 0.$$

Theses combined results, i.e. $\partial_+ j_- = \partial_- j_+ = 0$, combine to give us that

$$\partial_\alpha J_A^\alpha = 0, \tag{11.21}$$

or that the supercurrent is indeed conserved.

The next current of our theory is the current correspoding to translational symmetry of the RNS action. This current, as we have already seen, is called the stress-energy tensor and it is given by (HOW?)

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu + \frac{1}{4}\overline{\Psi}^\mu \rho_\alpha \partial_\beta \Psi_\mu + \frac{1}{4}\overline{\Psi}^\mu \rho_\beta \partial_\alpha \Psi_\mu - (\text{trace}). \tag{11.22}$$

We can rewrite this in terms of the worldsheet light-cone coordinates and spinor components as[§]

$$T_{++} = \partial_+ X_\mu \partial_+ X^\mu + \frac{i}{2}\psi_+^\mu \partial_+ \psi_{+\mu} \tag{11.23}$$

$$T_{--} = \partial_- X_\mu \partial_- X^\mu + \frac{i}{2}\psi_-^\mu \partial_- \psi_{-\mu} \tag{11.24}$$

$$T_{-+} = T_{+-} = 0, \tag{11.25}$$

where the last line follows from Weyl invariance of our theory.

Previously in the bosonic string theory we had that the Virasoro constraints were given by

$$T_{++} = T_{--} = 0,$$

which followed from the equation of motion for the worldsheet metric. The Virasoro constraints implied that all the components of the stress-energy tensor vanished, since the off-diagonal terms were already equal to zero. This, in turn, implied that, classically, all of the Virasoro generators, L_m, vanished - in particular L_0. However, when we quantized the theory we saw that the best we could say, for an open string, was that $(L_0 - a)|\phi\rangle = 0$, where a is due to normal ordering. This was the mass-shell condition and we saw that, via the spurious states, we were able to get rid of the ghost states by setting $a = 1$ and the central charge of the Virasoro algebra c equal to 26. Now, in the RNS theory, we have the analogous super-Virasoro constraints, which are given by

$$T_{++} = T_{--} = j_+ = j_- = 0. \tag{11.26}$$

[§]See Becker, Becker, and Schwarz problem 4.6 on page 121.

We will see, later on, that this constraint allows for us to remove the ghost states in the RNS theory just like before for the bosonic theory.

One should also note the Virasoro constraints implied that our theory was conformally invariant and that we could use this conformal invariance to remove the time-like components of X^μ, i.e. fixing the light-cone gauge, in order to remove the ghost states as we saw in the light-cone quantization. Likewise, the super-Virasoro constraints imply that the RNS superstring theory has a superconformal symmetry. And, analogous to the bosonic string theory, this superconformal symmetry allows for one to fix the light-cone gauge, which gives a manifestly positve-norm spectrum in the quantum theory (i.e. no ghost states).

The next step in the quantization process is to find the mode expansions of our fields. This is the topic of the next section.

11.4 Boundary Conditions and Mode Expansions

For the remainder of this chapter we will conform to the masses and write our integration measure as $d\tau d\sigma$ rather than $d\sigma^+ d\sigma^-$, while still writing the integrand as functions of the worldsheet light-cone coordinates. For example, instead of writing the action as we do in (11.8), we will write it in the form[‡]

$$S = \frac{1}{\pi} \int d\tau d\sigma \; \partial_+ X^\mu(\sigma^-, \sigma^+) \; \partial_- X_\mu(\sigma^-, \sigma^+) \tag{11.27}$$

$$+ \frac{i}{\pi} \int d\tau d\sigma \left(\psi_-^\mu(\sigma^-, \sigma^+) \partial_+ \psi_{-\mu}(\sigma^-, \sigma^+) + \psi_+^\mu(\sigma^-, \sigma^+) \partial_- \psi_{+\mu}(\sigma^-, \sigma^+) \right).$$

Also, since the boundary conditions and resulting field equations for the bosonic fields, X^μ, are the same as before we will not bother ourselves with rederiving everything and instead we refer the reader to section 3.3 and section 3.4 for review.

For the fermions we have the following action, after suppressing the Lorentz indices,

$$S_F \sim \int d^2\sigma \left(\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+ \right), \tag{11.28}$$

where $d^2\sigma \equiv d\tau d\sigma$. Varying this action gives

$$\delta S_F \sim \int d\tau \left(\psi_- \delta\psi_- - \psi_+ \delta\psi_+ \right)\Big|_{\sigma=\pi} - \int d\tau \left(\psi_- \delta\psi_- - \psi_+ \delta\psi_+ \right)\Big|_{\sigma=0}. \tag{11.29}$$

Now, as before for the bosonic theory, we want these surface terms to vanish. Thus leading to both open and closed RNS superstrings

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[‡]Also note that we are ignoring the overall constant appearing due to the Jacobian resulting from the coordinate change in the measure.

11.4.1 Open RNS Strings

In the case of open strings, the two boundary terms in the above expression for the variation of the action must vanish seperately. We are able to achieve this if we set

$$\psi_+^\mu = \pm \psi_-^\mu, \tag{11.30}$$

at both boundaries $\sigma = 0, \pi$, i.e. at $\sigma = 0$ we have that $\psi_+^\mu = \pm \psi_-^\mu$, while at $\sigma = \pi$ we have that $\psi_+^\mu = \pm \psi_-^\mu$. For example, if we take $\psi_+ = -\psi_-$ at both ends, then we have that

$$\int d\tau \big(\psi_- \delta\psi_- - (-\psi_-)\delta(-\psi_-) \big)|_{\sigma=\pi} - \int d\tau \big(\psi_- \delta\psi_- - (-\psi_-)\delta(-\psi_-) \big)|_{\sigma=0}$$

$$= \int d\tau (0) - \int d\tau (0) = 0 \,,$$

and so, $\delta S_F = 0$. The choice of which sign to take at one of the boundaries for σ is by convention, which we take to be

$$\psi_+^\mu|_{\sigma=0} = \psi_-^\mu|_{\sigma=0}. \tag{11.31}$$

However, one should note that once this choice for the sign has been made then the sign at the other end becomes relevant. There are two possible choices, also called sectors, for the sign.

- Ramond Sector: In this case one chooses the other end of the open string to obey

$$\psi_+^\mu|_{\sigma=\pi} = \psi_-^\mu|_{\sigma=\pi}. \tag{11.32}$$

  We will see later that the Ramond boundary condition, (11.32), induces fermions on the background spacetime. Now, the field equations for the fermionic fields were given by, see (11.9),

$$\partial_- \psi_+^\mu = 0, \qquad \text{and} \qquad \partial_+ \psi_-^\mu = 0, \tag{11.33}$$

  which implies that $\psi_- = \psi_-(\sigma^-)$ and $\psi_+ = \psi_+(\sigma^+)$. Imposing the Ramond boundary condition gives us, for the mode expansion of the fields[§],

$$\psi_-^\mu(\tau, \sigma) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)}, \tag{11.34}$$

$$\psi_+^\mu(\tau, \sigma) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} d_n^\mu e^{-in(\tau+\sigma)}. \tag{11.35}$$

[§]See problem 3.1 for an example of how to do the mode expansion.

Note that in the above expressions for the mode expansions we have switched back to the usual worldsheet coordinates, σ and τ, and also we have chosen $1/\sqrt{2}$ for future convenience. Now, since the original spinor, Ψ, was Majorana it implies that each of the components of the spinor must be real. This, in turn, implies that $d^\mu_{-n} = (d^\mu_n)^\dagger$.

The other boundary condition we will explore is that of the Neveu-Schwarz type.

- Neveu-Schwarz Sector: In this case one chooses the other end of the string to obey

$$\psi^\mu_+|_{\sigma=\pi} = -\psi^\mu_-|_{\sigma=\pi}. \tag{11.36}$$

The Neveu-Schwarz boundary condition gives rise to bosons living on the background spacetime. In the Neveu-Schwarz sector, the mode expansion of the fields are given by

$$\psi^\mu_-(\tau, \sigma) = \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z}+1/2} b^\mu_r e^{-ir(\tau-\sigma)}, \tag{11.37}$$

$$\psi^\mu_+(\tau, \sigma) = \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z}+1/2} b^\mu_r e^{-ir(\tau+\sigma)}. \tag{11.38}$$

We will use the convention where we only use an n or a m for integer valued numbers, and r or s for half-integer valued numbers.

Now that we have derived the mode expansions for the open superstrings in both the Ramond and Neveu-Schwarz sectors, let us consider the closed superstrings.

11.4.2 Closed RNS Strings

The closed-string boundary condition will, as before, give rise to two sets of fermionic modes, the so-called left-movers and right-movers. Once again, there are two possible periodicity conditions which make the boundary terms vanish, namely,

$$\psi^\mu_\pm(\tau, \sigma) = \pm\psi^\mu_\pm(\tau, \sigma + \pi), \tag{11.39}$$

where the positive sign describes periodic boundary conditions (Ramond or R boundary conditions) while the negative sign describes anti-periodic boundary conditions (Neveu-Schwarz or NS boundary conditions). One should note that it is possible to impose either the R or NS boundary conditions on the left and right-movers seperately. This leads to the two following choices for the mode expansion of the left-movers (here we

are writing the expansion in the R sector first, followed by the expansion for the field in the NS sector)

$$\psi_+^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(t+\sigma)} \qquad \text{or} \qquad \psi_+^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+1/2} \tilde{b}_r^\mu e^{-2ir(t+\sigma)}, \qquad (11.40)$$

while for the right-movers we have the following two choices

$$\psi_-^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(t-\sigma)} \qquad \text{or} \qquad \psi_-^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-2ir(t-\sigma)}. \qquad (11.41)$$

Now, since our true state is given by tensoring together a left-mover with a right-mover, and since there are two choices for the left-movers and two choices for the right-movers, we get a total of four different sectors; the R-R sector, the R-NS sector, the NS-R sector and finally the NS-NS sector. Note that the states in the R-R and NS-NS sectors are background spacetime bosons, while states in the NS-R and R-NS sectors are background spacetime fermions.

Now that we have the mode expansions for the fields we can canonically quantize the RNS superstring theory.

11.5 Canonical Quantization of the RNS Superstring Theory

In order to quantize the RNS theory we begin by promoting the modes α and $\tilde{\alpha}$, which come from the bosonic fields, and the modes $b, \tilde{b}, d$ and $\tilde{d}$, which come from the fermionic fields, to operators and we also introduce the following algebraic relations for these operators[‡]

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m,-n}\eta^{\mu\nu}, \qquad (11.42)$$

$$\{b_r^\mu, b_s^\nu\} = \{\tilde{b}_r^\mu, \tilde{b}_s^\nu\} = \delta_{r,-s}\eta^{\mu\nu}, \qquad (11.43)$$

$$\{d_m^\mu, d_n^\nu\} = \{\tilde{d}_m^\mu, \tilde{d}_n^\nu\} = \delta_{m,-n}\eta^{\mu\nu}, \qquad (11.44)$$

with the rest vanishing. One can see that since the spacetime metric, $\eta^{\mu\nu}$, appears on the RHS of the oscillator algebraic relations above, the time components of the bosonic oscillators as well as the fermionic oscillators give rise to ghost states. However, as was noted already, we will see that we can remove these ghost states by using the super-Virasoro constraints. Also, we will only consider the open string case in what follows.

[‡]In the following expressions one should recall that $[A,B] \equiv AB - BA$, while $\{A,B\} \equiv AB + BA$.

We now need to define the ground state of our RNS theory. Since there are two sectors in the case of open RNS superstrings we will have two oscillator ground states, one for the Ramond (R) sector, $|0\rangle_R$, and one for the Neveu-Schwarz (NS) sector, $|0\rangle_{NS}$. They are defined by

$$\alpha_m^\mu |0\rangle_R = d_m^\mu |0\rangle_R = 0 \qquad \text{for} \qquad m > 0, \tag{11.45}$$

for the R sector and

$$\alpha_m^\mu |0\rangle_{NS} = b_r^\mu |0\rangle_{NS} = 0 \qquad \text{for} \qquad m, r > 0, \tag{11.46}$$

for the NS sector. As before, the excited states are constructed by acting on the ground state with the negative mode oscillators, since these are the "creation operators" for the theory.

There are some important differences between the ground state in the R sector and the ground state in the NS sector.

11.5.1 R-Sector Ground State VS. NS-Sector Ground State

In the NS sector the ground state is unique and it corresponds to a state of spin 0, i.e. a boson, in the background spacetime. Now, since all the oscillators (α_n^μ and b_r^μ) transform under a Lorentz transformation as spacetime vectors, the excited states in the NS sector, which follow from acting on the vacuum by negative mode oscialltors, will also correspond to spacetime bosons. Also, acting with the negative mode oscialltors increases the mass of the state, as we have seen previously for the bosonic theory.

In the R sector the ground state is degenerate, which can be seen as follows. The operators d_0^μ can act without effecting the mass of the state since they commute with the number operator N, which, later on, we will see is defined by

$$N = \sum_{n=1}^\infty \alpha_{-n} \cdot \alpha_n + \sum_{r=1/2}^\infty r b_{-r} \cdot b_r, \tag{11.47}$$

whose eigenvalue determines the squared mass of the state. Now, from the oscillator algebra of the d oscillators, we see that the d_0 obey the same algebraic relations as the Clifford algebra[‡], up to a factor of 2,

$$\{d_0^\mu, d_0^\nu\} = \delta_{0,0}\eta^{\mu\nu} = \eta^{\mu\nu}.$$

[‡]Recall that the Clifford algebra relations are given by

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}.$$

Thus, since the Dirac algebra is isomorphic to the Clifford algebra it implies that the set of degenerate ground states in the R sector must furnish a representation of the Dirac algebra. This implies that there is a set of degenerate ground states, which can be written in the form $|a\rangle$ with a being a spinor index[§], such that

$$d_0^\mu |a\rangle = \frac{1}{\sqrt{2}} \Gamma_{ba}^\mu |b\rangle, \tag{11.48}$$

where Γ^μ is an a dimensional matrix representation of d_0^μ, i.e. a Dirac matrix. This expression defines how the oscillator acts on the spinor, i.e. it gives a representation for the d_0^μ in the spinor space. To get a further understanding of the above expression, note that if we have two values for a, say $+$ and $-$, then Γ^μ will be a 2×2 matrix and the above expression is saying that

$$d_0^\mu \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} = \begin{pmatrix} \Gamma_{++}^\mu & \Gamma_{+-}^\mu \\ \Gamma_{-+}^\mu & \Gamma_{--}^\mu \end{pmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}.$$

Now, since all of the oscillators (α_n^μ and d_n^μ) transform as spacetime vectors, and since every state in the R sector can be obtained by acting with negative mode oscillators on the ground state, $|0\rangle_R$, we see that all the states in the R sector are spacetime fermions.

11.5.2 Super-Virasoro Generators (Open Strings) and Physical States

The super-Virasoro generatros are the modes of the stress-energy tensor, $T_{\alpha\beta}$, and the supercurrent, J^α. A super-Virasoro generator, say L_3, will be given by the sum of the corresponding Virasoro generator, $L_3^{(b)}$, from the bosonic part with the corresponding Virasoro generator from the fermionic part, $L_3^{(f)}$. And so, in general, for an open string, the super-Virasoro generators are given by

$$L_m = \frac{1}{\pi} \int_{-\pi}^{\pi} d\sigma \; e^{im\sigma} T_{++} = L_m^{(b)} + L_m^{(f)}, \tag{11.49}$$

where the contribution from the bosonic modes is given by

$$L_m^{(b)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} : \alpha_{-n} \cdot \alpha_{m+n} : \qquad m \in \mathbb{Z}. \tag{11.50}$$

[§]Thus, we have that if a is a spinor with two components then it can be written as a coloumn matrix whose two entries are given by D dimensional states. For example, the Ψ spinor from before can be written as

$$|\Psi\rangle = \begin{pmatrix} |\psi_-\rangle \\ |\psi_+\rangle \end{pmatrix},$$

where $|\psi_-\rangle$ and $|\psi_+\rangle$ are D dimensional states in our Hilbert space.

Now, since the fermionic part of our RNS theory splits up into two sectors we will get different contributions to the super-Virasoro generators depending upon which sector we are in.

- NS Sector: For the NS sector the contribution of the fermionic modes to the super-Virasoro generators is given by

$$L_m^{(f)} = \frac{1}{2} \sum_{r \in \mathbb{Z}+1/2} \left(r + \frac{m}{2} \right) : b_{-r} \cdot b_{m+r} : \qquad m \in \mathbb{Z}, \qquad (11.51)$$

  while the modes of the suppercurrent are

$$G_r = \frac{\sqrt{2}}{\pi} \int_{-\pi}^{\pi} e^{ir\sigma} j_+ = \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot b_{r+n}, \qquad (11.52)$$

  where, as usual, $r \in \mathbb{Z} + 1/2$.

  Note that we can write the operator L_0 in the form

$$L_0 = \frac{1}{2} \alpha_0^2 + N, \qquad (11.53)$$

  where N is the number operator, defined by

$$N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r=1/2}^{\infty} r b_{-r} \cdot b_r, \qquad (11.54)$$

  whose eigenvalues determine the mass squared of an excited state.

- R Sector: For the R sector the contribution of the fermionic modes to the super-Virasoro generators is given by

$$L_m^{(f)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left(n + \frac{m}{2} \right) : d_{-n} \cdot d_{m+n} : \qquad m \in \mathbb{Z}, \qquad (11.55)$$

  while the modes of the suppercurrent in the R sector are

$$F_m = \frac{\sqrt{2}}{\pi} \int_{-\pi}^{\pi} e^{im\sigma} j_+ = \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot d_{m+n}. \qquad (11.56)$$

The Super-Virasoro Algebra

Previously in the bosonic string theory, which was defined by the Polyakov action, we had that the Virasoro algebra was defined by the set of Virasoro generators $\{L_m^{(b)}\}_{m \in \mathbb{Z}}$ along with the algebraic relation given by

$$[L_m^{(b)}, L_n^{(b)}] = (m - n)L_{m+n}^{(b)} + \frac{c}{12}m(m^2 - 1)\delta_{m,-n}.$$

Now, for the RNS theory we will get a super-Virasoro algebra whose elements consist of the super-Virasoro generators $\{L_m\}_{m \in \mathbb{Z}}$ and the modes of the supercurrent[§]. Thus, since there were two different expressions for the modes of the supercurrent, corresponding to the two different sectors, we will also get two different super-Virasoro algebras.

- Super-Virasoro Algebra in the NS Sector: In the NS sector the super-Virasoro algebra consists of the elements $\{L_m, G_r\}$, where $m \in \mathbb{Z}$ and $r \in \mathbb{Z} + 1/2$, along with the following algebraic relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{D}{8}m(m^2 - 1)\delta_{m,-n}, \tag{11.57}$$

$$[L_m, G_r] = \left(\frac{m}{2} - r\right)G_{m+r}, \tag{11.58}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}, \tag{11.59}$$

  where we are denoting the super-Virasoro central charge by D.

- Super-Virasoro Algebra in the R Sector: In the R sector the super-Virasoro algebra consists of the elements $\{L_m, F_n\}$, where $m, n \in \mathbb{Z}$, along with the following algebraic relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{D}{8}m^3\delta_{m,-n}, \tag{11.60}$$

$$[L_m, F_n] = \left(\frac{m}{2} - n\right)F_{m+n}, \tag{11.61}$$

$$\{F_m, F_n\} = 2L_{m+n} + \frac{D}{2}m^2\delta_{m,-n}. \tag{11.62}$$

[§]Remember that a super-Virasoro generator, L_m, is constructed by taking the corresponding generator from the bosonic Virasoro algebra, $L_m^{(b)}$, and adding to it the corresponding Virasoro generator from the fermionic Virasoro algebra, $L_m^{(f)}$, i.e. $L_m = L_m^{(b)} + L_m^{(f)}$.

11.5.3 Physical State Conditions

Recall that in the bosonic string theory we had that classically all the elements of the Virasoro algebra vanished. Then when we went to quantize the theory we saw that at best we could only say that all of the elements of the Virasoro algebra, now viewed as operators, for which $m > 0$ annihilated a physical state. Also, we saw that due to normal ordering ambiguities we had that $(L_0^{(b)} - a)|\phi\rangle = 0$, known as the mass-shell condition. Thus, in the bosonic string theory the physical states were characterized as states $|\phi\rangle$ such that

$$L_m^{(b)}|\phi\rangle = 0 \qquad m > 0, \tag{11.63}$$

$$(L_0^{(b)} - a)|\phi\rangle = 0. \tag{11.64}$$

In the RNS superstring theory we have analogous conditions. Once again, since there are two seperate sectors we will get two seperate physical state conditions. However, in both sectors one can only impose, as before, that the super-Virasoro generators with $m > 0$ annihilate the physical states, rather than all of them.

- Physical State Conditions in the NS Sector: In the NS sector the physical state condition is as follows. If $|\phi\rangle$ is a physical state, living in the NS sector, then it must satisfy

$$L_m|\phi\rangle = 0 \qquad m > 0, \tag{11.65}$$

$$G_r|\phi\rangle = 0 \qquad r > 0, \tag{11.66}$$

$$(L_0 - a_{NS})|\phi\rangle = 0, \tag{11.67}$$

  where a_{NS} is a constant which arises due to the normal ordering ambiguity of L_0. It can be shown that the last condition, the RNS mass-shell condition, implies that $\alpha' M^2 = N - a_{NS}$, where M is the mass of the state $|\phi\rangle$ and N is the eigenvalue of the number operator acting on the state $|\phi\rangle$.

- Physical State Conditions in the R Sector: In the R sector the physical state condition is as follows.

$$L_m|\phi\rangle = 0 \qquad m > 0, \tag{11.68}$$

$$F_n|\phi\rangle = 0 \qquad n \geq 0, \tag{11.69}$$

$$(L_0 - a_R)|\phi\rangle = 0, \qquad (11.70)$$

where a_R is a constant due to the normal ordering ambiguity of L_0.

Note that in the above expressions for the physical state conditions the constants a_{NS} and a_R arose out of the normal ordering ambiguity of the L_0 operator. Since the L_0 operator is different for different sectors we see that, in general, $a_{NS} \neq a_R$. As before for the bosonic string theory, having the right value for these constants (and for D) will ensure that there are no longer any ghost states in our theory.

11.5.4 Removing the Ghost States

As like everything preceeding this discussion, the correct value for the normal ordering constants which will remove the ghost states depends on which sector you are working. However, the critical dimension D, as we will see, turns out to be $D = 10$ which holds for both sectors. So, let us proceed as before with the zero-norm physical spurious approach to calculate the normal ordering constants and the critical dimension. Note that one could also use the OPE approach from before of finding the central charge (or critical dimension) which consisted of calculating the OPE of the stress-energy tensor with itself. For another example of doing this see problem 11.1.

- NS Sector: To fix the value of a_{NS} consider states in the NS sector of the form

$$|\phi\rangle = G_{-1/2}|\xi\rangle, \qquad (11.71)$$

where $|\xi\rangle$ satisfies the conditions

$$L_{m>0}|\xi\rangle = 0, \qquad (11.72)$$

and

$$G_{1/2}|\xi\rangle = G_{3/2}|\xi\rangle = \left(L_0 - a_{NS} + \frac{1}{2}\right)|\xi\rangle = 0, \qquad (11.73)$$

where the last equality follows from (11.67) (written in terms of $G_{-1/2}|\xi\rangle$)[§]. It is therefore sufficient to show that $G_{1/2}|\phi\rangle = G_{3/2}|\phi\rangle = 0$ in order for $|\phi\rangle$ to be physical[‡]. Now, since the $G_{3/2}$ condition holds, we only have to check the $G_{1/2}$

[§]This is due to the fact that $G_{-1/2}$ lowers the eigenvalue of L_0 from a_{NS} to $a_{NS} - 1/2$, just like before for spurious states.

[‡]This is because all G_r, for $r > 3/2$, can be written in terms of the generators $L_{m>0}$ and $G_{1/2}$ and $G_{3/2}$. For example, we have that (from (11.58))

$$G_{5/2} = G_{1+3/2} = \frac{1}{1/2 - 3/2}[L_1, G_{3/2}] = -[L_1, G_{3/2}].$$

condition. This is given by

$$G_{1/2}|\phi\rangle = G_{1/2}G_{-1/2}|\xi\rangle.$$

Now, since $G_{1/2}G_{-1/2} = \{G_{-1/2}, G_{1/2}\} - G_{-1/2}G_{1/2}$ and since $\{G_{-1/2}, G_{1/2}\} = 2L_0 + D/2(1/4 - 1/4)\delta_{1/2,1/2} = 2L_0$, we have that the above becomes

$$G_{1/2}|\phi\rangle = \left(\{G_{-1/2}, G_{1/2}\} - G_{-1/2}G_{1/2}\right)|\xi\rangle = 2L_0|\xi\rangle = 2\left(a_{NS} - \frac{1}{2}\right)|\xi\rangle.$$

For this to vanish we see that we need $a_{NS} = 1/2$.

In order to calculate the critical dimension we need to go up another level in the spurious state. Consider the state

$$|\phi\rangle = \left(G_{-3/2} + \lambda G_{-1/2}L_{-1}\right)|\xi\rangle. \tag{11.74}$$

Also, let us suppose that

$$G_{1/2}|\xi\rangle = G_{3/2}|\xi\rangle = (L_0 + 1)|\xi\rangle = 0. \tag{11.75}$$

Now, as before, we need to show that $G_{1/2}|\phi\rangle = 0$ and $G_{3/2}|\phi\rangle = 0$. For the $G_{1/2}$ condition we have that

$$G_{1/2}|\phi\rangle = \left(G_{1/2}G_{-3/2} + \lambda G_{1/2}G_{-1/2}L_{-1}\right)|\xi\rangle,$$

which from the NS-sector super-Virasoro algebra relations we get

$$G_{1/2}|\phi\rangle = (2 - \lambda)L_{-1}|\xi\rangle.$$

And so, if $G_{1/2}$ is to annihilate the $|\phi\rangle$ state then we need $\lambda = 2$. From the $G_{3/2}$ condition we get

$$G_{3/2}|\phi\rangle = (D - 2 - 4\lambda)|\xi\rangle,$$

which follows again from the super-Virasoro relations. Now, for $G_{3/2}$ to annihilate $|\phi\rangle$ we see that we must have $D = 2 + 4\lambda = 10$. Thus, the critical dimension is $D = 10$ and so, to remove the ghost states in the NS sector we need $a_{NS} = 1/2$ and $D = 10$.

- R Sector: It should be noted that for the R sector we don't even have to use spurious states. Instead we can find the value of a_R as follows. From $F_n|\phi\rangle = 0$ we get that, see B.B.S. "String Theory and M-Theory" exercise 4.8,

$$\left(p \cdot \Gamma + \frac{2\sqrt{2}}{l_s}\sum_{n=1}^{\infty}(\alpha_{-n} \cdot d_n + d_{-n} \cdot \alpha_n)\right)|\phi\rangle = 0. \tag{11.76}$$

Now, comparing this with the fact that $L_0 = F_0^2$ we see that $a_R = 0$. Thus, in the R sector, we see that the absence of ghost states requires one to set $a_R = 0$.

In order to calculate the critical dimension consider the state in the R sector given by

$$|\psi\rangle = F_0 F_{-1}|\chi\rangle, \tag{11.77}$$

where $|\chi\rangle$ satisfies

$$F_1|\chi\rangle = (L_0 + 1)|\chi\rangle = 0. \tag{11.78}$$

Also, note that $|\psi\rangle$ satisfies $F_0|\psi\rangle = 0$. Now, since we want physical states with zero norm all we need to show is that $L_1|\psi\rangle = 0$ (WHY?). Therefore, we have

$$L_1|\psi\rangle = \left(\frac{1}{2}F_1 + F_0 F_1\right) F_{-1}|\chi\rangle = \frac{1}{4}\left(D - 10\right)|\chi\rangle,$$

and so if and only if $D = 10$ is $|\psi\rangle$ a zero-norm spurious state. So, the conditions for the removal of the ghost states in the R sector is that we must have $a_R = 0$ and $D = 10$.

11.6 Light-Cone Quantization

Recall from the bosonic theory[‡] that even after we fixed the gauge symmetry there was still a residual symmetry left over. This residual symmetry made it possible for us to further impose the light-cone gauge condition, which states

$$X^+(\tau, \sigma) = x^+ + p^+\tau. \tag{11.79}$$

Now, this is also true in the RNS superstring theory. However, we also have a residual fermionic symmetry, along with the residual bosonic symmetry (which allows for the light-cone gauge choice). This residual fermionic symmetry will allow us to impose more conditions on our RNS theory. Namely, using the fermionic residual symmetry we can set, in the NS sector,

$$\Psi^+(\tau, \sigma) = 0, \tag{11.80}$$

while in the R sector we have to keep the zero mode, in the Ψ^+ expansion, which is a Dirac matrix. Thus, in the RNS theory we can choose the two "light-cone gauges"

$$X^+(\tau, \sigma) = x^+ + p^+\tau \qquad \text{and} \qquad \Psi^+(\tau, \sigma) = 0,$$

of course with the appropriate RHS for Ψ^+. In the light-cone gauge the coordinates X^- and Ψ^-, due to the super-Virasoro constraints, are not independent degrees of freedom. This implies that in the light-cone gauge all the indepedent physical states are given by acting with the transverse raising modes of the bosonic and fermionic fields, just as was the case for the physical states in the bosonic theory.

[‡]In order to refresh the memories of light-cone quantization see 5.3.

11.6.1 Open RNS String Mass Spectrum

We will now analyize some open RNS superstring states in the light-cone gauge. Not that anyone could forget, but we have two sectors and the mass spectrum depends on which spectrum you are working in. Thus, we need to seperate our analysis into the NS sector and the R sector.

NS Sector

The mass formula for the NS sector is given by

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1/2}^{\infty} r b_{-r}^i b_r^i - \frac{1}{2}, \tag{11.81}$$

where we have substituted in $1/2$ for the value of a_{NS}.

- NS Sector Ground State: The NS ground state is annihilated by all positive mode oscillators,

$$\alpha_n^i |0; k^\mu\rangle_{NS} = b_r^i |0; k^\mu\rangle_{NS} = 0 \qquad (n, r > 0), \tag{11.82}$$

along with

$$\alpha_0^\mu |0; k^\mu\rangle_{NS} = \sqrt{2\alpha'}\, k^\mu |0; k^\mu\rangle_{NS}, \tag{11.83}$$

where the $\sqrt{2\alpha'}$ is from normalization. Calculating the mass of the NS ground state we get, by using (11.82), that

$$\alpha' M^2 |0; k^\mu\rangle_{NS} = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i |0; k^\mu\rangle_{NS} + \sum_{r=1/2}^{\infty} r b_{-r}^i b_r^i |0; k^\mu\rangle - \frac{1}{2}|0; k^\mu\rangle_{NS}$$

$$= -\frac{1}{2}|0; k^\mu\rangle_{NS},$$

or that, the NS ground state has a mass given by $\alpha' M^2 = -1/2$. As a result of this, we see that the ground state for the NS sector is a tachyon, which is bad. We will see later that there is a way to project out this state from the spectrum.

- NS Sector First Excited State: In order to construct the excited states we need to act on the ground state with one of the negative mode oscillator, either α_{-1}^i or $b_{-1/2}^i$. But which one do we choose? It turns out that we act with the oscillator having the smallest frequency[§], which is given by $b_{-1/2}^i$. So, the first excited state in the NS sector is

$$b_{-1/2}^i |0; k^\mu\rangle_{NS}. \tag{11.84}$$

[§]The operator b_{-r}^i raises the value of $\alpha' M^2$ by r units while α_{-n}^i raises it by n units where r and n are positive and so, this is why the first excited state is given by $b_{-1/2}^i$ since it raises the value by $1/2$ rather than α_{-1}^i which raises the value by 1.

Now, since this operator is a vector in spacetime and since it is acting on a bosonic ground state that is a spacetime scalar, the resulting state is a spacetime vector. Also, since we are working in the light-cone gauge we have that i labels the $D - 2 = 8$ transverse directions and so the first excited state has a total of 8 polarizations, which is required for a massless[¶] vector in ten dimensions. To see that this state is indeed massless as required note that, in general (i.e. without choosing the value of a_{NS}), the mass of the above state is given by $\alpha' M^2 = 1/2 - a_{NS}$. And so, if the first excited, vector, state is to be massless then we need $a_{NS} = 1/2$, which is what we have it at.

R Sector

In the light-cone gauge the mass formula for an open RNS superstring in the Ramond sector is given by

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n + \sum_{n=1}^{\infty} n d^i_{-n} d^i_n. \tag{11.85}$$

- R Sector Ground State: The R sector ground state satisfies

$$\alpha^i_n |0; k^\mu\rangle_R = d^i_n |0; k^\mu\rangle_R = 0 \qquad (n > 0), \tag{11.86}$$

along with

$$F_0 |0; k^\mu\rangle_R = 0, \tag{11.87}$$

which implies that

$$0 = \left(\alpha^i_0 d^i_0 + \sum_{n=1}^{\infty} (\alpha^i_{-n} d^i_n + d^i_{-n} \alpha^i_n) \right) |0; k^\mu\rangle_R,$$
$$= \Gamma_\mu k^\mu |0; k^\mu\rangle_k,$$

$$\equiv \slashed{k} |0; k^\mu\rangle_R, \tag{11.88}$$

which is the Dirac equation in the momentum representation. Note that we are not writing the spinor index here even though these states are spinors.

How many degrees of freedom does the ground state have? As was discussed earlier, the R sector ground state is not unique due to the fact that the zero modes satisfy a D dimensional, here we have seen that $D = 10$, Dirac algebra.

[¶] Recall that since our theory has a Lorentz invariant background spacetime and since the first excited state is a vector representation of $SO(8)$, we see that this state must be massless. For more discussion of this spectrum analysis of the free bosonic theory given previously.

This implies that the ground state is, in general, a $spin(9,1)$ spinor. The operation of multiplying with the operator d_0^μ is then nothing more than multiplying by a 10 dimensional Dirac matrix, i.e. a 32×32 matrix. This, in turn, implies that that the ground state in the R sector is a spinor with 32 components[§]. However, in 10 dimensions we can impose the Majorana reality conditions and the Weyl condition which then reduces the number of independent components by a factor of $1/2$, i.e. our ground state is now described by a spinor with 16 independent components, which we call a Majorana-Weyl spinor. Now, this 16 component Majorana-Weyl spinor has to satisfy the Dirac equation which reduces the number of independent components to 8. Thus, the R sector ground state has 8 degrees of freedom corresponding to an irreducible spinor of $Spin(8)$.

Note that, as was already mentioned, the states in the R sector correspond to fermions in the background spacetime. Also, since the first excited state of the NS sector is a bosonic state with 8 degrees of freedom and the ground state of the R sector is a fermionic state with 8 degrees of freedom, if we could shift the first excited state of the NS sector to become its ground state then we could think of the total open RNS string theory as having 8 massless bosons and 8 massless fermions in the background spacetime which gives us a hope for a supersymmetry on the background spacetime, not just on the worldsheet. However, we quickly realize that even if we could do this we are still screwed due to the NS ground state being a tachyon and there is nothing corresponding to this in the R sector. All is not lost because, as we will see, we can impose a further condition on our states, called the GSO condition (or GSO projection), which will remove the NS tachyon state and shift the first excited NS state as the NS ground state. And so, we will have that the NS ground state has 8 bosonic degrees of freedom while the R ground state has 8 fermionic degrees of freedom, which gives us a necessary, albiet not sufficient, condition for a spacetime supersymmetry. Note that after this tachyon has been projected out it can be shown that there is indeed a background spacetime supersymmetry, at least in $D = 10$, in the ground state and in all other excited states as well. This, in turn, leads one to believe that the RNS approach to superstring theories is equivalent to the GS approach as was mentioned at the beginning; which in fact is true in $D = 10$.

- R Sector Excited State: The excited states in the R sector are obtained by acting on the R sector ground state with d_{-n}^i or α_{-n}^i. Since these operators are spacetime vectors, the resulting states are also spacetime spinors. Note that the possibilities

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[§]This is of no suprise since a spinor in a $2k$ dimensional space has 2^k components. And so, for our case $k = 5$ which then implies that a spinor in this space has 32 components.

for excited states are further reduced by the GSO projection, which will now be discussed.

We now want to remove the tachyon plaging the RNS theory. This is what we will do now.

11.6.2 GSO Projection

The previous section described the spectrum of states of the RNS string that survives the super-Virasoro constraints. But it is important to realize that this spectrum has several problems. For one thing, in the NS sector the ground state is a tachyon, that is, a particle with imaginary mass. Also, the spectrum is not spacetime supersymmetric. For example, there is no fermion in the spectrum with the same mass as the tachyon. Unbroken supersymmetry is required for a consistent interacting theory, since the spectrum contains a massless gravitino, which is the quantum of the gauge feld for local supersymmetry. This inconsistency manifests itself in a variety of ways. It is analogous to coupling massless Yang-Mills felds to incomplete gauge multiplets, which leads to a breakdown of gauge invariance and causality. This subsection explains how to turn the RNS string theory into a consistent theory, by truncating (or projecting) the spectrum in a very specifc way that eliminates the tachyon and leads to a supersymmetric theory in ten dimensional spacetime. This projection is called the GSO projection, since it was introduced by Gliozzi, Scherk and Olive.

We begin the discussion of the GSO projection proceedure by defining an operator which counts the number of b oscillators in a NS state, which is given by

$$F_{NS} = \sum_{r=1/2}^{\infty} b^i_{-r} b^i_r, \tag{11.89}$$

along with the operator

$$F_R = \sum_{r=1}^{\infty} d^i_{-n} d^i_n, \tag{11.90}$$

which counts the number of d oscillators in a R state.

Now, we can use these two operators to consruct another operator, called the G-parity operator, which, in the NS sector, is given by

$$G = (-1)^{F_{NS}+1} = (-1)^{\sum_{r=1/2}^{\infty} b^i_{-r} b^i_r + 1}. \tag{11.91}$$

In the R sector we have that

$$G = \Gamma^{11}(-1)^{F_R} = \Gamma^{11}(-1)^{\sum_{r=1}^{\infty} d^i_{-n} d^i_n}, \tag{11.92}$$

where Γ^{11} is defined by

$$\Gamma^{11} = \Gamma^0 \Gamma^1 \cdots \Gamma^{10}, \qquad (11.93)$$

which can be thought of as the ten dimensional analog of the γ^5 Dirac matrix in four dimensions[‡]. Spinors Ψ^μ which satisfy

$$\Gamma_{11} \Psi^\mu = \Psi^\mu, \qquad (11.95)$$

are said to have positive chirality, while spinors which satisfy

$$\Gamma_{11} \Psi^\mu = -\Psi^\mu, \qquad (11.96)$$

are said to have negative chirality. Also, if a given spinor has a definite chirality then we say that the spinor is a Weyl spinor.

Now, in the NS sector, we will impose the GSO projection which consists of keeping only the states with a positive G-parity or, equivalently, the GSO projection which projects out all of the states with a negative G-parity, i.e. we keep only the states $|\Omega\rangle$ such that

$$G|\Omega\rangle = (-1)^{F_{NS}+1}|\Omega\rangle = |\Omega\rangle.$$

This implies that we have

$$1 = (-1)^{F_{NS}+1}$$

$$= (-1)^{F_{NS}}(-1),$$

and so the only way to have this satisfied is if F_{NS} is equal to an odd number. Thus, in the NS sector, we are only keeping those states with an odd number of b oscillator excitations, i.e. we are projecting out all of the states with an even number of b oscillators. In the R sector we can project states with either an even or odd number of d oscillator excitations depending on the chirality of the spinor ground state. Which ever one we pick is purely by convention. Also, by projecting out different states in both of the sectors, we can get different string theories with different particles and properties in spacetime. For instance, Type IIA and Type IIB superstring theories, which will be mentioned briefly in a minute, have different GSO projections.

To recap, we have just seen that the ground state, under the GSO condition, in the RNS theory has an equal number of (physical on-shell) bosonic and fermionic degrees

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[‡]Just as the γ^5 Dirac matrix in four dimensions, the Γ^{11} matrix satisfies the following properties

$$(\Gamma^{11})^2 = 1 \qquad \text{and} \qquad \{\Gamma^{11}, \Gamma^\mu\} = 0. \qquad (11.94)$$

of freedom and having an equal number of degrees of freedom is a sufficient, but not necessary, condition for the two ground states to form a supersymmetry multiplet. Also, it can be shown (see Becker, Becker and Schwarz "String Theory and M-Theory" exercise 4.11) that the GSO projection leaves an equal number of bosons and fermions at each mass level in the RNS theory, which gives a strong evidence that the RNS theory has a spacetime supersymmetry. Remarkably, as was already mentioned, it can actually be proven that this supersymmetry does in fact exist. As an aside, note that the two ground states form two inequivalent real eight dimensional represenations of $Spin(8)$.

We will now show that there are the same number of physical degrees of freedom in the NS and R sectors at the first massive level after performing the GSO projection. So, to begin, note that at this level we have $N = 3/2$ for the NS sector states and $N = 1$ for the R sector states. Also, the G-parity constraint in the NS sector requires the states to have an odd number of b oscillators, while in the R sector the constraint correlates the number of d oscillator excitations with the chirality of the spinor. Now, in the NS sector, the states which survive the GSO projection, at this level, are given by

$$\alpha^i_{-1} b^j_{-1/2} |0; k^\mu\rangle_{NS}, \qquad b^i_{-1/2} b^j_{-1/2} b^k_{-1/2} |0; k^\mu\rangle_{NS}, \qquad b^i_{-3/2} |0; k^\mu\rangle_{NS}.$$

Counting[‡] the number of these states gives us a total of $64 + 56 + 8 = 128$. Since these states are massive they must, by Lorentz symmetry, combine into $SO(9)$ representations. It can be shown that they give two $SO(9)$ representations $\mathbf{128} = \mathbf{44} \oplus \mathbf{88}$. For the R sector we have the allowed states (here suppressing the momentum label)

$$\alpha^i_{-1} |\phi_-\rangle, \qquad d^i_{-1} |\phi_+\rangle,$$

where $|\phi_-\rangle$ and $|\phi_+\rangle$ denote a pair of Majorana-Weyl spinors of opposite chirality, each with 16 real components. Note, however, that there are only 8 degrees of physical freedom since these spinors must satisfy the Dirac equation. Also, counting the amount of independent states in the R sector gives us $64 + 64 = 128$ states. Thus, for the first massive excited state we see that, after the GSO projection, both the NS sector and R sector agree in degrees of freedom as was claimed. These 128 fermionic states form an irreducible spinor representation $Spin(9)$. Finally, note that this massive supermultiplet

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[‡]For example, since $i = 1, 8$ there are a total of $8 \times 8 = 64$ independent states which are built out of

$$\alpha^i_{-1} b^j_{-1/2} |0; k^\mu\rangle_{NS},$$

while for the expression

$$b^i_{-1/2} b^j_{-1/2} b^k_{-1/2} |0; k^\mu\rangle_{NS},$$

we get (HOW?) 54 independent states. Finally, for $b^i_{-3/2} |0; k^\mu\rangle_{NS}$ there are 8 independent states.

in ten dimensions, consisting of 128 bosons and 128 fermions, is identical to the massless supergravity multiplet in 11 dimensions.

At this point the GSO projection should seem as an *ad hoc* condition, but it is actually essential for a consistent theory. It is possible to derive this by demanding one-loop and two-loop modular invariance. A much simpler argument is to note that it leaves a supersymmetric spectrum. As has already been emphasized, the closed-string spectrum contains a massless gravitino (or two) and therefore the interacting theory wouldn't be consistent without supersymmetry. In particular, this requires an equal number of physical bosonic and fermionic modes at each mass level, which is satisfied by the GSO condition.

Let us now turn to the massless closed RNS superstring spectrum.

11.6.3 Closed RNS String Spectrum

As we have encountered before, the closed string has left-movers and right-movers. Also, there is the possibility of each mover either having R or NS boundary conditions. This tells us that in order to analyze the closed RNS string spectrum, we must consider the four possible sectors: R-R, R-NS, NS-R and NS-NS. As before, by projecting onto states with a positive G-parity in the NS sector, we can remove the tachyon state. For the R sector we can project onto states with positive or negative G-parity depending on the chirality of the ground state on which the states are built. Thus two different theories, the type IIA and type IIB superstring theories, can be obtained depending on whether the G-parity of the left- and right-moving R sectors is the same or opposite.

In the type IIB theory the left- and right-moving R-sector ground states have the same chirality, chosen to be positive for definiteness. Therefore, the two R sectors have the same G-parity. Let us denote each of them by $|+\rangle_R$. With these considerations the massless states in the type IIB closed string spectrum are (here also supressing the momentum label)

$$|+\rangle_R \otimes |+\rangle_R, \tag{11.97}$$

$$\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes b^i_{-1/2}|0\rangle_{NS}, \tag{11.98}$$

$$\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes |+\rangle_R, \tag{11.99}$$

$$|+\rangle_R \otimes b^i_{-1/2}|0\rangle_{NS}. \tag{11.100}$$

Now, since the state $|+\rangle_R$ is a spinor with eight components, we see that each of the four sectors, in the type IIB theory, contain $8 \times 8 = 64$ states.

In the type IIA theory the left- and right-moving ground states have opposite chirality, which we label as $|+\rangle_R$ and $|-\rangle_R$. The massless states in the type IIA closed string spectrum are given by

$$|-\rangle_R \otimes |+\rangle_R, \tag{11.101}$$

$$\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes b^i_{-1/2}|0\rangle_{NS}, \tag{11.102}$$

$$\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes |+\rangle_R, \tag{11.103}$$

$$|-\rangle_R \otimes b^i_{-1/2}|0\rangle_{NS}. \tag{11.104}$$

With similar arguments as the type IIB case, we see that each of the four sectors of the type IIA theory have 64 states. As an aside, in both type II theories, one has that the massless spectrum contains two Majorana-Weyl gravitinos. Thus, they form $\mathcal{N} = 2$ supergravity multiplets.

The different types of states in the massless sectors of the two theories are summarized as[¶]:

- R-R Sector: These states are bosons obtained by tensoring a pair of Majorana-Weyl spinors. In the IIA case, the two Majorana-Weyl spinors have opposite chirality, and one obtains a one-form (vector) gauge feld (eight states) and a three-form gauge feld (56 states). In the IIB case the two Majorana-Weyl spinors have the same chirality, and one obtains a zero-form (that is, scalar) gauge field (one state), a two-form gauge field (28 states) and a four-form gauge

  eld with a self-dual field strength (35 states).

- NS-NS Sector: This sector is the same for the type IIA and type IIB cases. The spectrum contains a scalar called the dilaton (one state), an antisymmetric two-form gauge field (28 states) called the Kalb-Ramond field and a symmetric traceless rank-two tensor, the graviton (35 states).

- NS-R and R-NS Sectors: Each of these sectors contains a spin 3/2 gravitino (56 states) and a spin 1/2 fermion called the dilatino (eight states). In the IIB case the two gravitinos have the same chirality, whereas in the type IIA case they have opposite chirality.

[¶]See problem 11.2 for further discussion of the predicted states of the closed string RNS theory.

So to recap, we began by imposing a worldsheet supersymmetry and we saw that this induced, in $D = 10$, a supersymmetry on the background spacetime (although we did not prove this, we just showed that there were sufficient conditions), as well as being able to define a theory with fermions and no tachyons, the RNS superstring theory.

One should note that this is not the only approach to superstring theories, there is the Green-Schwarz (GS) approach. As was mentioned earlier, at least in $D = 10$, the RNS superstring theory is equivalent to the GS superstring theory, which has manifest supersymmetry in the background spacetime. Other advantages of the GS theory, as compared to the RNS theory, are that the bosonic and fermionic strings are unified in a single Fock space as well as the GSO projection is automatically built into the formalism without having to truncate. However, the GS theory does have disadvantages, it is very difficult to quantize the worldsheet action in a way that maintains spacetime Lorentz invariance as a manifest symmetry. Although, it can be quantized in the light-cone gauge, which is sufficient for analyzing the physical spectrum it predicts, along with begin sufficient for studying tree level and one-loop amplitudes.

In the next chapter we will study D-branes and their physics along with T dualities. We begin the chapter by compactifying the $X^25(\sigma, \tau)$ spatial dimension. Then we will see that T duality allows for us to relate a theory with compactified extra dimension of radius R to one with compactified extra dimension of radius α'/R. We will then see how D-branes arise out of this T duality condition.

11.7 Exercises

Problem 1

The worldsheet fields in the RNS formalism consist of D target space coordinates X^μ and D worldsheet fermions ψ^μ, $\mu = 0, 1, \ldots, D-1$. The coordinates X^μ have exactly the same description as in the bosonic string. The OPE of the fermions is given by

$$\psi^\mu(z)\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{(z-w)} \tag{11.105}$$

where $\eta^{\mu\nu}$ is the ten dimensional Minkowski metric, and the corresponding energy-momentum tensor is

$$T_\psi = -\frac{1}{2} : \psi^\mu \partial \psi_\mu : \tag{11.106}$$

In BRST quantization one introduces in addition two pairs of ghost fields, the anti-commuting (b, c) fields with OPE

$$c(z)b(w) \sim \frac{1}{z-w}, \qquad c(z)c(w) \sim regular, \qquad b(z)b(w) \sim regular \tag{11.107}$$

and the commuting (β, γ) fields with OPE

$$\gamma(z)\beta(w) \sim \frac{1}{z-w}, \qquad \gamma(z)\gamma(w) \sim regular, \qquad \beta(z)\beta(w) \sim regular \tag{11.108}$$

The energy-momentum tensor of the ghost fields is given by

$$T_{gh} = -2 : b\partial c : + : c\partial b : -\frac{3}{2} : \beta\partial\gamma : -\frac{1}{2} : \gamma\partial\beta : \tag{11.109}$$

The total energy-momentum tensor T is the sum of these contributions (plus the energy-momentum tensor for X). In this problem we will ingore the antiholomorphic part.

(i) Compute the OPE's of T with $\psi^\mu, b, c, \beta, \gamma$. What are the conformal weights of these fields?

(ii) Compute the central charge, $c_\psi, c_{bc}, c_{\beta\gamma}$, due to ψ^μ, (b, c) and (β, γ), respectively, by computing the leading $(z-w)^{-4}$ singularity in the OPE of the energy-momentum tensor with itself.

(iii) Compute the critical dimension by requiring that the total central charge vanishes.

Problem 2

In this problem we want to work out the massless spectrum of IIA and IIB superstring theory and to verify that it has equal numbers of bosonic and fermionic degrees of freedom. As described in this chapter, the spectrum contains 4 sectors obtained by taking the tensor product of the NS and the R sectors of the left and the right movers.

In the NS sector, the ground state is a vector of $SO(1,9)$ and in the R sector it is either a positive or negative chirality spinor $\psi_\pm$ of $SO(1,9)$. Let us describe the spinors in more detail. Let

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad \mu = 0, \ldots, 9 \tag{11.110}$$

be the Clifford algebra of $SO(1,9)$ gamma matrices. The gamma matrices have the following hermiticity property,

$$(\gamma^\mu)^\dagger = -\gamma^0 \gamma^\mu (\gamma^0)^{-1}. \tag{11.111}$$

1. Verify that the matrix

$$\gamma^{11} = \gamma^0 \gamma^1 \cdots \gamma^9 \tag{11.112}$$

satisfies

$$(\gamma^{11})^2 = 1, \qquad \{\gamma^{11}, \gamma^\mu\} = 0. \tag{11.113}$$

Chiral spinors are now defined by

$$\gamma^{11} \psi_\pm = \pm \psi_\pm, \tag{11.114}$$

Show that

$$\bar\psi_\pm \gamma^{11} = \mp \bar\psi_\pm \tag{11.115}$$

where $\bar\psi_\pm = \psi_\pm^\dagger i \gamma^0$.

The IIB theory is chiral, having a positive chirality spinor both for the left and the right movers, while the IIA theory has a **negative** chirality spinor for the **left** movers and a **positive** chirality spinor for the **right** movers.

2. To count degrees of freedom in D dimensions it is useful to go to the lightcone gauge, so the vector indices, which are now denoted by i, take $(D-2)$ values.

   (i) The gravitational degrees of freedom are described by a transverse traceless tensor. How many independent components are in such a tensor?

   (ii) How many independent components are in an antisymmetric tensor $C_{i_1 \cdots i_d}$? Note that these are the components of a d-form, $\mathbf{C}_d = C_{i_1 \cdots i_d} dx^{i_1} \wedge \cdots \wedge dx^{i_d}$.

   (iii) The counting of degrees of freedom for fermions depends crucially on the spacetime dimension as the existence of Majorana, Weyl, Majorana-Weyl representations is dimension dependent, so we now restrict to $D = 10$. As we have seen in this chapter, in $D = 10$ a Majorana-Weyl spinor ψ_α has 8 independent real components after imposing the Dirac equation. The gravitino degrees of freedom are described by a Majorana-Weyl spinor field

with an additional vector index, $\hat{\psi}_\alpha^i$, which is gamma traceless, $(\gamma_i\hat{\psi}^i)_\alpha = 0$. It follows that the degrees of freedom are those of an unconstrained field ψ_α^i minus the degrees of freedom in $(\gamma_i\psi^i)_\alpha$. How many independent components are in the gravitino?

3. We now want to express the massless spectrum of IIA and IIB string theory in terms of gravitons, antisymmetric tensors, scalars, spinors and gravitini. This is done by decomposing the tensor product of the left and right movers into Lorentz representations.

   In the NS-NS sector we have the tensor product of two vectors. This decomposes into a symmetric traceless tensor (graviton), an anti-symmetric 2-tensor (the Kalb-Ramond field) and the trace part (dilaton), as in bosonic string theory.

   In the NS-R sector we have the tensor product of a vector with a spinor, ψ_α^i. This decomposes into a gamma-traceless field, $\hat{\psi}_\alpha^i$, (gravitino) and a spinor (dilatino), the gamma trace part $(\gamma_i\psi^i)_\alpha$ of ψ_α^i. The R-NS sector is similar.

   The most non-trivial part is to describe the R-R sector in terms of antisymmetric tensors. In the R-R sector we have the tensor product of two spinors, so we need to convert bi-spinors into antisymmetric tensors. This is done as follows. We identify the field strengths $F_{\mu_1\cdots\mu_{d+1}}$ with spinor bilinears,

   $$IIA: \quad F^{\mu_1\cdots\mu_{d+1}} = \bar{\psi}_-^L\gamma^{\mu_1\cdots\mu_{d+1}}\psi_+^R, \qquad IIB: \quad F^{\mu_1\cdots\mu_{d+1}} = \bar{\psi}_+^L\gamma^{\mu_1\cdots\mu_{d+1}}\psi_+^R, \tag{11.116}$$

   where $\bar{\psi}_\pm^L$ come from the left movers and $\psi_\pm^R$ come from the right movers and

   $$\gamma^{\mu_1\cdots\mu_{d+1}} = \gamma^{[\mu_1}\cdots\gamma^{\mu_{d+1}]} \tag{11.117}$$

   is the antisymmetric product of $(d+1)$ gamma matrices.

   (i) Use the chirality of the spinors in (11.114) to determine for which values of d the field strengths in (11.116) are non-zero.

   (ii) Note that the field strengths $F_{\mu_1\cdots\mu_{d+1}}$ are the components of $(d+1)$-forms $\mathbf{F}_{d+1} = \mathbf{C}_d$, with $\mathbf{C}_d$ the d-form RR potentials. Use (i) to compute the number of degrees of freedom in the RR sector. Note that because of special properties of gamma matrices, $\mathbf{F}_{d+1}$ is related with $\mathbf{F}_{9-d}$, so one should only count the cases $d \leq 4$. The $d = 4$ case is special because $d+1 = 9-d = 5$, so in this case one must divide the degrees of freedom by 2.

(iii) List the massless spectrum of IIA and IIB sting theory and verify that the bosonic and fermionic degrees of freedom are equal. What is the chirality of the fermionic fields?

12. T-Dualities and Dp-Branes

String theory is not only a theory of fundamental one-dimensional strings, but there are also a variety of other objects, called branes, of various dimensionalities. The list of possible branes, and their stability properties, depends on the specific theory and its vacuum configuration under consideration. The defining property of a brane is that they are the objects on which open strings can end. A string that does not land on a brane must be a closed loop.

One way of motivating the necessity of branes in string theory is based on T-duality, so this chapter starts with a discussion of T-duality in the bosonic string theory. We will see that, under T-duality transformations, closed bosonic strings transform into closed strings of the same type in the T-dual geometry. However, the situation is different for open strings. The key is to focus on the type of boundary conditions imposed at the ends of the open strings. We will see that even though the only open-string boundary conditions that are compatible with Poincaré invariance (in all directions) are of Neumann type, Dirichlet boundary conditions inevitably appear in the equivalent T-dual reformulation. Open strings with Dirichlet boundary conditions in certain directions have ends with specified positions in those directions, which means that they have to end on specified hypersurfaces. Although this violates Lorentz invariance, there is a good physical reason for them to end in this manner. The reason this is sensible is that they are ending on other physical objects that are also part of the theory, which are called Dp-branes. The letter D stands for Dirichlet, and p denotes the number of SPATIAL dimensions of the Dp-brane. Thus, a $D3$-brane is a four dimensional spacetime object living in the 26 dimensional background spacetime which, for example, could be specified by $\{X^0(\tau,\sigma), X^5(\tau,\sigma), X^{24}(\tau,\sigma), X^{25}(\tau,\sigma)\}$, while a $D4$-brane is a five dimensional spacetime object living in the background spacetime which could be specified by $\{X^0(\tau,\sigma), X^3(\tau,\sigma), X^8(\tau,\sigma), X^9(\tau,\sigma), X^{17}(\tau,\sigma)\}$.

Much of the importance of Dp-branes stems from the fact that they provide a remarkable way of introducing nonabelian gauge symmetries in string theory; we will see that nonabelian gauge fields naturally appear confined to the world volume of multiple coincident Dp-branes. Moreover, Dp-branes are useful for discovering dualities that relate apparently different string theories to one another.

12.1 T-Duality and Closed Bosonic Strings

Consider the bosonic string theory with one of its spatial dimensions compactified, i.e. we assume that this spatial dimension has periodic boundary conditions or, equivalently, we assume that the spatial dimension has the topology of a circle of some radius R. This implies that our background spacetime is topologically equivalent to the space

given by the Cartesian product of a 25-dimensional Minkowski spacetime and a circle of radius R,

$$\mathbb{R}^{24,1} \times S_R^1. \tag{12.1}$$

This proceedure is called compactifying on a circle of radius R. We will choose to compactify the $X^{25}(\tau, \sigma)$ coordinate. Pictorially, we can think of our background spacetime as given in figure 11. We will now investigate the changes in our bosonic string theory due to this compactification.

Previously, in the noncompact theory, a closed string was constrained by the following periodic boundary condition

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \pi). \tag{12.2}$$

Figure 11: An example of a compactified background spacetime where we take the X^{25} to be periodic (or compactified) and homeomorphic to a circle, S_R^1, of radius R.

This boundary condition was stated with the implicit assumption where the string was moving in a space-time with noncompact dimensions, but now our situation has changed due to the compactification of the $X^{25}(\tau, \sigma)$ coordinate. For instance, with the compactified direction, we can get winding numbers[‡] if the closed string wraps around the compactified direction. It is clear that we need to modify the above boundary conditions in order to take into account this new phenomena of winding numbers. Thus, the new boundary conditions for the closed string are given by changing the boundary conditions for $X^{25}(\tau, \sigma)$ to

$$X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma) + 2\pi R W \qquad (W \in \mathbb{Z}), \tag{12.3}$$

where W is the winding number of the string, while leaving the other $X^i(\tau, \sigma)$, for $i = 0, ..., 24$, as

$$X^i(\tau, \sigma + \pi) = X^i(\tau, \sigma). \tag{12.4}$$

So, the bondary conditions for the closed string, compactified on a circle of radius R, are given by

$$X^i(\tau, \sigma + \pi) = X^i(\tau, \sigma) \qquad (i = 0, ..., 24),$$

$$X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma) + 2\pi R W \qquad (W \in \mathbb{Z}).$$

12.1.1 Mode Expansion for the Compactified Dimension

The modified boundary conditions for $X^{25}(\tau, \sigma)$ lead to a modified mode expansion for $X^{25}(\tau, \sigma)$, while the mode expansions for the $X^i(\tau, \sigma)$ $(i = 0, ..., 24)$ fields remain

[‡]The winding number indicates how many times a closed string winds around the circle. For each positive oriented winding, which we take to be counter-clockwise, the winding number increases by 1, while for each clockwise winding the winding number decreases by 1.
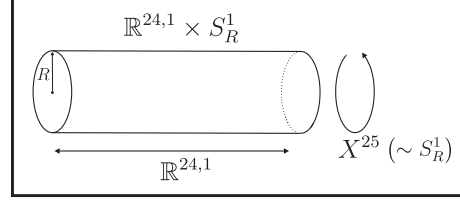
unchanged. The new mode expansion for the compactified dimension, $X^{25}(\tau, \sigma)$, is given by adding a term linear in σ in order to incorporate the boundary condition given by (12.3). In particular,

$$X^{25}(\tau, \sigma) = x^{25} + 2\alpha' p^{25}\tau + 2RW\sigma + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^{25}e^{2in\sigma} + \tilde{\alpha}_n^{25}e^{-2in\sigma}\right)e^{-2in\tau}, \quad (12.5)$$

where σ is chosen to satisfy (12.3). Also, note that since one dimension is compact, the momentum eigenvalue along that direction, p^{25}, has to be quantized since, from quantum mechanics, we see that the wave function contains the factor $\exp(ip^{25}x^{25})$. As a result, if we increase x^{25} by the amount $2\pi R$, which corresponds to a winding number $W = 1$ (i.e. going around the circle once), the wave function should be mapped back to the initial value, i.e. the wave function should be single-valued on the circle. This implies that the momentum in the 25 direction has to be of the form

$$p^{25} = \frac{K}{R}, \quad (12.6)$$

where K is called the Kaluza-Klein excitation number. Thus, without the compactified dimension the center of mass momentum of the string is continuous, while compactifying one of the dimensions quantizes the center of mass momentum along that direction.

Now, we can split the above expansion for $X^{25}(\tau, \sigma)$ into left- and right-movers. This gives us

$$X^{25}(\tau, \sigma) = X_L^{25}(\tau + \sigma) + X_R^{25}(\tau - \sigma), \quad (12.7)$$

with

$$X_L^{25}(\tau + \sigma) = \frac{1}{2}(x^{25} + \tilde{x}^{25}) + \left(\alpha'\frac{K}{R} + WR\right)(\tau + \sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^{25}e^{-2in(\tau+\sigma)},$$

$$X_R^{25}(\tau - \sigma) = \frac{1}{2}(x^{25} - \tilde{x}^{25}) + \left(\alpha'\frac{K}{R} - WR\right)(\tau - \sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-2in(\tau-\sigma)},$$

where $\tilde{x}^{25}$ is some constant which cancels in the sum to form $X^{25}(\tau, \sigma)$. We can further rewrite the expressions for the left- and right-movers, in terms of the zero modes

$$\sqrt{2\alpha'}\tilde{\alpha}_0^{25} = \left(\alpha'\frac{K}{R} + WR\right), \quad (12.8)$$

$$\sqrt{2\alpha'}\alpha_0^{25} = \left(\alpha'\frac{K}{R} - WR\right), \quad (12.9)$$

as

$$X_L^{25}(\tau + \sigma) = \frac{1}{2}(x^{25} + \tilde{x}^{25}) + \sqrt{2\alpha'}\tilde{\alpha}_0^{25}(\tau + \sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^{25}e^{-2in(\tau+\sigma)} \quad (12.10)$$

$$X_R^{25}(\tau - \sigma) = \frac{1}{2}(x^{25} - \tilde{x}^{25}) + \sqrt{2\alpha'}\alpha_0^{25}(\tau - \sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-2in(\tau-\sigma)} \quad (12.11)$$

Note, in general we have that $\alpha_0^{25} \neq \tilde{\alpha}_0^{25}$.

12.1.2 Mass Formula

The mass formula for the string with one dimension compactified on a circle can be interpreted from a 25-dimensional viewpoint in which one regards each of the Kaluza-Klein excitations, which are given by K, as distinct particles. In general, the mass formula is given by

$$M^2 = \sum_{\mu=0}^{24} p_\mu p^\mu. \quad (12.12)$$

Note that we only perform the sum over the non-compact dimensions.

Now, we still have the requirement that all on-shell physical states are annihilated by the operators, $L_0 - 1$ and $\tilde{L}_0 - 1$, where by 1 we really mean the number 1 times the identity operator[‡], which we leave out for simplicity. Also, note that L_0 and $\tilde{L}_0$ do include contributions from the compactified dimension. As a result, the equations $L_0 = 1$ and $\tilde{L}_0 = 1$ become

$$\frac{1}{2}\alpha' M^2 = \left(\tilde{\alpha}_0^{25}\right)^2 + 2N_L - 2 = \left(\alpha_0^{25}\right)^2 + 2N_R - 2, \quad (12.13)$$

where N_L and N_R are the levels for the left- and right-movers, respectively. Taking the sum and difference of the above two expressions for $\alpha' M^2$ and using (12.8) and (12.9) gives, for the modified level matching condition,

$$N_R - N_L = WK, \quad (12.14)$$

while, for the mass formula of a string with one compactified dimension, we get

$$\alpha' M^2 = \alpha'\left[\left(\frac{K}{R}\right)^2 + \left(\frac{WR}{\alpha'}\right)^2\right] + 2N_L + 2N_R - 4. \quad (12.15)$$

One should note that previously, in the non-compact theory, we had that the level matching condition was given by $N_R = N_L$. Thus, compactifying a spatial dimension leads to the modified level matching condition given by (12.14).

12.1.3 T-Duality of the Bosonic String

[‡]Recall that, for the bosonic string theory, we saw that inorder to remove the ghost states from our spectrum one of the requirements was that $a = 1$. Thus, the general constraint satisfied by physical states, namely

$$(L_0 - a)|\phi\rangle = 0,$$

becomes $(L_0 - 1)|\phi\rangle = 0$.

It can be shown that the level matching condition, (12.14), and the mass formula, (12.15), are invariant under sending the radius of the compactified dimension, R, to $\tilde{R} = \alpha'/R$ as long as we interchange W and K. This symmetry of the bosonic string theory is called T-duality. It tells us that compactification on a circle of radius R has the same mass spectrum as a theory which is compactified on a circle of radius $\tilde{R} = \alpha'/R$, see figure 12.

In the example considered here, T-duality maps two theories of the same type, one with a compactified dimension of radius R, and one with the same dimension compactified but now with radius $\tilde{R}$, into one another. Also, note that the interchange of W and K implies that the momentum excitations, K, in one description correspond to winding-mode excitations in the dual description, and vice versa.

string wrapped around X^{25} with radius R

string wrapped around X^{25} with radius $\tilde{R}$

Figure 12:

The T-duality transformation,

$$T: \ \mathbb{R}^{24,1} \times S_R^1 \longleftrightarrow \mathbb{R}^{24,1} \times S_{\tilde{R}}^1 \ ,$$

$$T: \ W \longleftrightarrow K \ ,$$

has the following action on the modes in the expansion of the compactified dimension,

$$\alpha_0^{25} \mapsto -\alpha_0^{25}, \tag{12.16}$$

while

$$\tilde{\alpha}_0^{25} \mapsto \tilde{\alpha}_0^{25}, \tag{12.17}$$

which can be seen from (12.8) and (12.9). In fact, it is not just the zero mode, but the entire right-moving part of the compact coordinate that flips sign under the T-duality transformation

$$T: X_R^{25} \mapsto -X_R^{25} \qquad \text{and} \qquad T: X_L^{25} \mapsto X_L^{25} \ . \tag{12.18}$$

Thus, we have that $X^{25}(\tau, \sigma)$ is mapped to, under the T-duality transformation,

$$T: X^{25}(\tau, \sigma) \mapsto \tilde{X}^{25}(\tau, \sigma) = X_L^{25}(\tau + \sigma) - X_R^{25}(\tau - \sigma), \tag{12.19}$$

which has an expansion

$$\tilde{X}^{25}(\tau, \sigma) = \tilde{x}^{25} + 2\alpha' \frac{K}{R} \sigma + 2RW + \frac{i}{2} \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \left(\tilde{\alpha}_n^{25} e^{-2in\sigma} - \alpha_n^{25} e^{2in\sigma} \right) e^{-2in\tau}. \tag{12.20}$$
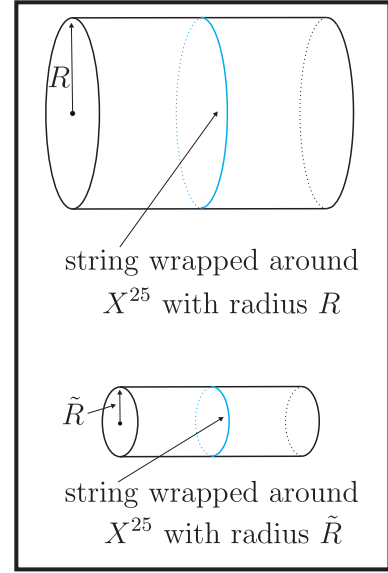
One should note that the coordinate x^{25}, which parameterizes the original circle with periodicity $2\pi R$, has been replaced by a coordinate $\tilde{x}^{25}$. It is clear that this parameterizes the dual circle with periodicity $2\pi\tilde{R}$, because its conjugate momentum is $\tilde{p}^{25} = RW/\alpha' = W/\tilde{R}$.

We will now see that T-duality interchanges $X^{25}(\tau, \sigma)$ and $\tilde{X}^{25}(\tau, \sigma)$ from the viewpoint of the worldsheet. To begin, consider the following worldsheet action[‡]

$$S = \int d\tau d\sigma \left(\frac{1}{2} V^\alpha V_\alpha - \epsilon^{\alpha\beta} X^{25}(\tau, \sigma) \partial_\beta V_\alpha \right). \tag{12.21}$$

Varying the action, S, with respect to X^{25} gives us

$$\epsilon^{\alpha\beta} \partial_\beta V_\alpha = 0,$$

$$\Rightarrow \partial_1 V_2 - \partial_2 V_1 = 0,$$

for the field equation, whose solution is given by $V_\alpha = \partial_\alpha \tilde{X}^{25}$, where $\tilde{X}^{25}$ is an arbitrary function. Alternatively, varying S with respect to V_α gives the equation of motion

$$V_\alpha = -\epsilon_\alpha{}^\beta \partial_\beta X^{25}.$$

Now, comparing the two expressions for V_α gives us

$$\partial_\alpha \tilde{X}^{25} = -\epsilon_\alpha{}^\beta \partial_\beta X^{25},$$

which implies that[¶]

$$\partial_+ \tilde{X}^{25} = \partial_+ X^{25},$$

$$\partial_- \tilde{X}^{25} = -\partial_- X^{25}.$$

The first equation implies that

$$\tilde{X}_L^{25} = X_L^{25},$$

while the second equation tells us that

$$\tilde{X}_R^{25} = -X_R^{25}.$$

[‡]Here we will ingnore the overall constant in front of the integral. This is because we are only going to treat this action as a classical quantity and so the overall constants do not matter.

[¶]Remember that you have to lower (raise) the β (α) index on $\epsilon_\alpha{}^\beta$ before you can use the antisymmetric properties of ϵ since the properties are defined with both indices lowered (raised).

And so, we see that, under a T-duality transformation,

$$X_R^{25} \mapsto -X_R^{25} \qquad \text{and} \qquad X_L^{25} \mapsto X_L^{25},$$

as was stated previously. Furthermore, if we plug back into S, (12.21), the equation of motion for V_α, namely $V_\alpha = -\epsilon_\alpha{}^\beta \partial_\beta X^{25}$, while noting that $\epsilon^{\alpha\beta}\epsilon_\alpha{}^\gamma = -\eta^{\beta\gamma}$, we get

$$\frac{1}{2}\int d\tau d\sigma \ \partial^\alpha X^{25}\partial_\alpha X^{25},$$

which is the original Polyakov action for the X^{25} component.

We will now repeat the previous arguments for the case of open strings.

12.2 T-Duality and Open Strings

Previously, we have seen, see (3.15), that when we vary the Polyakov action, written here in conformal gauge

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu, \tag{12.22}$$

we get a bulk term which upon vanishing gives the equations of motion and the boundary term

$$\delta S = \left[-\frac{1}{2\pi\alpha'}\int d\tau \ \partial_\sigma X_\mu \delta X^\mu\right]_{\sigma=0}^{\sigma=\pi}. \tag{12.23}$$

Now, as before, we need for this boundary term to vanish. This is achieved by forcing the ends of the open string to obey either Neumann boundary conditions,

$$\frac{\partial}{\partial\sigma}X^\mu(\tau,\sigma) = 0 \qquad (\sigma = 0, \pi), \tag{12.24}$$

or Dirichlet boundary conditions,

$$X(\tau,\sigma) = \underbrace{X_0(\tau,\sigma)}_{\text{constant}} \qquad (\sigma = 0, \pi). \tag{12.25}$$

However, only the Neumann boundary conditions preserve the Poincaré invariance for all 26 dimensions. Thus, we will assume that our open string theory has Neumann boundary conditions.

Now, we will assume that we compactify, as before, along the $X^{25}(\tau,\sigma)$ coordinate. Then, we want to perform a T-duality transformation on this open string theory in the X^{25} direction. But before we get into the quantitative properties which arise from the transformation, lets see what kind of qualitative changes we will get. The first thing to

notice is that when we apply the T-duality transformation on the bosonic string which has been compactified on a circle we see that the winding number is meaningless. This is due to the fact that an open string is homotopy equivalent to a point, which has $W = 0$. Since the winding modes were crucial to relate the closed-string spectra of two bosonic theories using T-duality, we then expect to not have open strings transform in this way. But how do they transform under the T-duality transformation?

In order to see how the open string transforms under a T-duality transformation we first start with the mode expansion for the coordinates $X^\mu(\tau, \sigma)$, which are given by, in the Neumann boundary conditions,

$$X^\mu(\tau, \sigma) = x^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma), \qquad (12.26)$$

where we have set $l_s = 1$ or, equivalently, $\alpha' = 1/2$. We can further split the mode expansions into two parts, the left- and right-moving pieces, which are given by

$$X_R^\mu(\tau - \sigma) = \frac{1}{2}(x^\mu - \tilde{x}^\mu) + \frac{1}{2}p^\mu(\tau - \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in(\tau - \sigma)}, \qquad (12.27)$$

$$X_L^\mu(\tau + \sigma) = \frac{1}{2}(x^\mu + \tilde{x}^\mu) + \frac{1}{2}p^\mu(\tau + \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in(\tau + \sigma)}. \qquad (12.28)$$

Now, when we compactify the $X^{25}(\tau, \sigma)$ coordinate and apply the T-duality transformation in this direction we get, as was shown already, that

$$T : X_L^{25} \mapsto X_L^{25},$$

$$T : X_R^{25} \mapsto -X_R^{25}.$$

So, under the T-duality map we get for the X^{25} coordinate

$$T : X^{25} \mapsto X_L^{25} - X_R^{25} (\equiv \tilde{X}^{25})$$

$$= \frac{1}{2}(x^{25} + \tilde{x}^{25}) + \frac{1}{2}p^{25}(\tau + \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^{25} e^{-in(\tau + \sigma)}$$

$$- \left(\frac{1}{2}(x^{25} - \tilde{x}^{25}) + \frac{1}{2}p^{25}(\tau - \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^{25} e^{-in(\tau - \sigma)} \right)$$

$$= \tilde{x}^{25} + p^{25}\sigma + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^{25} e^{-in\tau} \left(e^{-in\sigma} - e^{in\sigma} \right)$$

$$= \tilde{x}^{25} + p^{25}\sigma + \sum_{n \neq 0} \frac{1}{n}\alpha_n^{25}e^{-in\tau}\frac{i}{2}\left(e^{-in\sigma} - e^{in\sigma}\right)$$

$$= \tilde{x}^{25} + p^{25}\sigma + \sum_{n \neq 0} \frac{1}{n}\alpha_n^{25}e^{-in\tau}\frac{1}{2i}\left(-e^{-in\sigma} + e^{in\sigma}\right)$$

$$= \tilde{x}^{25} + p^{25}\sigma + \sum_{n \neq 0} \frac{1}{n}\alpha_n^{25}e^{-in\tau}\sin(n\sigma).$$

Thus, under a T-duality transformation, we have that

$$T : X^{25} \mapsto \tilde{X}^{25} = \tilde{x}^{25} + p^{25}\sigma + \sum_{n \neq 0} \frac{1}{n}\alpha_n^{25}e^{-in\tau}\sin(n\sigma). \tag{12.29}$$

Ok, so what do we see for the properties of X^{25} under a T-duality from (12.29)? First, note that since, in (12.29), there is no linear term in τ we see that the T-dual open string has no momentum in the X^{25} direction. This implies that the open string can only oscillate in this direction. Also, from (12.29), we see that the T-dual open string has fixed endpoints at $\sigma = 0$ and $\sigma = \pi$ since when we plug in these values for σ the oscillator term vanishes. But wait, this is equivalent to the Dirichlet boundary conditions for an open string. Thus, we see that the T-duality transformation maps the X^{25} coordinate, of the open bosonic string with Neumann boundary conditions to Dirichlet boundar conditions, and vice versa. Explicitly, the boundary conditions of the X^{25} coordinate of the T-dual string are given by

$$\tilde{X}^{25}(\tau, 0) = \tilde{x}^{25} \qquad \text{and} \qquad \tilde{X}^{25}(\tau, \pi) = \tilde{x}^{25} + \frac{\pi K}{R} = \tilde{x}^{25} + 2\pi K\tilde{R}, \tag{12.30}$$

where we have used $p^{25} = K/R$ and $\tilde{R} = \alpha'/R = 1/(2R)$ for the dual radius. Also, observe that this string wraps around the dual circle K times. This winding mode is topologically stable, i.e. not nessecarily homotopic to a point, since the endpoints of the string are fixed by the Dirichlet boundary conditions which implies that the sring cannot unwind without breaking.

So, we have seen that T-duality has transformed a bosonic open string with Neumann boundary conditions on a circle of radius R to a bosonic open string with Dirichlet boundary conditions on a circle of radius $\tilde{R}$. We started with a string that had momentum and no winding in the circular direction and ended up with a string that has winding but no momentum in the dual circular direction. The ends of the dual open string are attached to the hyperplane $\tilde{X}^{25} = \tilde{x}^{25}$ and they can wrap around the circle an integer number of times. This hyperplane is an example of a Dp-brane, here D is for Dirichlet while p specifies the number of spatial dimensions of the hyperplane. In

general, a Dp-brane is defined as a hypersurface on which an open string can end. One should note that Dp-branes are also physical objects, i.e. they have their own dynamics. In the example above, the hypersurface $\tilde{X}^{25} = \tilde{x}^{25}$ is an example of a $D24$-brane. Note that if we start from an open string on

$$\mathbb{R}^{25-n,1} \times T^n,$$

where T^n is the n-torus‡, and T-dualize in T^n then we end up with a $D(25-n)$-brane. In particular, for $n = 0$ we can think that the whole spacetime is filled by a $D25$-brane.

So, to summarize: the general rule that we learn from the previous discussion is that if a Dp-brane wraps around a circle that, then it doesn't wrap around the T-dual circle, and vice versa.

- Aside: In the preceding construction a single Dp-brane appeared naturally after applying T-duality to an open string with Neumann boundary conditions. It can be shown that, when several Dp-branes are present instead of a single one, something rather interesting happens, namely nonabelian gauge symmetries emerge in the theory.

  An open string can carry additional degrees of freedom at its end points, called Chan-Paton charges. These Chan-Paton factors associate N degrees of freedom with each of the end points of the string. For the case of oriented open strings, the two ends of the string are distinguished, and so it makes sense to associate the fundamental representation N with the $\sigma = 0$ end and the antifundamental representation $\overline{N}$ with the $\sigma = \pi$ end. In this way one describes the gauge group $U(N)$. For strings that are unoriented, such as type I superstrings, the representations associated with the two ends have to be the same, and this forces the symmetry group to be one with a real fundamental representation, specifically an orthogonal or symplectic group.

12.2.1 Mass Spectrum of Open Strings on Dp-Branes

Consider a configuration for our bosonic string theory which has several Dp-branes, namely a configuration such that the coordinates $X^0, X^1, ..., X^p \equiv X^\mu$ have Neumann boundary conditions while the coordinates $X^{p+1}, X^{p+2}, ..., X^{25} \equiv X^I$ have Dirichlet boundary conditions. Then the mode expansion for X^μ is given by (HOW?)

$$X^\mu = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma). \tag{12.32}$$

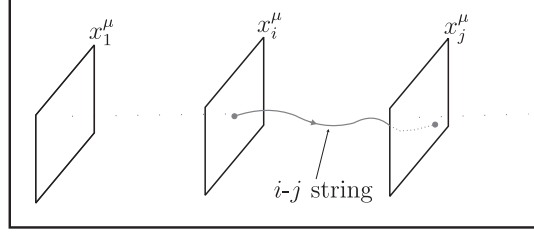---

‡The n-torus is defined by

$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}}. \tag{12.31}$$

while for X^I we have

$$X^I = x_i^I + \frac{\sigma}{\pi}(x_j^I - x_i^I) + l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^I e^{-in\tau} \sin(n\sigma), \tag{12.33}$$

where i and j label the different Dp-branes. Now, from these mode expansions, we can derive the expressions for the mass-shell condition. Namely, we have that

i-j string

$$L_0 = H = \frac{T}{2} \int_0^\pi \left((\dot{X})^2 + (X')^2 \right)$$

$$= \frac{T}{2} \int_0^\pi \left((\dot{X}^\mu)^2 + (X^{\mu'})^2 + (\dot{X}^I)^2 + (X^{I'})^2 \right)$$

$$= \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{n\mu} + \alpha' p^\mu p_\mu + \sum_{n=1}^{\infty} \alpha_{-n}^I \alpha_{nI} + \frac{1}{4\alpha'} \left(\frac{x_i^I - x_j^I}{\pi} \right)^2.$$

And so, we get

$$L_0 = N - \alpha' M^2 + \frac{1}{4\alpha'} \left(\frac{x_i^I - x_j^I}{\pi} \right)^2, \tag{12.34}$$

where we have defined N as

$$N = \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{n\mu} + \sum_{n=1}^{\infty} \alpha_{-n}^I \alpha_{nI}$$

$$= \sum_{n=1}^{\infty} \alpha_{-n}^\nu \alpha_{n\nu}, \tag{12.35}$$

with $\nu = 0, ..., 25$. Furthermore, imposing the physical state condition, $L_0 - 1 = 0$, leads to

$$M^2 = \frac{N - 1}{\alpha'} + T^2 (x_i^I - x_j^I)^2, \tag{12.36}$$

where T is the tension, which is defined by $T = 1/(2\pi\alpha')$. Note that we can think of $(x_i^I - x_j^I)^2$ as the energy stored in tension of string streching between a Dp-brane and x_i and a Dp-brane at x_j. Let us now explore the spectrum for our theory.

Massless States

- One Dp-brane: For the case of one Dp-brane the term given by $T^2 (x_i^I - x_j^I)^2$ vanishes because $x_i^I = x_j^I$. This leaves us with the following expression for the mass,

$$M^2 = \frac{N - 1}{\alpha'}.$$

Thus, we see that for massless states of the bosonic string theory with one Dp-brane, i.e. $i = j$, we must have that $N - 1$ or, the only massless states with one Dp-brane are level 1 states. So, what kind of states can we have for the $N = 1$ level? Theses states are given by

$$\alpha^{\mu}_{-1}|0; k^{\mu}\rangle, \tag{12.37}$$

which corresponds to a $p + 1$-dimensional vector (which we denote by A_{μ}), and also

$$\alpha^{I}_{-1}|0; k^{\mu}\rangle, \tag{12.38}$$

which corresponds to $25 - p$ scalars (which we denote by X^I). Note that we can interprete A_{μ} as a gauge field and X^I as the position of the Dp-brane. This implies that on the Dp-brane the lives a transverse $U(1)$ gauge theory with some scalars X^I. For further explanation of this see Becker, Becker and Schwarz "String Theory and M-Theory".

- Two Dp-branes: For this case the mass formula is given by, as before,

$$M^2 = \frac{N - 1}{\alpha'} + T^2(x_i^I - x_j^I)^2,$$

where, in this case, $i, j = 1, 2$. Now, if the two Dp-branes are at the same position, i.e. if $x_i = x_j$ then there are extra massless, than before for one brane, modes which arise from $1 - 2$ and $2 - 1$ strings[‡]. So, we get, for the states,

$$(A_{\mu})_{\alpha\beta} \qquad \text{and} \qquad X^I_{\alpha\beta}, \tag{12.39}$$

where $\alpha, \beta = 1, 2$. The four component enitity denoted by $(A_{\mu})_{\alpha\beta}$ describes a $U(2)$ gauge theory. Thus, in this case, when the two branes are coincident, we get a $U(2)$ gauge theory living on them. Note that, in general, if you have N coincident Dp-branes then you will get a $U(N)$ gauge theory localized on them and the fields are given by

$$(A_{\mu})_{\alpha\beta} \qquad \text{and} \qquad X^I_{\alpha\beta}, \tag{12.40}$$

where $\alpha, \beta = 1, 2, ..., N$. Also note that as soon as one of the Dp-branes moves apart from the others, your $U(N)$ will split into $U(N - 1) \times U(1)$. Thus, in the case of two Dp-branes, if they were not coincident then our gauge symmetry would be $U(1) \times U(1)$.

Dp-branes are not restricted to the bosonic theory only. In fact, Dp-branes can also exist in superstring theories as well.

[‡]Recall that we call a string which goes from a brane at x_i to a brane at x_j a $i - j$ string. Also, since our strings are orientated there is a difference between an $i - j$ string and a $j - i$ string.

12.3 Branes in Type II Superstring Theory

As an aside, note that a point particle, i.e. a 0-brane, couples naturally to a guage field, A_μ, via

$$S_A = \int_{\mathcal{C}} A_\mu \dot{X}^\mu d\tau = \int_{\mathcal{C}} A_\mu \frac{dX^\mu}{d\tau} d\tau = \int_{\mathcal{C}} A_{(1)}, \qquad (12.41)$$

where in the above $\mathcal{C}$ is the worldline of the particle and we have also used the fact that if given a gauge field A_μ, one can construct from it a one-form[§], which is denoted $A_{(1)}$. For example, we now that a charge point particle will couple to an electric or magnetic field, which can be described by a gauge field A_μ, as it moves through spacetime. Also, if given a one-form $A_{(1)}$, which is defined by a gauge field A_μ, then one can construct an object called the field strength $F_{(2)}$, associated to the one-form, by defining $F_{(2)}$ as

$$F_{(2)} = dA_{(1)}, \qquad (12.44)$$

where d is the exterior derivative. Note that since the field strength $F_{(2)}$ is defined as the exterior derivative of a one-form, we see that the field strength, associated to a one-form, is a two-form. So, to recap, a 0-brane couples naturally to a one-form gauge potential, $A_{(1)}$, which in turn can be used to desribe a two-form field strength $F_{(2)}$.

This can be generalized to 1-branes. Namely, a 1-brane couples naturally to a two-form gauge potential, $B_{(2)} = B_{\mu\nu} dx^\mu \wedge dx^\nu$, via

$$S_B = \int_{\mathcal{M}} d\tau d\sigma B_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu \equiv \int_{\mathcal{M}} d\tau d\sigma B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta}, \qquad (12.45)$$

where $\mathcal{M}$ is the worldsheet mapped out by the 1-brane (or string). Indeed, the spectrum of closed superstrings contains an anti-symmetric two-form $B_{\mu\nu}$, called the Kolb-Ramond field. Also, it is known that the fundamental string couples to $B_{(2)}$ via the above action, S_B. Summary, a 1-brane couples naturally to a two-form potential, which in turn can be used to define its associated field strength, $H_{(3)} = dB_{(2)}$, which is given by a three-form.

[§]Note that if given a gauge field, A_μ, one can construct a one-form, $A_{(1)}$, from it by defining the one-form as,

$$A_{(1)} = A_\mu dx^\mu. \qquad (12.42)$$

Similarily, if one is given a gauge field with n component, $A_{\mu_1 \mu_2 \cdots \mu_n}$, then they can construct a n-form, $A_{(n)}$, from it by defining the n-form as

$$A_{(n)} = \frac{1}{n!} A_{\mu_1 \mu_2 \cdots \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}. \qquad (12.43)$$

Now, a good question to ask is if this can be generalized to p-branes? The answer to this question is yes of course. Why else would we have taken the time to do all of this developement if it could not be generalized? So, to generalize: a p-brane couples naturally to a $p + 1$-form gauge potential, $c_{(p+1)}$, via the action

$$S_c = \int_{\mathcal{M}} d\tau d\sigma^1 \cdots d\sigma^p \; c_{i_1 \cdots i_{p+1}} \partial_{\alpha_1} X^{i_1} \cdots \partial_{\alpha_{p+1}} X^{i_{p+1}} \epsilon^{\alpha_1 \cdots \alpha_{p+1}} \tag{12.46}$$

where $\mathcal{M}$ is the worldvolume mapped out by the p-form as it moves through the background spacetime. Indeed, the spectrum of superstrings contains p-forms in the Ramond-Ramond sector and it is also known that D-branes couple to the R-R sector gauge potentials via S_c, (12.46).

We can use the above information about the coupling of branes to gauge potentials in order to see what kind of stable branes we expect in a particular string theory. For example, since we know what kind of gauge fields are present in type IIA and type IIB superstring theories, see last chapter, we can use this knowledge to see what kind of branes we expect to see in these two theories. We will begin with the type IIA superstring theory.

- Type IIA: Recall that in the type IIA theory there exists 1-form and 3-form gauge potentials. Now, since $D0$-branes couple to 1-form gauge potentials and since $D2$-branes couple to 3-form gauge potentials, we see that there exists $D0$-branes and $D2$-branes in the type IIA superstring theories. But is this all? The answer to this question is no. This is because, as in electrodynamics, if one is given a field strength, which remember is associated to a form potential, then they can construct another field strength by taking the Hodge dual[‡] of the previous one, i.e. if we have a gauge potential $c_{(1)}$ then to this there corresponds a field strength $F_{(2)} = dc_{(1)}$ and by taking the Hodge dual of $F_{(2)}$ we get another field strength $*F_{(2)}$. Thus, in our case, we have 1- and 3-form potentials which defines 2- and 4- form field strengths and, since $D = 10$ for type IIA theories, we see that corresponding to the 2-form field strength there is a dual 8-form field strength and also, corresponding to the 4-form field strength there is a dual 6-form field strength. Now, associated to the dual 8-form field strength will be a 7-form

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[‡]In D-dimensions, given a p-form field strength, $F_{(p)}$, the Hodge dual of this form, $*F_{(D-p)}$, will be a $D - p$-form whose components are given by

$$(*F_{(D-p)})^{\mu_1 \mu_2 \cdots \mu_D} = \frac{\epsilon^{\mu_1 \mu_2 \cdots \mu_D}}{2\sqrt{-g}} F_{\mu_{D-p+1} \mu_{D-p+2} \cdots \mu_D}, \tag{12.47}$$

where $g \equiv det(\text{metric})$, which in our case is the flat metric, so $g = -1$.

gauge potential[§] which couples naturally to a $D6$-brane. Similarly, we see that associated to the 6-form field strength there will be a 5-form gauge potential which couples to a $D4$-brane. Thus, along with the $D0$-branes and $D2$-branes, we get additional $D4$-branes and $D6$-branes. So, in conclusion, the type IIA superstring theory can have $D0$–branes, $D2$–branes, $D4$–branes and $D6$-branes.

- Type IIB: Recall that in the type IIB theory there exists 0-form (scalars) gauge potentials, 2-form gauge potentials and 4-form gauge potentials. Which gives rise to couple $D(-1)$-branes, $D1$-branes and $D3$-branes, respectively. Before we proceed with the dual branes, let us figure what in the hell a $D(-1)$-brane is. The $D(-1)$-brane is an object which is localized in time as well as in space, i.e. it is interpretated as a D-instanton. There is also another point that should be made here. Recall from electrodynamics that the field strength described the electric field part, while the dual field strength described the magnetic field part. Thus, technically, we have that a 0-form gauge potential couples electrically to a $D(-1)$-brane, a 2-form gauge potential couples electrically to a $D1$-brane, etc.. Also, this is also true for the type IIA case. In particular, we see that a 3-form gauge potential couples electrically to a $D2$-brane, while a 7-form gauge potential couples magnetically to a $D6$-brane. Now, back to the type IIB case. Along with the $D(-1)$-branes, $D1$-branes and $D3$-branes, we also get the dual $D7$-branes and the dual $D5$-branes. Note that before in the type IIA case we had two types of branes and they gave rise to two types of dual branes. But now, in the type IIB case, we have 3 types of branes which only give rise to two types of dual branes. Why is this? The answer to this question is as follows: The $D3$-brane couples electrically to a 4-form gauge potential which, in turn, defines a 5-form field strength. Now if one computes the Hodge dual of this field strength they will end up with a dual 5-form field strength, i.e. the degree of the form does not change under the Hodge star operator, which gives a 4-form gauge potential that couples magnetically to a $D3$-brane. Thus, by Hodge dualizing the $D3$-brane we do not get any new branes, rather, we only see that the $D3$-brane can couple both electrically and magentically to a 4-form gauge potential. We say that $D3$-branes are self-dual, with respect to the Hodge star operator.

So, to recap: For the type IIA superstring theory we can have $D0$–branes, $D2$–branes, $D4$–branes and $D6$-branes, while for the type IIB superstring theory we can

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[§]This is because a field strength, $F_{(p)}$ is defined by

$$F_{(p)} = dc_{(p-1)},$$

where $c_{(p-1)}$ is some arbitrary $p-1$-form.

have $D(-1)$-branes (the D-instanton), $D1$-branes, $D3$-branes, $D5$-branes and $D7$-branes. Note the type II superstring theories also admit Dp-branes with "wrong" values of p, meaning that p is odd in the IIA theory or even in the IIB theory. These Dp-branes do not carry conserved charges and are unstable, meaning that they decay into other branes. They break all of the supersymmetry and give an open-string spectrum that includes a tachyon. The features of these branes are the same as those of Dp-branes with any value of p in the bosonic string theory. In the context of superstring theories, Dp-branes of this type are sometimes referred to as non-BPS Dp-branes. As a final remark, T-duality maps a type IIA theory compactified on a circle(s) of radius R to a type IIB theory compactified on a circle(s) of radius α'/R, sends the Neumann boundary conditions to Dirichlet and maps various Dp-branes (p even) into other Dq-branes[‡] (q is odd), and vice versa (i.e. T-duality will map type IIB theories of radius R to type IIA theories of radius α'/R etc.).

As has been mentioned on several occaissons, Dp-branes are physical objects and thus it should be possible to describe there physics by some action. This is the topic of the last section of this chapter.

12.4 Dirac-Born-Infeld (DBI) Action

The goal of this section is to construct a Dp-brane action. But how do we go about doing this? Well, recall from earlier, in particular (6.1)-(2.24), that the worldvolume action for a p-brane is given by generalizing the Nambu-Goto to

$$S_p = -T_p \int d^{p+1}\sigma \sqrt{-det(G_{\mu\nu})}, \tag{12.48}$$

where $d^{p+1}\sigma \equiv d\tau d\sigma^1 d\sigma^2 \cdots d\sigma^p$ and where $G_{\mu\nu}$ is the induced metric which is given by, in flat space,

$$G_{\mu\nu} = \eta_{MN} \frac{\partial X^M}{\partial \sigma^\mu} \frac{\partial X^N}{\partial \sigma^\nu}, \tag{12.49}$$

with $M, N = 0, 1, ..., 25$ and $\mu, \nu = 0, ..., p$. Also, since, as we have seen for the particular case of $p = 1$, the S_p action is invariant under diffeomorphisms (i.e. it possesses gauge symmetries) we get a redundancy in our description which we can fix by picking a particular gauge. Whereas before we chose the flat metric gauge, here we will choose to work in the static gauge, which says that

$$X^\mu(\tau, \sigma) = \sigma^\mu. \tag{12.50}$$

[‡]To be more precise, T-duality transforms a Dp-brane into a $D(p-1)$-brane if a direction along the brane is T-dualized, while it transforms a Dp-brane into a $D(p+1)$-brane if a direction orthogonal to the brane is T-dualized.

Plugging this back into the expression for the induced metric, $G_{\mu\nu}$, yeilds

$$G_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu X^I \tag{12.51}$$

where as before $I = p+1, p+2, ..., 25$, which gives us for our action

$$S_p = -T_p \int d^{p+1}\sigma \sqrt{-det(\eta_{\mu\nu} + \partial_\mu X^I)}. \tag{12.52}$$

Now, we are forgetting one important thing, Dp-branes couple with gauge fields. Thus, we need to figure out how to incorporate gauge fields into the p-brane action. To do this we replace the previous induced metric, $G_{\mu\nu}$, with

$$G_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}, \tag{12.53}$$

where $F_{\mu\nu}$ is the field strength given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A is some gauge field. Plugging all of this back into the p-brane action gives our desired result, i.e. a Dp-brane action,

$$S_{Dp} = -T_{Dp} \int d^{p+1}\sigma \sqrt{-det\left(\eta_{\mu\nu} + \partial_\mu X^I + 2\pi\alpha' F_{\mu\nu}\right)}, \tag{12.54}$$

written here in the static gauge, where T_{Dp} is given by[§]

$$T_{Dp} = \frac{1}{(2\pi)^2 (\alpha')^{(p+1)/2} g_s}, \tag{12.55}$$

with g_s the strings coupling constant. Note that if $g \approx 0$ then the Dp-brane is extremely heavy. So, for free strings, i.e. $g_s = 0$, we can describe the Dp-branes in this theory simply as Dirichlet boundary conditions for strings since the Dp-brane is infinitely heavy, which implies it does not move at all, i.e. the strings endpoints are fixed in spacetime (which is what Dirichlet boundary conditions say).

This concludes our discussion of T-duality and Dp-branes. In the next chapter we will discuss the string low energy effective action. Then we will discuss T-dualities and S-dualities in curved spacetime. Finally, we will discuss M-theory.

[§]See Becker, Becker and Schwarz "String Theory and M-Theory" pages 233 - 234 for this derivation.

12.5 Exercises

Problem 1

Consider the DBI action

$$S = -T_p \int d^{p+1}\sigma\sqrt{-M},$$

where

$$T_p = \frac{1}{g_s(2\pi)^p(\alpha')^{(p+1)/2}}, \quad M \equiv \det M_{\mu\nu}, \quad M_{\mu\nu} = \partial_\mu X^P \partial_\nu X^Q \eta_{PQ} + kF_{\mu\nu},$$

with $\mu, \nu = 0, \ldots, p$, $P, Q = 0, \ldots, 9$ and $k = 2\pi\alpha'$.

(1) Show that the field equations are given by

$$\partial_\mu(\sqrt{-M}\theta^{\mu\nu}) = 0, \tag{12.56}$$

$$\partial_\mu(\sqrt{-M}G^{\mu\nu}\partial_\nu X^P) = 0, \tag{12.57}$$

where $G^{\mu\nu}$ and $\theta^{\mu\nu}$ is the symmetric and antisymmetric part of $M^{\mu\nu}$, respectively, and

$$M^{\mu\nu}M_{\nu\kappa} = \delta^\mu{}_\kappa.$$

(2) Show that in the static gauge, i.e. with $X^\mu = \sigma^\mu$ ($\mu = 0, \ldots, p$), and for the constant transverse scalars X^i, ($i = p+1, \ldots, 9$), equation (12.56) becomes the Maxwell equations to leading order in k as $k \to 0$.

Problem 2

Consider the DBI action for a 2-brane

$$S = -T_2 \int d^3\sigma\sqrt{-M}, \tag{12.58}$$

where M is defined as in the previous exercise.

(1) Show that the action S in the static gauge $X^\mu = \sigma^\mu$ ($\mu = 0, \ldots, 2$), and for the constant transverse scalars X^i, ($i = 3, \ldots, 9$) becomes

$$S = -T_2 \int d^3\sigma\sqrt{1 + \frac{k^2}{2}F_{\mu\nu}F^{\mu\nu}}.$$

(2) Show that in the limit $k \to 0$ the leading order terms are a cosmological constant Λ plus the Maxwell action,

$$S = \int d^3\sigma\left(\Lambda - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}\right). \tag{12.59}$$

What is the coupling constant e in terms of g_s and α'? Explain the α' dependence using dimensional analysis.

Problem 3

Two D-branes intersect orthogonally over a p-brane if they share p directions with the remaining directions wrapping different directions. For example a D5 brane extending in the directions $x^0, x^1, x^2, x^3, x^4, x^5$ and a D1 brane extending in x^0, x^1 intersect orthogonally over an 1-brane. In such cases we can divide the spacetime directions into 4 sets, $\{NN, ND, DN, DD\}$ according to whether the coordinate X^μ has Neumann (N) or Dirichlet (D) boundary conditions on the first or second brane. In the example of the D1-D5 system for a string stretching from the D1 brane to the D5 brane: $NN = \{x^0, x^1\}$, $DN = \{x^2, x^3, x^4, x^5\}$, $DD = \{x^6, x^7, x^8, x^9\}$ and there are no ND coordinates.

(1) Show that the numbers $(\#NN + \#DD)$ and $(\#ND + \#DN)$ are invariant under T-duality, where $\#NN$ is the number of NN directions, etc.

(2) List all orthogonal intersections in IIB string theory that have $(\#ND + \#DN) = 4$ and contain at least one $D3$ brane. Show that all these configurations are T-dual to the D1-D5 configuration listed above.

13. Effective Actions, Dualities, and M-Theory

During the "Second Superstring Revolution," which took place in the mid-1990s, it became evident that the five different ten-dimensional superstring theories are related through an intricate web of dualities. In addition to the T-dualities that were discussed before, there are also S-dualities that relate various string theories at strong coupling to a corresponding dual description at weak coupling. Moreover, two of the superstring theories (the type IIA superstring and the $E_8 \times E_8$ heterotic string) exhibit an eleventh dimension at strong coupling and thus approach a common eleven-dimensional limit, a theory called M-theory. In the decompactifcation limit, this eleven-dimensional theory does not contain any strings, so it is not a string theory.

13.1 Low Energy Effective Actions

As we have seen on several occasions, a string propagating through a flat background spacetime is described by

$$S_{\sigma,F} = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} \; h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \tag{13.1}$$

where, recall that, h is the metric on the worldsheet and η is the (flat) metric describing the background spacetime, through which the string propagates. Now, we can generalize this action to describe a string propagating through a background spacetime which no longer must be flat. This, generalized, action is given by

$$S_\sigma = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} \; h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X), \tag{13.2}$$

where h is as before and g is the background metric which need not be flat, i.e. we do not necessarily have that $g = \eta$.

Although it is obvious that (13.2) describes a string moving through an arbitrary background there is a problem that we need to adress. The problem is that the quantization of the closed string already gave us a graviton. If we want to build up some background metric $g_{\mu\nu}(X)$, it should be constructed from these gravitons, in much the same manner that a laser beam is made from the underlying photons. How do we see that the metric in (13.2) has anything to do with the gravitons that arise from the quantization of the string?

To answer this question we procede as follows. First, let us expand the spacetime metric, $g_{\mu\nu}(X)$, around the flat metric, $\eta_{\mu\nu}$,

$$g_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(X). \tag{13.3}$$

Then, the partition function which we build from the generalized action, (13.2), is related to the partition function which we build the "flat" action, (13.1), by

$$Z = \int \mathcal{D}X\mathcal{D}h \; e^{-(S_{\sigma,F}+V)},$$

$$= \int \mathcal{D}X\mathcal{D}h \; e^{-S_{\sigma,F}}e^{-V},$$

$$= \int \mathcal{D}X\mathcal{D}h \; e^{-S_{\sigma,F}}\left(1 - V + \frac{1}{2}V^2 + \cdots\right), \tag{13.4}$$

where $S_{\sigma,F}$ is the flat action given in (13.1) and V is given by

$$V = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}\; h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu h_{\mu\nu}(X). \tag{13.5}$$

Now, the expression for V is called the vertex operator associated to the graviton state of the string. Thus, we know that inserting a single copy of V in the path integral corresponds to the introduction of a single graviton state. Inserting e^V into the path integral corresponds to a coherent state of gravitons, changing the metric from $\eta_{\mu\nu}$ to $\eta_{\mu\nu} + h_{\mu\nu}(X)$. In this way we see that the background arbitrary metric in (13.2) is indeed built of the quantized gravitons which arose from quantizing the closed string.

13.1.1 Conformal Invariance of S_σ and the Einstein Equations

We have seen that, in conformal gauge, the Polyakov action in a flat spacetime reduces to a free theory. However, in a curved spacetime this is no longer true. In conformal gauge, the worldsheet theory is described by an interacting two-dimensional field theory,

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma g_{\mu\nu}(X)\partial_\alpha X^\mu \partial^\alpha X^\nu. \tag{13.6}$$

To understand these interactions in more detail, lets expand around a classical solution which we take to simply be a string sitting at a point x^μ,

$$X^\mu(\sigma) = x^\mu + \sqrt{\alpha'}Y^\mu(\sigma). \tag{13.7}$$

Here $Y^\mu(\sigma)$ are the dynamical flucations around the point x^μ, which we assume to be small, and the factor of $\sqrt{\alpha'}$ is there for dimensional reasons. Now, exanding the Lagragian gives

$$g_{\mu\nu}(X)\partial_\alpha X^\mu \partial^\alpha X^\nu = \alpha'\left(g_{\mu\nu}(x) + \sqrt{\alpha'}g_{\mu\nu,\omega}(x)Y^\omega + \frac{\alpha'}{2}g_{\mu\nu,\omega\rho}(x)Y^\omega Y^\rho + \cdots\right)\partial_\alpha Y^\mu \partial^\alpha Y^\nu,$$

where $g_{\mu\nu,i_1\cdots i_n}(x) \equiv \partial_{i_n}\cdots\partial_{i_1}(g_{\mu\nu})(x)$. Note that each of the $g_{\mu\nu,\ldots}$'s appearing in the Taylor expansion above are coupling constants for the interactions of the Y's. Also, there are an infinite number of these coupling constants which are contained in $g_{\mu\nu}(X)$.

Now, classical, the theory defined by (13.6) is conformally invariant. However, this is not true when we quantize the theory. To regulate divergences we will have to introduce a UV cut-off and, typically, after renormalization, physical quantities depend on the scale of a given process ξ. If this is the case, the theory is no longer conformally invariant. There are plenty of theories which classically possess scale invariance which is broken quantum mechanically. The most famous of these is Yang-Mills.

As has been shown throughout, conformal invariance in string theory is a very good property, it is in fact a gauge symmetry. Thus, we need to see under what conditions our theory, defined by (13.6), remains conformally invariant after quantization. These conditions will be given by the coupling constants and whether they depend on ξ or not, i.e. if the coupling constants do not depend on ξ then our theory will be conformally invariant under the quantization process. And so, we need to see how the coupling constants depend on ξ.

The object which describes how a coupling constant depend on a scale ξ is called the β-function. Since we have a functions worth of couplings, we should really be talking about a β-functional, schematically of the form

$$\beta_{\mu\nu}(g) = \frac{\partial g_{\mu\nu}(X;\xi)}{\partial \ln(\xi)}. \tag{13.8}$$

Now, there is an easy way to express whether the quantum theory will be invariant or not. It is defined by

$$\beta_{\mu\nu}(g) = 0, \tag{13.9}$$

i.e. if the β-function(al) vanishes then the quantum version of (13.6) will remain conformally invariant. The only thing left to do is to see what restriction this constraint of having the β-function(al) vanish imposes on the coupling constants, $g_{\mu\nu}$, and, as we will see, this restriction is more than incredible.

The strategy is as follows. We will isolate the UV-divergence of (13.6) and then use this to see what kind of term we should add. The β-function will then vanish if this term vanishes. So, let us now proceed.

To begin, note that around any point x, we can always pick Riemann normal coordinates such that the expansion in $X^\mu(\sigma) = x^\mu + \sqrt{\alpha'}Y^\mu(\sigma)$ gives

$$g_{\mu\nu}(X) = \delta_{\mu\nu} - \frac{\alpha'}{3}R_{\mu\lambda\nu\kappa}(x)Y^\lambda Y^\kappa + \mathcal{O}(Y^\alpha Y^\gamma Y^\delta). \tag{13.10}$$

Plugging this back into the action, (13.6), gives (up to quadratic order in the flucuations Y)

$$S = \frac{1}{4\pi} \int d^2\sigma \left(\partial Y^\mu \partial Y^\nu \delta_{\mu\nu} - \frac{\alpha'}{3} R_{\mu\lambda\nu\kappa} Y^\lambda Y^\kappa \partial Y^\mu \partial Y^\nu \right). \tag{13.11}$$

Now, we can now treat this as an interacting quantum field theory in two dimensions. The quartic interaction gives a vertex with the Feynman rule,

$$\sim R_{\mu\lambda\nu\kappa}(k^\mu \cdot k^\nu), \tag{13.12}$$

where k_α^μ is the two-momentum ($\alpha = 1, 2$ is a worldsheet index) for the scalar field Y^μ.

Now that we have reduced the problem to a simple interacting quantum field theory, we can compute the β-function using whatever method we like. The divergence in the theory comes from the one-loop diagram

$$. \tag{13.13}$$

Let us now think of this diagram in position space. The propagator for a scalar particle is given by

$$\langle Y^\lambda(\sigma) Y^\kappa(\sigma') \rangle = -\frac{1}{2} \delta^{\lambda\kappa} \ln \left(|\sigma - \sigma'|^2 \right). \tag{13.14}$$

For the scalar field running in the loop, the beginning and end point coincide. The propagator diverges as $\sigma \mapsto \sigma'$, which is simply reflecting the UV divergence that we would see in the momentum integral around the loop. To isolate this divergence, we choose to work with dimensional regularization, with $d = 2 + \epsilon$. The propagator then becomes,

$$\langle Y^\lambda(\sigma) Y^\kappa(\sigma') \rangle = 2\pi \delta^{\lambda\kappa} \int \frac{d^{2+\epsilon}k}{(2\pi)^{2+\epsilon}} \frac{e^{ik(\sigma-\sigma')}}{k^2}, \tag{13.15}$$

and so we see that

$$\lim_{\sigma \to \sigma'} \langle Y^\lambda(\sigma) Y^\kappa(\sigma') \rangle \longrightarrow \frac{\delta^{\lambda\kappa}}{\epsilon}. \tag{13.16}$$

The necessary counterterm for this divergence can be determined simply by replacing $Y^\lambda Y^\kappa$ in the action with $\langle Y^\lambda Y^\kappa \rangle$. To subtract the $1/\epsilon$ term, we add the counterterm given by

$$R_{\mu\lambda\nu\kappa} Y^\lambda Y^\kappa \partial Y^\mu \partial Y^\nu \longrightarrow R_{\mu\lambda\nu\kappa} Y^\lambda Y^\kappa \partial Y^\mu \partial Y^\nu - \frac{1}{\epsilon} R_{\mu\nu} \partial Y^\mu \partial Y^\nu. \tag{13.17}$$

It can be shown that this can be absorbed by a wavefunction renormalization $Y^\mu \to Y^\mu - (\alpha'/6\epsilon)R^\mu{}_\nu Y^\nu$, along with the renomalization of the coupling constants

$$g_{\mu\nu} \to g_{\mu\nu} + \frac{\alpha'}{\epsilon}R^{\mu\nu}. \tag{13.18}$$

Now, recall that for the original theory, (13.6), to remain invariant under conformal transformations after quantization we need for $\beta(g) = 0$, i.e. we need

$$\beta(g) = \alpha' R^{\mu\nu} = 0. \tag{13.19}$$

This is none other than the Einstein equations in vacuum. Thus, we see a remarkable result. Namely, for the Polyakov action to remain conformally invariant we need for the background spacetime to be flat, $R_{\mu\nu} = 0$. Or, in other words, the background spacetime in which the string moves must obey the vacuum Einstein equations! We see that the equations of general relativity also describe the renormalization group flow of two-dimensional sigma (Polyakov) models.

13.1.2 Other Couplings of the String

We have just seen how strings couple to a background spacetime metric, $g_{\mu\nu}$. Namely, the metric appears as the background field in the action

$$S_\sigma = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h}\; g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu h^{\alpha\beta}.$$

But what about the other modes of the string? Previously we have seen that a string has further massless states which are associated to the Kolb-Ramond Field $B_{\mu\nu}$ and the dilaton $\phi(X)$. We will now see how the string reacts if these fields are turned on in spacetime.

String Moving in a $B_{\mu\nu}$ Field

We will begin by seeing how the strings couple to the antisymmetric Kolb-Ramond field, $B_{\mu\nu}$. To do this we need to construct an action to describe the interaction, or coupliing. To begin, note that the vertex operator, V_B, for the Kolb-Ramond field, $B_{\mu\nu}$, is of the form

$$V_B \sim \int d^2z : e^{ip\cdot X}\partial X^\mu \overline{\partial} X^\nu : \epsilon_{\mu\nu}, \tag{13.20}$$

where $\zeta_{\mu\nu}$ is the antisymmetric part of a constant tensor, which we also denote as $\zeta_{\mu\nu}$. This follows from the observation that the vertex operator, V, given by

$$V_{g,B} \sim \int d^2z : e^{ip\cdot X}\partial X^\mu \overline{\partial} X^\nu : \zeta_{\mu\nu}$$

is the correction vertex operator for the graviton field, $g_{\mu\nu}$, if we take $\zeta_{\mu\nu}$ to be the traceless symmetric part of $\zeta_{\mu\nu}$, or it is the correct vertex operator for the Kolb-Ramond field if $\zeta_{\mu\nu}$ is antisymmetric. While the correct vertex operator for the dilaton field, $\phi(X)$, is given by

$$V_\phi \sim \int d^2z : e^{ip\cdot X}\partial X^\mu \overline{\partial} X^\nu : \zeta,$$

where ζ is the trace of the constant tensor $\zeta_{\mu\nu}$. Now, it is a simple matter to exponentiate this, to get an expression for how strings propagate in a background $B_{\mu\nu}$ field. The general action describing a string propagating through a Kolb-Ramond background field is given by

$$S_B = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h}\Big(iB_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta}\Big), \qquad (13.21)$$

where $\epsilon^{\alpha\beta}$ is the antisymmetric 2-tensor, normalized such that $\sqrt{-h}\epsilon^{12} = 1$. Note that if we had of taken $\zeta_{\mu\nu}$ to be the traceless and symmetric part of the constant tensor then the above action describing how the string couples to a Kolb-Ramond field would change to

$$S_g = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h}\Big(g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu h^{\alpha\beta}\Big),$$

which, as no suprise, describes how a string couples to a graviton field. It should also be obvious that the action desrcibing a string propagating through a background which comprises of a graviton field and a Kolb-Ramond field is given by

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h}\Big(g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu h^{\alpha\beta} + iB_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta}\Big). \qquad (13.22)$$

Also, it can be shown that this new action, (13.22), still retains invariance under worldsheet reparameterizations and Weyl scailing. As a side note, the gauge field strength associated to $B_{\mu\nu}$ is a 3-form, which we denote by H (i.e. $H_{(3)} = dB_{(2)} = \partial_\mu B_{\nu\rho} + \partial_n u B_{\rho\mu} + \partial_\rho B_{\mu\nu}$), which plays the same role as torsion in general relativity, providing an anti-symmetric component to the affine connection. For this reason we call H the torsion as well.

So what is the interpretation of this new field? Well, one should think of the field $B_{\mu\nu}$ as analogous to the gauge potential A_μ in electromagnetism. Thus the action (13.22) is telling us that the string is electrically charged under $B_{\mu\nu}$.

Now, let us see what happens to the string if we turn on the dilaton field in the background spacetime.

String Moving in a Dilaton Field

Let us now construct the action which describes how a string couples to the dialton field, $\phi(X)$. A first guess might be to simply exponentiate the vertex operator, V_ϕ, for the dilaton field[¶], however the dilaton vertex operator is not primary and one must work a little harder. It turns out that the correct expression for the action which describes the coupling of the string with the dilaton field, $\phi(X)$, is given by

$$S_\phi = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h}\left(\alpha'\phi(X)R^{(2)}\right), \tag{13.23}$$

where $R^{(2)}$ is the two-dimensional Ricci scalar of the worldsheet.

The coupling to the dilaton is surprising for several reasons. Firstly, we see that the dilaton coupling vanishes on a flat worldsheet, $R^{(2)} = 0$. This is one of the reasons that its a little trickier to determine this coupling using vertex operators. However, the most surprising thing about the coupling to the dilaton is that it does not respect Weyl invariance. So, why on earth are we willing to throw away Weyl invariance now after basically requiring it before? The answer, of course, is that were not. Although the dilaton coupling does violate Weyl invariance, there is a way to restore it. We will explain this shortly. But firstly, lets discuss one crucially important implication of the dilaton coupling, (13.23).

There is an exception to the statement that the classical coupling to the dilaton violates Weyl invariance. This arises when the dilaton is constant. For example, suppose

$$\phi(X) = \lambda,$$

where λ a constant. Then the dilaton coupling reduces to

$$S_\phi = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h}\left(\alpha'\lambda R^{(2)}\right),$$

$$= \lambda\left(\frac{1}{4\pi} \int d^2\sigma \sqrt{-h}\, R^{(2)}\right),$$

$$= \lambda\chi, \tag{13.24}$$

[¶]Recall that the vertex operator for the dialton field is given by

$$V_\phi \sim \int d^2z : e^{ip\cdot X}\partial X^\mu \overline{\partial} X^\nu : \zeta,$$

where ζ is the trace of the constant 2-tensor $\zeta_{\mu\nu}$.

where χ is the Euler characteristic. This implies that the constant mode of the dilaton, $\langle\phi\rangle$, determines the string coupling constant, g_s, i.e. we have that

$$g_s = \langle e^\phi \rangle,$$

where ϕ is constant. So the string coupling is not an independent parameter of string theory: it is the expectation value of a field. This means that, just like the spacetime metric $g_{\mu\nu}$ (or, indeed, like the Higgs vev) it can be determined dynamically.

We now return to understanding how we can get away with the violation of Weyl invariance in the dilaton coupling (13.23). The key to this is to notice the presence of α' in front of the dilaton coupling. Its there simply on dimensional grounds. However, note that α' also plays the role of the loop-expansion parameter inthe non-linear sigma (Polyakov) model. This means that the classical lack of Weyl invariance in the dilaton coupling can be compensated by a one-loop contribution arising from the couplings to $g_{\mu\nu}$ and $B_{\mu\nu}$. To see this explicitly, one can compute the beta-functions for the two-dimensional field theory (13.23). In the presence of the dilaton coupling, its best to look at the breakdown of Weyl invariance as seen by $\langle T^\alpha{}_\alpha \rangle$. There are three different kinds of contributions that the stress-tensor can receive, related to the three different spacetime fields. Correspondingly, we define three different beta functions,

$$\langle T^\alpha{}_\alpha \rangle = -\frac{1}{2\alpha'}\beta_{\mu\nu}(g)h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{i}{2\alpha'}\beta_{\mu\nu}(B)\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2}\beta\phi R^{(2)},$$

with

$$\beta_{\mu\nu}(g) = \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\phi - \frac{\alpha'}{4}H_{\mu\lambda\kappa}H_\nu{}^{\lambda\kappa},$$

$$\beta_{\mu\nu}(B) = -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha'\nabla^\lambda H_{\lambda\mu\nu}, \tag{13.25}$$

$$\beta(\phi) = -\frac{\alpha'}{2}\nabla^2\phi + \alpha'\nabla_\mu\phi\nabla^\mu\phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda}.$$

Now, a consistent background of string theory must preserve Weyl invariance, and so we must require $\beta_{\mu\nu}(g) = \beta_{\mu\nu}(B) = \beta(\phi) = 0$, i.e. conformal invariance implies the above equations for $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ. In the next section we will look at these actions in the low-energy limit.

13.1.3 Low Energy Effective Action for the Bosonic String Theory

Recall, or scan ahead to (13.46), that the mass spectrum for an open string with Dirichlet boundary conditions is given by

$$M^2 = \frac{1}{\alpha'}(N - 1).$$

Now, by low energies, E, we mean energies much smaller than α', i.e. $E << \alpha'^{\ddagger}$, which is equivalent to fixing E and sending $\alpha' \mapsto 0$. In this limit massive modes decouple and only the massless modes are of importance. In this regime the interactions between the massless modes are described the Einstein equations coupled to matter and, for example, the low-energy effective actions for the type II superstring theories are given by supergravity (SUGRA) actions[§]. We will begin with the low energy effective action for the bosonic string theory.

As was mentioned previously, the β equations, $\beta_{\mu\nu}(g) = \beta_{\mu\nu}(B) = \beta(\phi) = 0$, can be viewed as the equations of motion for the background spacetime, comprising of the three fields, through which the string propagates. Now, we would like to write down an action which is a function of the three fields, $S = S(g, B, \phi)$, such that the variation of this action with respect to each field yields one of the β equations. The action is called the low-energy effective action for our bosonic string theory. Also mentioned before, when we do this for the superstring theories we will see that these effective actions are precisely the corresponding SUGRA actions, i.e. the low-energy effective action for type IIB superstring theory is given by the type IIB SUGRA action.

The $D = 26$-dimensional low-energy effective action for the bosonic string theory is given by

$$S = \frac{1}{2\kappa_0^2} \int d^{26}X \sqrt{-g} \, e^{-2\phi} \left(R + 4\left(\partial\phi\right)^2 - \frac{1}{12}|H_{(3)}|^2 \right), \qquad (13.26)$$

where κ_0 is a constant, $|H_{(3)}|^2 \equiv H_{\mu\nu\lambda}H^{\mu\nu\lambda}$ and R is the Ricci scalar. The action (13.26) governs the low-energy dynamics of the spacetime fields. The caveat low-energy refers to the fact that we worked with the one-loop beta functions only which requires large spacetime curvature.

To see that this is the indeed low-energy effective action for the bosonic string theory consider that varying this action with respect to the graviton field, Kolb-Ramond field and the dilaton field gives

$$\delta S = \frac{1}{2\kappa_0^2\alpha'} \int d^{26}X \sqrt{-g} \, e^{-2\phi} \Big[\delta g_{\mu\nu}\beta^{\mu\nu}(g) - \delta B_{\mu\nu}\beta^{\mu\nu}(B)$$

$$- \left(2\delta\phi + \frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu} \right)\left(\beta^\lambda{}_\lambda(g) - 4\beta(\phi) \right) \Big]. \qquad (13.27)$$

And so, we see that the action given in (13.26) does indeed reproduce the desired β functions upon variation, which was the requirement.

[‡]This implies that we cannot excite the massless modes to create massive states.

[§]Supergravity is a supersymmetric theory of gravity which has both gravitons and gravitinos.

Einstein Frame vs. String Frame

The term in the Lagrangian, (13.26), of the form

$$R + 4(\partial\phi)^2 - \frac{1}{12}|H_{(3)}|^2,$$

describes the Einstein equations coupled to matter. Usually, however, the Einstein equations follow from the Einstein-Hilbert action,

$$S_E = \int \sqrt{-g} \left(R + \text{Matter Term} \right),$$

but our action has

$$S_S = \int \sqrt{-g} \, e^{-2\phi} \left(R + \text{Matter Term} \right).$$

The factor of $e^{-2\phi}$ reflects the fact that the action has been computed at tree level in string perturbation theory. Also, due to this overall factor, the kinetic terms are not canonically normalized.

We can put the action, (13.26), in the familiar Einstein-Hilbert form by redefining the fields. First, let us distinguish between the constant part of the dilaton field ϕ, which we denote by ϕ_0, and the part which varies, which we denote by $\tilde{\phi}$. As we saw before, the constant part of the dilaton field is defined by

$$\langle e^{-2\phi} \rangle = g_s, \tag{13.28}$$

while $\tilde{\phi}$ is defined by

$$\tilde{\phi} = \phi - \phi_0. \tag{13.29}$$

Next, we define a new metric, $\tilde{g}_{\mu\nu}$, by

$$\tilde{g}_{\mu\nu} = e^{-\frac{4\tilde{\phi}}{D-2}} g_{\mu\nu}, \tag{13.30}$$

where D represents the dimension of the background spacetime in which our theory is defined, i.e. $D = 26$ for the bosonic theory or $D = 10$ for the superstring theories. Note that this isnt to be thought of as a coordinate transformation or symmetry of the action. Its merely a relabeling, a mixing-up, of the fields in the theory. We could make such redefinitions in any field theory. Typically, we choose not to because the fields already have canonical kinetic terms. The point of the (Weyl) transformation, (13.30), is to get the fields in (13.26) to have canonical kinetic terms as well.

Now, in terms of the new rescaled metric, $\tilde{g}_{\mu\nu}$, the Ricci scalar, R, becomes[‡]

$$\tilde{R} = \left(R - 2(D-1)\nabla^2\left(-2\phi/(D-2)\right) - (D-2)(D-1)\partial_\mu\left(-2\phi/(D-2)\right)\partial^\mu\left(-2\phi/(D-2)\right)\right).$$
(13.32)

Restricting $D = 26$, the low-energy effective bosonic action, (13.26), can be written as

$$S = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{-\tilde{g}}\left(\tilde{R} - \frac{1}{6}\left(\partial\tilde{\phi}\right)^2 - \frac{1}{12}e^{-\tilde{\phi}/3}|H_{(3)}|^2\right),$$
(13.33)

where κ is constant which is defined so as to make the previous action, (13.26), reduce to the one above, (13.33). Note that the kinetic terms are now canonical. Also, notice that there is no potential term for the dilaton, and therefore nothing that dynamically sets its expectation value in the bosonic string. However, there do exist backgrounds of the superstring in which a potential for the dilaton develops, fixing the string coupling constant. Finally, notice that this new form of the action does indeed correspond with the form of the Einstein-Hilbert action.

The original metric $g_{\mu\nu}$ is usually called the string metric or sigma-model metric. It is the metric that strings see, as reflected in the action (13.2). In contrast, $\tilde{g}_{\mu\nu}$ is called the Einstein metric. Of course, the two actions, (13.26) and (13.33), describe the same physics: we have simply chosen to package the fields in a different way in each. The choice of metric $g_{\mu\nu}$ or $\tilde{g}_{\mu\nu}$ is referred to as a choice of frame: the string frame, or the Einstein frame. Also, we will denote the string frame metric, $g_{\mu\nu}$, by

$$g_{\mu\nu}^S \equiv g_{\mu\nu},$$
(13.34)

while we denote the Einstein frame metric, $\tilde{g}_{\mu\nu}$, by

$$g_{\mu\nu}^E \equiv \tilde{g}_{\mu\nu}.$$
(13.35)

As a final remark, note that the possibility of defining two metrics really arises because we have a massless scalar field ϕ in the game. Whenever such a field exists, theres nothing to stop us measuring distances in different ways by including ϕ in our ruler. Said another way, massless scalar fields give rise to long range attractive forces

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[‡]The preceeding expression for $\tilde{R}$ follows from the fact that if two metrics, $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$, are related to each other via a general conformal transformation, $\tilde{g}_{\mu\nu} = e^{-2\omega}g_{\mu\nu}$, then there corresponding Ricci scalars are related via

$$\tilde{R} = \left(R - 2(D-1)\nabla^2\omega - (D-2)(D-1)\partial_\mu\omega\partial^\mu\omega\right).$$
(13.31)

Now, simply replace ω with $-2\phi/(D-2)$ to get (13.32).

which can mix with gravitational forces and violate the principle of equivalence. Ultimately, if we want to connect to Nature, we need to find a way to make ϕ massive. Such mechanisms exist in the context of the superstring.

We will now construct low-energy effective actions, or rather SUGRA actions, for some of the superstring theories.

13.1.4 Low Energy Effective Action for the Superstring Theories

We have previously looked at the massless spectrum of the four superstring theories: Het $E_8 \times E_8$, Het $SO(32)$, Type IIA and Type IIB. Namely, each of the four theories contain the graviton field, the Kolb-Ramond field and the dilaton along as did the bosonic theory along with other massless fields depending upon which superstring theory you are in. For each, the low-energy effective action describes the dynamics of these fields in $D = 10$ dimensional spacetime. It naturally splits up into three pieces,

$$S = S_1 + S_2 + S_F. \tag{13.36}$$

Here, S_F describes the spacetime fermionic interactions which we will not bother with here. Thus, we will set $S_F = 0$.

The S_1 part of the action is the same for all superstring theories, and is (alomst!) the same as the low-energy effective action we had before for the bosonic theory. Namely, we have that

$$S_1 = \frac{1}{2\kappa_0^2} \int d^{10}X \sqrt{-g}\, e^{-2\phi} \left(R - \frac{1}{2}|\tilde{H}_{(3)}|^2 + 4(\partial\phi)^2 \right). \tag{13.37}$$

There is one small difference, which is that the field $\tilde{H}_{(3)}$ that appears here for the heterotic low-energy effective action is not quite the same as the original $H_{(3)}$; well explain this further shortly.

The second part of the action, S_2, describes the dynamics of the extra fields which are specific to each different theory. We will now go through the four theories in turn, explaining S_2 in each case.

- Type IIA: First, for type IIA theory the $\tilde{H}_{(3)}$ appearing in (13.37) is given by $\tilde{H}_{(3)} \equiv H_{(3)} = dB_{(2)}$ just like for the bosonic effective action. Also, in the type IIA theory we have seen that there exists the antisymmetric fields $C_{(1)}$ and $C_{(3)}$. The dynamics of these field is governed by the so-called Ramond-Ramond part of the action which is written as

$$S_2 = -\frac{1}{4\kappa_0^2} \int d^{10}X \left\{ \sqrt{-g}\left(|F_{(2)}|^2 + |\tilde{F}_{(4)}|^2\right) + B_{(2)} \wedge F_{(4)} \wedge F_{(4)} \right\}, \tag{13.38}$$

  where $F_{(2)} = dC_{(1)}$, $F_{(4)} = dC_{(3)}$ and $\tilde{F}_{(4)} = F_{(4)} - C_{(1)} \wedge H_{(3)}$. Also, note that the last term in the action, (13.38), is called a Chern-Simons term.

- Type IIB: Once again, for type IIB $\tilde{H}_{(3)} \equiv H_{(3)}$. For type IIB theory we have the antisymmetric fields $C_{(0)}$, $C_{(2)}$, and $C_{(4)}$. The dynamics of these fields is given by

$$S_2 = -\frac{1}{4\kappa_0^2} \int d^{10}X \left\{ \sqrt{-g}\left(|F_{(1)}|^2 + |\tilde{F}_{(3)}|^2 + \frac{1}{2}|\tilde{F}_{(5)}|^2 \right) + C_{(4)} \wedge H_{(3)} \wedge F_{(3)} \right\},$$

(13.39)

with $F_{(1)} = dC_{(0)}$, $F_{(3)} = dC_{(2)}$ and $F_{(5)} = dC_{(4)}$ along with $\tilde{F}_{(3)} = F_{(3)} - C_{(0)} \wedge H_{(3)}$ and $\tilde{F}_{(5)} = F_{(5)} - \frac{1}{2}C_{(2)} \wedge H_{(3)} + \frac{1}{2}B_{(2)} \wedge F_{(3)}$. Along with the above action, (13.39), one must also impose that the field $\tilde{F}_{(5)}$ be self-dual, i.e. we need that

$$\tilde{F}_{(5)} = \star\tilde{F}_{(5)}.$$

(13.40)

And so, strictly speaking, one should say that the low-energy dynamics of type IIB theory is governed by the equations of motion that we get from the action, supplemented with this self-duality requirement.

- Heterotic: Both heterotic theories have just one further massless bosonic ingredient: a non-Abelian gauge field strength $F_{(2)}$, with gauge group $SO(32)$ or $E_8 \times E_8$. The dynamics of this field is simply the Yang-Mills action in ten dimensions,

$$S_2 = \frac{\alpha'}{8\kappa_0^2} \int d^{10}X \sqrt{-g}\, Tr\left(|F_{(2)}|^2 \right).$$

(13.41)

The one remaining subtlety is to explain what $\tilde{H}_{(3)}$ means in (13.37): it is defined as $\tilde{H}_{(3)} = dB_{(2)} - \alpha'\omega_{(3)}/4$ where $\omega_{(3)}$ is the Chern-Simons three form constructed from the non-Abelian gauge field $A_{(1)}$

$$\omega_{(3)} = Tr\left(A_{(1)} \wedge dA_{(1)} + \frac{2}{3}A_{(1)} \wedge A_{(1)} \wedge A_{(1)} \right)$$

(13.42)

The presence of this strange looking combination of forms sitting in the kinetic terms is tied up with one of the most intricate and interesting aspects of the heterotic string, known as anomaly cancelation.

The low-energy effective actions that we have written down here probably look a little arbitrary. But they have very important properties. In particular, the full action superstring of the Type II theories are invariant under $N = 2$ spacetime supersymmetry (that means 32 supercharges), hence the relation to $N = 2$ SUGRA. They are the unique actions with this property. Similarly, the heterotic superstring actions are invariant under $N = 1$ supersymmetry and, crucially, do not suffer from anomalies.

We will now see how the background fields in the type II superstring theories behave under T-duality. By defining there transformations we can then see how solutions to the type II SUGRA actions behave under T-duality. We then introduce a new duality, S-duality, and see how the solutions behave under this duality.

13.2 T-Duality on a Curved Background

Last week we saw that T-duality maps a theory compactified on a circle, S_R^1, of radius R to another theory compactified on a circle, $S_{\tilde{R}}^1$, of radius $\tilde{R} = \alpha'/R$. For example, upon compactification, the two type II theories and the two heterotic are equivalent via the T-duality map,

$$[IIA]_R \xleftrightarrow{\;T\;} [IIB]_{\tilde{R}} \,, \tag{13.43}$$

$$[Het\ SO(32)]_R \xleftrightarrow{\;T\;} [Het\ E_8 \times E_8]_{\tilde{R}} \,. \tag{13.44}$$

We now want to see how T-duality acts on the background fields present in a string theory. In order to do this let us breifly review the types of background fields which are present in the string theories.

Recall from ealier that the mass spectrum for the quantized closed bosonic string is given by

$$M_{\text{closed}}^2 = \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right), \tag{13.45}$$

where N the left-moving level, while $\tilde{N}$ denotes the right-moving level. Along with the closed string spectrum there is the open string spectrum. However, in this case - the open string - there are two different types of theories; the open string with Neumann boundary conditions (N) and the open string with Dirichlet boundary conditions (N). Corresponding to quantized Neumann strings there is the mass spectrum given by

$$M_{\text{open},N}^2 = \frac{1}{\alpha'}\left(N - 1\right), \tag{13.46}$$

whereas for the Dirichlet we have

$$M_{\text{open},D}^2 = \left(\frac{l}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}\left(N - 1\right). \tag{13.47}$$

Note that $l/2\pi\alpha'$ gives the energy of a string which is streched between two D-branes.

From (13.45), we have seen that the massless spectrum of the closed strings consists of a dilaton, ϕ, a graviton, $g_{\mu\nu}$, and an antisymmetric rank 2 tensor, $B_{\mu\nu}$, known as the Kolb-Ramond field. The massless spectrum of open (N) strings consists of a photon, A_μ, while the massless spectrum of a string that ends on a Dp-brane[‡] consists of a vector field, A_m ($m = 0, 1, 2..., p$), and $(25 - p)$ scalar fields. Note that these scalar fields describe position of the Dp-brane.

[‡]Recall that this implies that the string's endpoints are confined to the $p + 1$-dimensional world-volume of the D-brane.

Finally, there is the string coupling constant, which we denote by g_s. This constant is not a new object, but rather it is given by the expectation value of the dilaton field. Namely, we have that

$$g_s = \left\langle e^\phi \right\rangle. \tag{13.48}$$

Now that we have reviewed massless spectrum for the bosonic string theory, we will next review the spectrum of the superstring theories.

We have seen that there exists five different superstring theories: Type I, Type IIA, Type IIB, Het $SO(32)$, and Het $E_8 \times E_8$. For this chapter we will basically only focus on Type II superstring theories and thus we will only review the spectrum of these two theories.

For the type IIA superstring theory, the bosonic sector of the massless spectrum consists of the graviton, $g_{\mu\nu}$, the dilaton, ϕ, and the Kolb-Ramond field, $B_{\mu\nu}$, for the $NS-NS$ sector and the antisymmetric gauge fields, $C_\mu^{(1)}$ and $C_{\mu\nu\lambda}^{(3)}$ for the $R-R$ sector. Note that the superscript on the antisymmetric gauge fields corresponds to the degree of the corresponding differential form, i.e. for $C_\mu^{(p)}$ the corresponding form would be a p-form. While for the type IIB superstring theory, the massless bosonic sector consists of, once again, the graviton, $g_{\mu\nu}$, the dilaton, ϕ, and the Kolb-Ramond field, $B_{\mu\nu}$, for the $NS-NS$ sector and the antisymmetric gauge fields $C^{(0)}$, $C_{\mu\nu}^{(2)}$, and $C_{\kappa\lambda\mu\nu}^{(4)+}$, for the $R-R$ sector. Here by $+$ we denote the fact that the field strength associated to $C^{(4)+}$, $dC^{(4)+}$, is self-dual, i.e. the field strength, $dC^{(4)+}$, is invariant under the Hodge map. The fields in the NS-NS sector - namely the graviton, dilaton, and the Kolb-Ramond field - couple naturally to perturbative strings, while the antisymmetric fields, i.e. the fields in the R-R sector, couple to Dp-branes. For example, fundamental strings couple to the Kolb-Ramond field via

$$\int_\Sigma B_{\mu\nu} \; dx^\mu \wedge dx^\nu, \tag{13.49}$$

where Σ is the worldsheet of the string, while the antisymmetric field $C^{(p+1)}$ couples to a Dp-brane via

$$\int_\mathcal{M} C^{(p+1)}_{\mu_1\mu_1\cdots\mu_{p+1}} \; dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_{p+1}}, \tag{13.50}$$

where $\mathcal{M}$ is the $(p+1)$-dimensional worldvolume of the Dp-brane. Thus, since the type IIA theory has antisymmetric fields of type $C^{(1)}$, $C^{(3)}$, and their Hodge duals $C^{(5)}$, $C^{(7)}$, and $C^{(9)}$, we see that there are also $D0$-branes, $D2$-branes, $D4$-branes, $D6$-branes, and $D8$-branes living in the type IIA theory. While, from the exact same reasoning, we see that there exist $D(-1)$-branes, $D(1)$-branes, $D(3)$-branes, $D(5)$-branes, $D(7)$-branes, and $D(9)$-branes living in the type IIB theory.

In addition, strings can carry momentum. This corresponds to gravitational waves, which we denote by W. As before, these gravitational waves have their own Hodge dual objects, which turns out to be kaluza-Klein monopoles (KK).

So, to recap, both type II theories have fundamental strings (F1) and their duals, the solitonic 5-branes (NS5), along with gravitational waves and their duals, the Kaluza-Klein charges (KK). However, type IIA string theory has $D0$-branes, $D2$-branes, $D4$-branes, $D6$-branes, and $D8$-branes, while type IIB string theory has $D(-1)$-branes, $D1$-branes, $D3$-branes, $D5$-branes, $D7$-branes, and $D9$-branes, i.e. we have that

	Background Fields								
Type IIA	F1	NS5	W	KK	D0	D2	D4	D6	D8
Type IIB	F1	NS5	W	KK	D(-1)	D1	D3	D5	D7	D9

Now that we have reviewed the background fields in the type II theories, let us see how T-duality acts on them. In particular, we want to see if T-duality equates string theories in generalized backgrounds with $U(1)$ isometries (i.e. not necessarily compactified on a circle) to one another, as we saw before with the theories compactified on a circle. To proceed, consider the following action:

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \; \sqrt{h} \left[\left(h^{\alpha\beta} g_{\mu\nu} + i\frac{\epsilon^{\alpha\beta}}{\sqrt{h}} B_{\mu\nu} \right) \partial_\alpha X^\mu \partial_\beta X^\nu + \alpha' R^{(2)} \phi \right], \qquad (13.51)$$

where h and $R^{(2)}$ are the worldsheet metric and curvature form, respectively, g is the background metric and B is a potential for the torsion 3-form, $H = dB$. This action is invariant under the transformation given by

$$\delta X^\mu = \kappa V^\mu, \qquad (13.52)$$

provided V^μ is a Killing vector, the Lie derivative of B is a total derivative and the dilaton is invariant,

$$\mathcal{L}_V g_{ij} = V_{i;j} - V_{j;i} = 0,$$

$$\mathcal{L}_V B = \imath_V(dB) + d\imath_V(B) = d(k + \imath_V B), \qquad (13.53)$$

$$\mathcal{L}_V \phi = V^\mu \partial_\mu \phi = 0 \,.$$

Now, let us choose coordinates $\{X^\mu\} = \{x, x^i\}$ such that the fields ϕ, B and g are independent of x and the isometry acts by translation along x. Next we add a Lagrange

multiplier, λ, to the action in order to force the connection to be flat. The new action, in the conformal gauge and excluding the dilaton term, is given by

$$S = \frac{1}{2\pi\alpha'} \int dz d\bar{z} \left[\left(g_{\mu\nu} + B_{\mu\nu} \right) \partial X^\mu \bar{\partial} X^\nu + \left(J_V - \partial\lambda \right) \bar{A} + \left(\bar{J}_V + \bar{\partial}\lambda \right) A + V \cdot V A \bar{A} \right],$$
(13.54)

where $J_V = (V + k)_\mu \partial X^\mu$ and $\bar{J}_V = (V - k)_\mu \bar{\partial} X^\mu$ are the currents of the Noether current associated to the aforementioned symmetry. Integrating out the gauge fields, A and $\bar{A}$, gives the following action

$$\tilde{S} = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \ \sqrt{h} \left[\left(h^{\alpha\beta} \tilde{g}_{\mu\nu} + i\frac{\epsilon^{\alpha\beta}}{\sqrt{h}} \tilde{B}_{\mu\nu} \right) \partial_\alpha X^\mu \partial_\beta X^\nu + \alpha' R^{(2)} \tilde{\phi} \right],$$
(13.55)

where, in terms of the $\{x, x^i\}$ coordinates,

$$\tilde{g}_{xx} = \frac{1}{g_{xx}}, \qquad \tilde{g}_{xi} = \frac{B_{xi}}{g_{xx}}, \qquad \tilde{g}_{ij} = g_{ij} - \frac{g_{xi}g_{xj} - B_{xi}B_{xj}}{g_{xx}},$$

$$\tilde{B}_{xi} = \frac{g_{xi}}{g_{xx}}, \qquad \tilde{B}_{ij} = B_{ij} + \frac{g_{xi}B_{xj} - B_{xi}g_{xj}}{g_{xx}}$$
(13.56)

$$\tilde{\phi} = \phi - \frac{1}{2} ln(g_{xx}) \,.$$

Thus, we have arrived at an identical theory, see (13.51), with the background fields replaced by the new ones. And so, we see that T-duality maps the theory defined by S (13.51) to the theory defined by $\tilde{S}$ (13.55),

$$g \overset{T}{\longleftrightarrow} \tilde{g} \,,$$
(13.57)

$$\phi \overset{T}{\longleftrightarrow} \tilde{\phi} \,,$$
(13.58)

$$B \overset{T}{\longleftrightarrow} \tilde{B} \,.$$
(13.59)

The truly remarkable thing is that even though the two metrics, g and $\tilde{g}$, of the different theories, S and $\tilde{S}$, are different, we still end up with the same physics! Thus, we indeed see that T-duality relates different theories living in backgrounds which posses $U(1)$ isometries.

One should note that there is another way to derive the transformations rules for the background fields, (13.56). This is done by putting the metric, of your theory, into the form given by

$$ds^2 = g_{xx} \left(dx + A_i dx^i \right)^2 + \bar{g}_{ij} dx^i dx^j,$$
(13.60)

where $A_i = g_{xi}/g_{xx}$. Then, under the T-duality mapping we see that the above theory, defined by (13.60), is T-dual to the theory defined by

$$ds^2 = \tilde{g}_{xx}\left(dx + \tilde{A}_i dx^i\right)^2 + \tilde{\bar{g}}_{ij}dx^i dx^j, \tag{13.61}$$

where

$$\tilde{g}_{xx} = \frac{1}{g_{xx}}, \qquad \tilde{A}_i = B_{xi}, \qquad \tilde{B}_{xi} = A_i, \qquad \tilde{B}_{ij} = B_{ij} - 2A_{[i}B_{j]x}, \tag{13.62}$$

$$\tilde{\bar{g}}_{ij} = \bar{g}_{ij}, \qquad \tilde{\phi} = \phi - \frac{1}{2}ln(g_{xx}), \qquad e^{-2\tilde{\phi}} = e^{-2\phi}g_{xx}.$$

As a final remark to T-dualities, note that the above transformations, (13.56)/(13.62), where for the NS-NS fields only. Thus, we still need to specify how the R-R fields (antisymmetric fields) transform under T. This is as follows (here written in the $\{x, x^i\}$ coordinates):

$$C_{\mu_1\cdots\mu_{p+1}} \overset{T}{\longmapsto} C_{\mu_1\cdots\mu_{p+1}x} \qquad \text{if } x \notin \{x^{\mu_1}, ..., x^{\mu_{p+1}}\}, \tag{13.63}$$

$$C_{x\mu_1\cdots\mu_{p+1}} \overset{T}{\longmapsto} C_{\mu_1\cdots\mu_{p+1}}. \tag{13.64}$$

Although, as we have seen, T-duality is a symmetry of type IIB superstring theories it is not the only one. Another duality of type IIB theory, namely S-duality, is the topic of the next section.

13.3 S-Duality on the Type IIB Superstring Theories

S-duality has the effect of sending the dilaton, ϕ, to minus itself, i.e. $S : \phi \mapsto -\phi$. This has the effect of transforming the string coupling constant, $g_s = \langle e^\phi \rangle$, to $S : g_s \mapsto \langle e^{-\phi} \rangle = 1/g_s$. Thus, if we have a theory with small coupling constant, which is required for perturbation theory, then the S-dual theory has a large coupling constant, which implies that S-duality is a non-perturbative symmetry. In particular, we have that

$$[IIB] \overset{S}{\longleftrightarrow} [IIB], \tag{13.65}$$

$$[Het\ SO(32)] \overset{S}{\longleftrightarrow} [\text{Type I}]. \tag{13.66}$$

Through S-duality one can get a handle on the strong coupling limit of three of the superstring theories. It turns out that that the strong coupling limit of type I and Het $E_8 \times E_8$ are more exotic in nature. We will see later on that in the strong coupling

limit of these two theories we get an eleven-dimensional theory, the M-theory. More on this M-theory will come later on, and so, for now, let us see how S-duality acts on the background fields of the type IIB superstring theory.

To begin, S-duality leaves the Einstein frame metric, g_E (see (13.35)), invariant. While mapping the Kolb-Ramond field, B_μ, into the antisymmetric field of degree two, $C^{(2)}_{\mu\nu}$ and leaving invariant the antisymmetric field $C^{(4)+}_{\mu\nu\lambda k}$ of degree four. Also, as we have previously seen, S-duality maps $\phi \mapsto -\phi$ which implies that $e^\phi \mapsto e^{-\phi}$.

Now, how does S-duality act on the D-branes present in our type IIB theory? Recall that that D-branes are massive objects which, via Einstein's theory of relativity, implies that curve the background spacetime through which they move. This, in turn, implies there must exist solutions of IIA/IIB supergravity (SUGRA) which desribe the long-range fields produced by the D-brane. Thus, we need to see how S-duality acts on the solutions in order to get an understanding of how it acts on the branes.

13.3.1 Brane Solutions of Type IIB SUGRA

For our purposes, the relevant bosonic part of the type IIB SUGRA action is given by

$$S = \frac{1}{128\pi^7 g_s^2(\alpha')^4} \int d^{10}x \ \sqrt{-g} \left[e^{-2\phi}\left(R + 4(\partial\phi)^2 - \frac{1}{12}|H_{(3)}|^2 \right) - \sum_p \frac{1}{2(p+2)!}|F_{(p+2)}|^2 \right],$$

(13.67)

where $H_{(3)} = dB_{(2)}$, $F_{(p+2)} = dC_{(p+1)}$ and where the sum over p makes sense for type IIB theory, i.e. we should only have forms of even degree for the type IIB theory. Note that we have omitted several of the other bosonic terms in the action since they are not important for the solutions we will be describing.

The equations of motion resulting from (13.67) have solutions which have the interpretation of describing the long range fields produced by fundamental strings (F1), Dp-branes and the solitonic 5-branes (NS5). These solutions are given by[‡]

$$ds^2 = H_i^\alpha \left[H^{-1} \underbrace{\left(-dt^2 + dx_1^2 + \cdots + dx_p^2 \right)}_{\equiv \ ds^2(\mathbb{E}^{(p,1)})} + \underbrace{\left(dx_{p+1}^2 + dx_{p+2}^2 + \cdots + dx_9^2 \right)}_{\equiv \ ds^2(\mathbb{E}^{(9-p)})} \right]$$

$$e^\phi = H_i^\beta \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (13.68)$$

$$A^{(p+1)}_{01\cdots p} = H_i^{-1} - 1 \qquad (\text{``electric''}), \qquad \text{or} \qquad F_{8-p} = \star dH_i \qquad (\text{``magnetic''}),$$

where α and β are numbers, $A^{(p+1)}_{01\cdots p}$ is either the R-R potential $C^{(p+1)}$ or the NS-NS two-form $B^{(2)}$ depending on the solution, $\star$ is the Hodge dual map and the subscript $i =$

[‡]See G.T. Horowitz and A. Strominger, *Black strings and p-branes*, Nucl. Phys. B360 (1991) 197.

$\{F1, p, NS5\}$ on H_i denotes which solution, fundamental string, Dp-brane or solitonic branes, repsectively, we are describing. For the case of a Dp-brane solution one takes $\alpha = 1/2$ and $\beta = (3 - p)/4^{\S}$. Thus, the Dp-brane solution is given by

$$ds^2 = H_i^{-1/2} ds^2(\mathbb{E}^{(p,1)}) + H_i^{1/2} ds^2(\mathbb{E}^{(9-p)})$$

$$e^{-2\phi} = H_i^{\frac{p-3}{2}} \tag{13.69}$$

$$A_{01\cdots p}^{(p+1)} = H_i^{-1} - 1, \qquad \text{or} \qquad F_{8-p} = \star dH_i \,.$$

In order for (14.34) to be a solution we must have that H_i be a harmonic function on $\mathbb{E}^{(9-p)}$, i.e. we must impose that

$$\partial_\alpha \partial^\alpha H_i = 0, \tag{13.70}$$

for $\alpha \in \{P + 1, ..., 9\}$.

Now, if we take r to be the distance from the origin of $\mathbb{E}^{(9-p)}$, then by defining H_i to be

$$H_i = 1 + \frac{Q_i}{r^{7-p}}, \tag{13.71}$$

yields the long-range fields of N infinite parallel planar p-branes near the origin. The constant part was chosen equal to one in order the solution to be asymptotically flat. The different values for Q_i are given as follows: for fundamental strings we take $Q_{F1} = d_1 N g_s^2 l_s^6$, for Dp-branes we take $Q_{Dp} = d_p N g_s l_s^{7-p}$ and for solitonic branes we take $Q_{NS5} = N l_s^2$, where we are using

$$d_p = (2\sqrt{\pi})^{5-p} \, \Gamma\left(\frac{7-p}{2}\right). \tag{13.72}$$

Apart from the previous solutions for fundamental strings, Dp-branes and solitonic branes, there are also purely gravitational solutions. Namely there is a solution which describes the long range field produced by momentum modes carried by a string, i.e. a gravitational wave solution. This solution is given by

$$ds^2 = -K^{-1}dt^2 + K\left(dx_1 - (K^{-1} - 1)dt\right)^2 + dx_2^2 + \cdots + dx_9^2, \tag{13.73}$$

where $K = 1 + Q_K/r^6$ is again a harmonic function and $Q_K = d_1 g_s^2 N \alpha'/R^2$. Now that we have a Dp-brane solution to the type IIB SUGRA action, namely (14.34), let us see how T- and S-duality acts on the solution.

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§For fundamental strings one takes $\alpha = 0$ and $\beta = -1/2$, while for solitonic branes one takes $\alpha = 1$ and $\beta = 1/2$.

13.3.2 Action of the Brane Solutions Under the Duality Maps

T-Duality

We want to see how the Dp-brane solution of the type IIB SUGRA action, see (13.67), which is given by

$$ds^2 = H_i^{-1/2}ds^2(\mathbb{E}^{(p,1)}) + H_i^{1/2}ds^2(\mathbb{E}^{(9-p)})$$

$$e^{-2\phi} = H_i^{\frac{p-3}{2}} \tag{13.74}$$

$$C_{01\cdots p}^{(p+1)} = H_i^{-1} - 1\,,$$

transforms under a T-duality map. So, to begin, assume that the x_p coordinate is periodic. Next, let us rewrite the metric given above, (13.74), as

$$ds^2 = H^{-1/2}dx_p^2 + H^{-1/2}\Big(- dt^2 + \cdots + dx_{p-1}^2 \Big) + H^{1/2}\Big(dx_{p+1}^2 + \cdots + dx_9^2 \Big). \tag{13.75}$$

Now, comparing this metric with

$$ds^2 = g_{xx}\Big(dx + A_i dx^i \Big)^2 + \overline{g}_{ij}dx^i dx^j,$$

we see that $g_{x_p x_p} = H^{-1/2}$, $A_i = 0$, and $\overline{g}_{ij} = H^{1/2}\delta_{ij}$, where δ_{ij} is the Kronecker delta. Now, reading off from the rules for the transformation of the background fields under a T-duality map, namely (13.62), we see that

$$\tilde{g}_{x_p x_p} = H^{1/2},$$

$$e^{-2\tilde{\phi}} = e^{-2\phi}g_{x_p x_p} = H^{\frac{p-3}{2}}H^{-1/2},$$

$$\tilde{B}_{\mu\nu}^{(2)} = 0,$$

$$C_{01\cdots p}^{(p+1)} \mapsto C_{01\cdots p-1}^{(p)} = H^{-1} - 1\,.$$

Thus, the previous solution defined by (13.74) is T-dual to the solution given by

$$ds^2 = H^{-1/2}\Big(- dt^2 + \cdots + dx_{p-1}^2 \Big) + H^{1/2}\Big(dx_p + \cdots dx_9^2 \Big), \qquad (13.76)$$

$$e^{-2\phi} = H^{-\frac{p-4}{2}}, \qquad (13.77)$$

$$B_{\mu\nu}^{(2)} = 0, \qquad (13.78)$$

$$C_{01\cdots p-1}^{(p)} = H^{-1} - 1. \qquad (13.79)$$

From the expression for $e^{-2\tilde{\phi}}$ we see that this new solution to the SUGRA action describes a $D(p-1)$-brane solution[¶], i.e. under a T-dual map we change p to $p-1$. However, if we had T-dualized along a coordinate x_q with $q \neq p$ then we would have arrived at a solution describing a $D(P+1)$-brane. Also, we have that

$$H = 1 + \frac{Q}{r^{7-(p-1)}}. \qquad (13.80)$$

We will now see how S-duality acts on the SUGRA Dp-brane solutions.

S-Duality

In order to see how Dp-brane solutions of the type IIB SUGRA action behave under S-duality let us consider the case for a $D3$-brane. This solution is given by

$$ds^2 = H^{-1/2}\Big(- dt^2 + \cdots + dx_3^2 \Big) + H^{1/2}\Big(dx_4 + \cdots + dx_9^2 \Big), \qquad (13.81)$$

$$e^{-2\phi} = 1 \quad (\Rightarrow \phi = 0), \qquad (13.82)$$

$$C_{0123}^{(4)+} = H^{-1} - 1. \qquad (13.83)$$

Now, as we have seen before, S-duality leaves the metric in the Einstein frame invariant. Thus, we should map our metric, which is implicitely assumed to be in the string frame, into the Einstein frame. This is acheived by acting on the metric ds^2 in the string frame with a Weyl transformation, $e^{\phi/2}$. However, since our solution has $\phi = 0$ we can see that the Weyl transformation is nothing more, in our solution desribed above, than

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[¶]This is due to the fact that, as was stated earlier (see (14.34)), for a Dp-brane solution we require $e^{-2\phi} = H^{(p-3)/2}$ and since this T-dual theory has $e^{-2\tilde{\phi}} = H^{(p-4)/2} = H^{((p-1)-3)/2}$ we can see that it describes a $D(p-1)$-brane.

multiplying by unity, i.e. $ds_E^2 = e^{\phi/2} ds^2 = ds^2$, where ds^2 is as above and ds_E^2 is the metric in the Einstein frame. The remaining fields in the above solution transform under the S-duality map as follows: the dilaton field, ϕ, maps to negative itself, $\phi \mapsto -\phi$[§], the quantity $e^{-2\phi}$ goes to $e^{-2\tilde{\phi}}$ and the antisymmetric field $C_{0123}^{(4)+}$ is invariant under the S-duality mapping. Thus, combining these results we see that the above SUGRA solution for a $D3$-brane is S-dual to the solution defined by

$$ds_E^2 = H^{-1/2}\Big(-dt^2 + \cdots + dx_3^2 \Big) + H^{1/2}\Big(dx_4 + \cdots + dx_9^2 \Big),$$

$$e^{-\tilde{\phi}} = e^{\phi} = 1,$$

$$\tilde{C}_{0123}^{(4)+} = C_{0123}^{(4)+} = H^{-1} - 1 \,,$$

where the subscript E on ds_E^2 is there to remind ourselves that this is the metric in the Einstein frame. Now, in order to compare this S-dual solution to the original $D3$-brane solution we need to transform the metric back to the string frame. This is done by acting on ds_E^2 with the inverse of the Weyl transformation which we used earlier, i.e. we need to multiply the metric ds_E^2 by $e^{-\tilde{\phi}/2}$. Note that here we must use the S-dualized dilaton, $\tilde{\phi} = -\phi$, which amounts to multiplying the metric ds_E^2 by $e^{\phi/2}$. However, since $\phi = 0$ we once again see that changing from the metric in the Einstein frame to the metric in the string frame is nothing more than multiplying by unity. Thus, the $D3$-brane type IIB SUGRA solution is S-dual to the solution given by

$$ds^2 = H^{-1/2}\Big(-dt^2 + \cdots + dx_3^2 \Big) + H^{1/2}\Big(dx_4 + \cdots + dx_9^2 \Big),$$

$$e^{-\tilde{\phi}} = e^{\phi} = 1,$$

$$\tilde{C}_{0123}^{(4)+} = C_{0123}^{(4)+} = H^{-1} - 1 \,.$$

But wait. This is the same solution as the $D3$-brane solution, and so we see that the $D3$-brane solution is self-dual, i.e. S-duality maps the $D3$-brane solution back to itself.

[§]Recall that this implies that the string coupling constant goes to its inverse since $g_s = \langle e^{\phi} \rangle \mapsto \langle e^{-\phi} \rangle = 1/g_s$.

One can see that $NS5$-brane solution, which is defined by:

$$ds^2 = \left(- dt^2 + \cdots + dx_5^2 \right) + H \left(dx_6^2 + \cdots + dx_9^2 \right), \qquad (13.84)$$

$$H_{(3)} = dB_{(2)} = \star dH, \qquad (13.85)$$

$$e^{-2\phi} = H^{-1}, \qquad (13.86)$$

is S-dual to the type IIB SUGRA solution which describes a $D5$-brane. Also, one can show that the $D1$-brane solution is S-dual to the fundamental string ($F1$) solution.

We next explore the relationship between S-duality of type IIA superstring theories and M-theory.

13.4 M-Theory

The term M-theory was introduced by Witten to refer to the "mysterious" or "magical" quantum theory in 11 dimensions whose leading low-energy effective action is eleven-dimensional supergravity. M-theory is not yet fully formulated, but the evidence for its existence is very compelling. It is as fundamental (but not more) as type IIB superstring theory, for example. In this section we will look at the relationship between the strong coupling limit of the type IIA superstring theory and M-theory.

M-theory in low energies is described by eleven-dimensional supergravity. Now, eleven-dimensional SUGRA consists of the metric G_{MN} and the 3-form antisymmetric field A_{MNP}, where $M, N, P = 0, 1, ..., 9, 11$. We will now see that there is a relation between these fields and the fields of type IIA superstring theory. So, note that if we have an M-theory compactified on a circle $S_{x^{11}}^1$ then the dimensional reduction from M-theory to type IIA SUGRA is given by

$$ds_{11}^2 = e^{-2/3\phi} g_{\mu\nu} dx^\mu dx^\nu + e^{4/3\phi} \left(dx_{11} + C_\mu^{(1)} dx^\mu \right)^2, \qquad (13.87)$$

$$A_{\mu\nu k} = C_{\mu\nu k}^{(3)} \qquad (\mu, \nu, k = 0, 1, ..., 9), \qquad (13.88)$$

$$A_{\mu\nu(11)} = B_{\mu\nu}^{(2)}, \qquad (13.89)$$

which are the background fields from type IIA superstring theory. Thus, we see that fields, G_{MN} and A_{MNP}, of eleven-dimensional SUGRA gives us the fields of type IIA superstring theory, and so, in the strong coupling limit, we have that M-theory accounts for the background fields of type IIA superstring theory.

Now, what about the branes in the type IIA superstring theory? Well, first, note that since there is an antisymmetric field, $C^{(3)}_{\mu\nu k}$, of degree 3 in the eleven-dimensional SUGRA then it will couple to a 2-brane, known as the $M2$-brane. Also, since we have the field strength $F^{(4)} = dA^{(3)}$ we get its dual $(\star F)^{(7)} = dA^{(6)}$ and so we also see that there exists $M5$-branes. Now, it can be shown that all of the branes in the type IIA superstring theories follow from these Mp-branes. Thus, we have shown that M-theory has the ability to describe the type IIA superstring theory and, in fact, the claim is much stronger than this. Namely, in the strong coupling limit type IIA superstring theory is really M-theory. But, wait a minute. M-theory is a theory which lives in eleven dimensions, while type IIA superstrings live in ten dimensions. How is this possible?

We will now show that the radius of the eleventh-dimension is proportional to the string coupling constant, g_s. Thus, only for extremely large values of g_s does the eleventh-dimension become visible, i.e. only at strong coupling does the type IIA theory show its extra dimension. Now, in order to prove the previous claim about the radius of x_{11}, which we denote by R_{11}, let us look at $D0$-branes in type IIA superstring theory.

It can be shown that the mass of a $D0$-brane is given by $1/l_s g_s$. Also, it has been proposed that there exists bound states of N $D0$-branes, which have a mass of $N/l_s g_s$. Moreover, these states are BPS states, which means that (up to a factor) their charge under the type IIA one-form field is equal to their mass. Thus, we have a tower of evenly spread massive states, which is usually the sign of a theory compactified on a circle.

To see this, suppose that we have some eleven-dimensional theory which contains gravity and that we compactify this theory on a circle. Then, in eleven-dimensions, the graviton satisfies the mass-energy relation given by

$$m_{11}^2 - p_M^2 = 0,$$

where $M = 0, 1, ..., 9, 11$. However, one should note that in ten dimensions this graviton has a mass given by

$$m_{10}^2 = -p_\mu^2 = p_{11}^2, \tag{13.90}$$

where $\mu = 0, 1, ..., 9$. Also, since the eleventh dimension is periodic we have that (see (12.6))

$$p_{11} = \frac{N}{R_{11}}, \tag{13.91}$$

which implies that

$$m = \frac{N}{R_{11}}. \tag{13.92}$$

So, from (13.92) we see that we do indeed get the tower of states which were mentioned before.

Now, compare the two coefficients that give the distance between two consecutive masses in the tower of states. In the case of $D0$-branes, we have that $m = N/l_s g_s$, whereas from dimensional reduction of the eleven dimensional theory, we have that $m = N/R_{11}$. And so, for the claim to be correct we must have that

$$R_{11} = l_s g_s, \tag{13.93}$$

or that the radius of the compactified dimension is directly proportional to the string coupling constant. Thus, as was stated earlier, even though it comes from an eleven dimensional theory, namely M-theory, perturbative (i.e. in the limit $g_s \to \infty$) type IIA superstring theory is ten-dimensional.

So, to recap, we have seen that in the strong coupling limit we see that type IIA superstring theory transforms into M-theory or, said another way, we can recover type IIA superstring theory by compactifying eleven-dimensional M-theory on a circle. This is not the only superstring theory which we can recover from M-theory. It turns out that by compactifying M-theory on a line segment one recovers the $Het\ E_8 \times E_8$ superstring theory. Also, we have previously seen that we can move from type IIA to type IIB via the T-duality map and then from type IIB type I to by taking states which are invariant under interchanging left-movers with right-movers. From type I we can go $Het\ SO(32)$ via an S-duality map and then from here to $Het\ E_8 \times E_8$ via a T-duality map. Thus, we see that all 5 types of superstring theories are really derived from the same theory, M-theory:
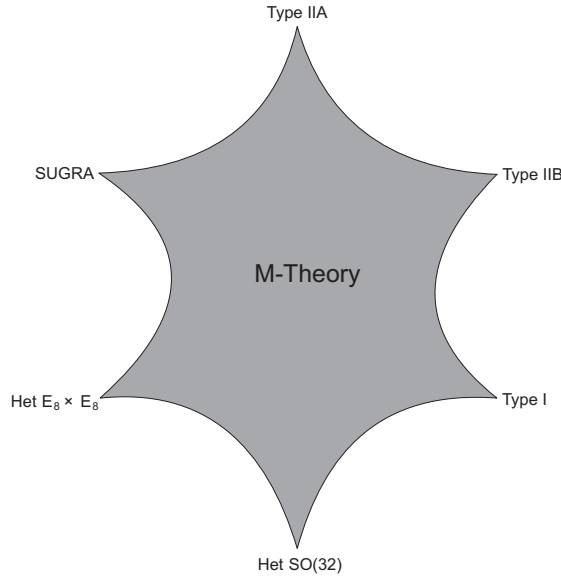
This concludes our discussion of dualities in curved backgrounds and M-theory. In the next chapter we will look at black holes in string theory. This will begin with an overview of the properties of black holes in classical physics and then move into the quantum theory. We will see that a number of puzzles arise in the quantum theory due to Hawking radiation. One we view black holes in terms of branes and strings we will see that some of these puzzles are solved. The chapter will conclude with an introduction to the holographic principle and the AdS/CFT (anti-de Sitter space/Conformal Field Theory) correspondence.

13.5 Exercises

Problem 1

All gravitational theories contain vacuum solutions describing gravitational waves. In this exercise we will derive a solution which describes a gravitational wave traveling in the x_1 direction (Brinkmann (1923)).

Consider the line element

$$ds^2 = dx^+ dx^- + (H(x^i, x^+) - 1)(dx^+)^2 + \sum_{i=2}^{D-1} dx^i dx^i, \qquad (13.94)$$

where $x^\pm = x^1 \pm t$ and $H(x^i, x^+)$ is a function of x^i and x^+. Show that the Einstein equations ($\mu, \nu = \{+, -, i\}$),

$$R_{\mu\nu} = 0, \qquad (13.95)$$

imply

$$\partial^i \partial_i H = 0. \qquad (13.96)$$

Useful formulae:

$$\Gamma^\kappa_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}),$$
$$R_{\mu\nu\kappa}{}^\lambda = \partial_\nu \Gamma^\lambda_{\mu\kappa} + \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\kappa} - (\mu \leftrightarrow \nu), \qquad (13.97)$$
$$R_{\mu\nu} = R_{\mu\kappa\nu}{}^\kappa.$$

Problem 2

Consider a gravitational wave in 10 dimensions traveling along the x_1 direction with H independent of x^+. Then, the metric (13.94) does not depend on x^1 and we may compactify this direction.

(1) T-dualize along x^1 to obtain the solution describing the long range fields produced by a fundamental string,

$$ds^2 = H^{-1}(-dt^2 + (dx^1)^2) + \sum_{i=2}^{9} dx^i dx^i, \qquad (13.98)$$

$$B_{01} = H^{-1} - 1,$$
$$e^{-2\phi} = H.$$

Note that this is a solution of both IIA and IIB supergravity.

(2) In the case of IIB supergravity, S-dualize this solution to obtain the D1-brane
solution

$$ds^2 = H^{-1/2}(-dt^2 + (dx^1)^2) + H^{1/2} \sum_{i=2}^{9} dx^i dx^i,$$

$$C_{01} = H^{-1} - 1, \tag{13.99}$$

$$e^{-2\phi} = H^{-1}.$$

Note that the solutions in (13.98) and (13.99) are in the string frame. Recall
that S-duality acts as follows: (i) the metric in the Einstein frame is invariant,
(ii) $B_{\mu\nu}$ becomes $C_{\mu\nu}$ and vice versa, (iii) $C^+_{\mu_1\mu_2\mu_3\mu_4}$ is invariant, (iv) the dilaton ϕ
becomes $-\phi$. The Einstein frame metric g_E is related to the string frame metric
g_S by $g_E = e^{-\phi/2}g_S$.

(3) Now consider the x^2 direction to be periodic as well and T-dualize along x^2 to
obtain the D2 solution of IIA supergravity,

$$ds^2 = H^{-1/2}(-dt^2 + (dx^1)^2 + (dx^2)^2) + H^{1/2} \sum_{i=3}^{9} dx^i dx^i,$$

$$C_{012} = H^{-1} - 1, \tag{13.100}$$

$$e^{-2\phi} = H^{-1/2}.$$

Problem 3

The M2 brane solution of 11d supergravity is given by

$$ds^2 = H^{-2/3}(-dt^2 + (dx^1)^2 + (dx^2)^2) + H^{1/3} \sum_{i=3}^{10} dx^i dx^i,$$

$$A_{012} = H^{-1} - 1. \tag{13.101}$$

(1) Consider the x^2 direction as the M-theory direction. Show that the M2 solution
reduces to the fundamental string solution (13.98) of IIA supergravity.

(2) Consider the x^3 direction as the M-theory direction. Show that the M2 solution
reduces to the D2 solution (13.100) of IIA supergravity.

14. Black Holes in String Theory and the AdS/CFT Correspondence

Over the course of this manuscript we have seen that string theory has many good properties of a final[‡] theory. For instance, we have seen that one can quantize string theory and that out of string theory pops the Einstein field equations. Also, although not explicitly shown here, the scattering of gravitons around flat spacetimes is well-defined in string theory, while purtabative quantization of gravity is non-renormalizable. Thus, if string theory is to be a consistent quantum theory of gravity then it should also explain non-perturbative processes in gravity. In particular, string theory should explain black holes.

14.1 Black Holes

Black holes, the collapsed remnants of large stars or the massive central cores of many galaxies, represent an arena where a quantum theory of gravity becomes important. The possibility of the existence of black holes was recognized long ago by the great physicist and mathematician Laplace[§], but it wasnt until the Schwarzschild solution in general relativity was put forward that these objects and their truly bizarre properties really came into their own. In recent decades the existence of black holes has been established without doubt from observational evidence. Classically, black holes are remarkably simple objects that can be described by just three properties:

- Mass

- Charge

- Angular Momentum

Then Stephen Hawking made the remarkable discovery that black holes radiate. But this was only the beginning of the story. Black holes have remarkable characteristics that connect them directly it turns out to the science of thermodynamics. Black holes have entropy and temperature, and the laws of thermodynamics have analogs that Hawking and his colleagues dubbed the laws of black hole mechanics.

One of the most dramatic results of Hawkings work was the implication that black holes are associated with information loss. Physically speaking, we can associate information with pure states in quantum mechanics. In ordinary quantum physics, it is not

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[‡]By "final" here we mean a unified theory of the four known fundamental forces. We by no means mean that a unified theory will be the "final" theory of physics.

[§]For a great explanation of Laplace's ideas see "The Large Scale Structure of Space-Time (Cambridge Monographs on Mathematical Physics)" by Hawking and Ellis.

possible for a pure quantum state to evolve into a mixed state. This is related to the unitary nature of time evolution. What Hawking found was that pure quantum states evolved into mixed states. This is because the character of the radiation emitted by a black hole is thermal - its purely random so a pure state that falls into the black hole is emitted as a mixed state. The implication is that perhaps a quantum theory of gravity would drastically alter quantum theory to allow for nonunitary evolution. This is bad because nonunitary transformations do not preserve probabilities. Either black holes destroy quantum mechanics or we have not included an aspect of the analysis that would maintain the missing information required to keep pure states evolving into pure states.

However, it is important to realize that the analysis done by Hawking and others in this context was done using semiclassical methods. That is, a classical spacetime background with quantum fields was studied. Given this fact, the results can not necessarily be trusted.

String theory is a fully quantum theory so evolution is unitary. And it turns out that the application of string theory to black hole physics has produced one of the theories most dramatic results to date. Using string theory, it is possible to count the microscopic states of a black hole and compare this to the result obtained using the laws of black hole mechanics (which state that entropy is proportional to area, $S = A/4G$). It is found that there is an exact agreement using the two methods.

Let us now review the appearance of black holes in general relativity.

14.1.1 Classical Theory of Black Holes

To sum it up: black holes aare solutions of the Einstein equations that posess an event horizon, where the event horizon acts as a one-way membrane - things can fall into the black hole while nothing can escape. Classically, black holes are stable objects, whose mass can only increase as matter (or radiation) crosses the horizon and becomes trapped forever. Quantum mechanically, black holes have thermodynamic properties, and they can decay by the emission of thermal radiation.

To begin, recall that the Einstein equations, in D dimensions without sources, are given by varying the Einstein-Hilbert action

$$S = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g}\ R, \tag{14.1}$$

where G_D is the D-dimensional Newton constant and R is the Ricci scalar. The resulting equations of motion, or Einstein equations, are given by the vanishing of the Einstein tensor, i.e.

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \tag{14.2}$$

or, equivalently (if $D > 2$), $R_{\mu\nu} = 0$. Thus, the solutions are Ricci-flat space-times. Straightforward generalizations are provided by adding electromagnetic fields, spinor fields or tensor fields of various sorts, such as those that appear in supergravity theories. Some of the most interesting solutions describe black holes.

Schwarzchild Black Hole

The simplest example of a black hole solution to Einstein's equaitons in four-dimensions is the Schwarzchild solution. This solution is given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2 d\Omega_2^2. \tag{14.3}$$

The Schwarzchild solution appears to have two singularities, one when $r = 0$ and one when $r = 2m$. Are both of these singularities in the actual physical theory or are they do to a bad choice of coordinates? In order to answer this question we first need a precise definition of a singularity. So, we define a singularity as a point, r_s, in spacetime where, at this point, the metric, ds^2, goes to infinity in all coordinate frames, i.e. there does not exist a coordinate change such that $ds^2|_{r_s} \neq \infty$. Now, to see whether there exists a singularity we should look at scalars, constructed out of objects which contain information about the curvature, since they do not change under coordinate transformations. Thus, if our scalar is infinite at r_s then it will be infinite for all coordinate frames and thus we would have that r_s is a singularity. But, now the question of which scalar should we use arises? Let us consider some choices:

- $g_{\mu\nu}g^{\mu\nu}$: For the scalar given by $g_{\mu\nu}g^{\mu\nu}$ we get

$$g_{\mu\nu}g^{\mu\nu} = Dim(spacetime) = 4. \tag{14.4}$$

  Thus, we see that the scalar $g_{\mu\nu}g^{\mu\nu}$ is not very informative.

- $R \equiv g^{\mu\rho}g^{\nu\sigma}R_{\mu\nu\rho\sigma}$: We see that the Ricci scalar, R, of the Schwarzchild metric is equal to zero since it solves the Einstein equations in vaccum, $R_{\mu\nu} = 0$. Thus, the scalar R does not prove or disprove whether or not we have a singularity just like the previous scalar.

- $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$: For this scalar we get, in the Schwarzchild metric,

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2m^2}{r^6} = \frac{12r_H}{r^6}, \tag{14.5}$$

  where in the last equality we have defined $r_H = 2m$ and we are using units such that $m = 1$.

So, finally we see that the only singularity in the Schwarzchild metric comes from $r_s = 0$. The "fake" singularity $r = 2m$ is called the event horizon

Note that $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ does not prove or disprove that $r = 2m$ is a singularity. We could try to find another scalar that blows up at $r = 2m$ and then we would know that $2m$ is a singularity, but this would take a long long time. It turns out that there does exist a coordinate change that at the point $r = 2m$ the metric is regular, i.e. $ds^2|_{r=2m} \neq \infty$. Classically, the event horizon is the point of no return for a particle, i.e. after it crosses this point it cannot move to a distance greater than $2m$ from the singularity again. Thus, classically we see that black holes do not radiate and hence have no temperature. This, which we will see, changes in the quantum theory.

14.1.2 Quantum Theory of Black Holes

Black holes pose very significant theoretical challenges. As we just saw, in Einsteins theory of general relativity black holes appear as classical solutions which represent matter that has collapsed down to a point with infinite density: a singularity. Although dealing with classical singularities is already a theoretical challenge, the real puzzles of black holes arise at the quantum level. Quantum mechanically, black holes radiate energy. They also have thermodynamical temperature and entropy.

In the early 1970s, James Bardeen, Brandon Carter, and Stephen Hawking found that there are laws governing black hole mechanics which correspond very closely to the laws of thermodynamics. The zeroth law states that the surface gravity κ at the horizon of a stationary black hole is constant. The first law relates the mass m, horizon area A, angular momentum J, and charge Q of a black hole as follows:

$$dm = \frac{\kappa}{8\pi}dA + \Omega dJ + \phi dQ, \tag{14.6}$$

where Ω is given by, in D dimensions,

$$\Omega = \frac{2\pi^{(D+1)/2}}{\Gamma(\frac{D+1}{2})}. \tag{14.7}$$

This law is analogous to the law relating energy and entropy.

The second law of black hole mechanics tells us that the area of the event horizon does not decrease with time. This is quantified by writing:

$$dA \geq 0. \tag{14.8}$$

This is directly analogous to the second law of thermodynamics which tells us that the entropy of a closed system is a nondecreasing function of time. A consequence of (14.8)

is that if black holes of areas A_1 and A_2 coalesce to form a new black hole with area A_3 then the following relationship must hold:

$$A_3 > A_1 + A_2. \tag{14.9}$$

The third law states that it is impossible to reduce the surface gravity κ to 0.

The correspondence between the laws of black hole mechanics and thermodynamics is more than analogy. We can go so far as to say that the analogy is taken to be real and exact. That is, the area of the horizon A is the entropy S of the black hole and the surface gravity κ is proportional to the temperature of the black hole. We can express the entropy of the black hole in terms of mass or area. In terms of mass the entropy of a black holes is proportional to the mass of the black hole squared. In terms of the area of the event horizon, the entropy is $1/4$ of the area of the horizon in units of Planck length:

$$S = \frac{A}{4l_p^2}, \tag{14.10}$$

or,

$$S = \frac{A}{4G}, \tag{14.11}$$

where G is the Newton constant.

Before moving to the discussion of black holes in string theory, let us compute the temperature of a Schwarzschild black hole. To begin we Wick rotate the metric, $t \mapsto i\tau$, and write

$$ds^2 = -\left(1 - \frac{2Gm}{r}\right)d\tau^2 + \left(1 - \frac{2Gm}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2. \tag{14.12}$$

Now, define

$$R d\alpha = \left(1 - \frac{2Gm}{r}\right)^{1/2} d\tau,$$

$$dR = \left(1 - \frac{2Gm}{r}\right)^{-1/2} dr,$$

and integrate over the values

$$\alpha: \qquad 0 \le \alpha \le 2\pi,$$

$$\tau: \qquad 0 \le \tau \le \beta,$$

$$r: \qquad 2Gm \le r' \le r.$$

Doing this gives us the following relations:

$$2\pi R = (2Gm)^{-1/2}(r - 2gm)^{1/2}\beta, \tag{14.13}$$

$$R = 2(2Gm)^{1/2}(r - 2Gm)^{1/2}. \tag{14.14}$$

And so, dividing (14.13) by (14.14) we see that

$$\beta = 8\pi Gm. \tag{14.15}$$

Thus, we see that (since $\beta \propto T^{-1}$)

$$T = \frac{1}{8\pi Gm} = \frac{1}{8\pi m}, \tag{14.16}$$

where in the last equality we switched back to the $G = 1$ units.

Now, to recap, we have just seen that although classically black holes have no temperature due to the one-way membrane known as the event horizon, quantum mechanically speaking black holes actually radiate (Hawking radiation) a thermal spectrum with temperature $T = 1/(8\pi m)$. Along with temperature, black holes have entropy and mass.

These thermodynamic properties of black holes lead to new puzzles:

(1) What are the degrees of freedom responsible for the black hole entropy? People have found that black holes have "no hair" which in turn implies that given a mass (and/or other conserved charges) there is a unique solution to the field equations. Thus, since there is only one unique solution there should not be any entropy associated to the black hole solution. However, as we have just seen, a black holes has an entropy given by $S = A/4G$. How is this possible?

(2) In thermodynamics the entropy, S, is an extensive quantity, i.e. it scales as the volume with which the object in question occupies. However, why for black holes does their entropy scale as the area with which its event horizon occupies?

(3) Information loss paradox: As we discussed, the genial idea of Hawking was to realize that the black hole emits a radiation that has the black body spectrum of a body having a temperature proprotional to the mass of the black hole. This thermal radiation can convey no information (it is in a mixed quantum state). However, as a body falls into the black hole, it brings information with it (comes in a pure quantum state). The big question is what happens with the information?

In the limit when the black hole completely evaporates, the pure quantum state has transformed to a mixed quantum state and this is forbidden by the quantum theory, whose unitarity demands a pure quantum state to remain pure and thus conserve information. Thus, either black holes destroy quantum mechanics or we have not included an aspect of the analysis that would maintain the missing information required to keep pure states evolving into pure states. This is the information loss paradox.

The preceding puzzles are not the only ones arising from the study of black holes and it is one of the hopes of string theory to answer these and other puzzles. We will now focus on the study of black holes in string theory.

14.2 Black Holes in String Theory

Black holes arise in string theory as solutions of the corresponding low-energy supergravity theory. Recall that string theory lives in 10 dimensions (or 11 from the M-theory perspective). Now, suppose the theory is compactified on a compact manifold down to d spacetime dimensions. Then branes, which are wrapped in the compact dimensions, will look like pointlike objects in the D-dimensional spacetime. So, the idea to construct a black hole is to construct a configuration of intersecting wrapped branes which upon dimensional reduction yields a black hole spacetime. Also, note that if the brane intersection is supersymmetric then the black hole will be an extremal supersymmetric[‡] black hole. On the other hand, non-extremal intersections yield non-extremal black holes.

In general, the regime of the parameter space in which supergravity is valid is different from the regime in which weakly coupled string theory is valid. Thus, even if we know that a given brane configuration becomes a black hole when we go from a weak to a strong coupling, it would seem difficult to extract information about the black hole from this fact.

For supersymmetric black holes, however, the BPS property of the states allows one to learn certain things about black holes from the weakly coupled D-brane system. For example, one can count the number of states at weak coupling and extrapolate the result to the black hole phase. We will see that in this way, one derives the Bekenstein-Hawking entropy formula (including the precise numerical coefficient) for this class of black holes.

In the absence of supersymmetry, we cannot, in general, follow the states from weak to strong coupling. However, one could still obtain some qualitative understanding of

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[‡]An extremal black hole is a black hole with the minimal possible mass that can be compatible with the given charges and angular momentum, while a black hole is supersymmetric if it is invariant under supercharges.

the black hole entropy. On general grounds, one might expect that the transition from weakly coupled strings to black holes happens when the string scale becomes approximately equal to the Schwarzschild radius (or more generally to the curvature radius at the horizon). This point is called the correspondence point. Demanding that the mass and the all other charges of the two different configurations match, one obtains that the entropies also match. These considerations correctly provide the dependence of the entropy on the mass and the other charges, but the numerical coefficient in the Bekenstein-Hawking entropy formula remains undetermined.

There is a different approach which abandons supersymmetry in favor of U-duality[§]. Instead of trying to determine the physics of black holes using the fact that at weak coupling they become a set of D-branes, the symmetries of M-theory are used in order to map the black hole configuration to another black hole configuration. Since the U-duality group involves strong/weak transitions one does not, in general, have control over the microscopic states that make up a generic configuration. However, the situation is better when it comes to black holes! U-duality maps black holes to black holes with the same thermodynamic characteristics, i.e. the entropy and the temperature remain invariant. This implies that the number of microstates that make up the black hole configuration remains the same. Notice that to reach this conclusion we did not use supersymmetry, but the fact that the area of the horizon of a black hole (divided by Newtons constant) tell us how many degree of freedom the black hole contains.

The effect of the U-duality transformations mentioned above is to remove the constant part from certain harmonic functions (and also change the values of some moduli). One can achieve a similar result by taking the low-energy limit, $\alpha' \to 0$, while keeping fixed the masses of strings stretched between different D-branes. We will see that considerations involving this limit lead to the adS/CFT correspondence.

14.2.1 Five-Dimensional Extremal Black Holes

The simplest nontrivial example for which the entropy can be calculated involves supersymmetric black holes in five dimensions that carry three different kinds of charges. These can be studied in the context of compactifications of the type IIB superstring theory on a five-torus, T^5. The analysis is carried out in the approximation that five of the ten dimensions of the IIB theory are sufficiently small and the black holes are sufficiently large so that a five-dimensional supergravity analysis can be used.

Three-charge black holes in five dimensions can be obtained by taking N_1 $D1$-branes wrapped around x^1, which is periodic with period (radius) R_1, inside the T^5, N_5 $D5$-branes wrapped around $\{x^1, ..., x^5\}$, which are periodic with periods R_i $i = 1, .., 5$,

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[§]U-duality is the combination of both T- and S-duality.

and N_K units of Kaluza-Klein momentum along x^1. Each of these objects breaks half of the supersymmetry, so altogether 7/8 of the supersymmetry is broken, and one is left with solutions that have four conserved supercharges. So, just to recap, we are compactifying our type IIB superstring theory on a five torus $T^5 = S^1_{R_1} \times T^4$ whose coordinates are $\{x^1\} \times \{x^2, ..., x^5\}$. And we are wrapping $D1$-branes around x^1 and $D5$-branes around $x^1, ..., x^5$, while also adding N_K KK momentum modes in the x^1 direction.

Since the branes and momentum break 7/8 of the supersymmetry we will be considering a 1/8 BPS solution of the strings equations of motion. This ten-dimensional solution is given by

$$ds^2 = H_1^{1/2} H_5^{1/2} \left[H_1^{-1} H_5^{-1} \left(-K^{-1} dt^2 + K \left(dx_1 - (K^{-1} - 1) dt \right)^2 \right) \right.$$

$$\left. + H_5^{-1} \left(dx_2^2 + \cdots + dx_5^2 \right) + dx_6^2 + \cdots dx_9^2 \right], \tag{14.17}$$

$$e^{-2\phi} = H_1^{-1} H_5, \tag{14.18}$$

$$C_{01}^{(2)} = H_1^{-1} - 1, \tag{14.19}$$

$$H_{ijk}^{(3)} = \frac{1}{2} \epsilon_{ijkl} \partial_l H_5 \qquad (i, j, k, l = 6, ..., 9), \tag{14.20}$$

$$r^2 = x_6^2 + \cdots x_9^2. \tag{14.21}$$

The harmonic functions are equal to

$$H_1 = 1 + \frac{Q_1}{r^2} \qquad \text{with} \qquad Q_1 = \frac{N_1 g_s \alpha'^3}{V}, \tag{14.22}$$

$$H_5 = 1 + \frac{Q_5}{r^2} \qquad \text{with} \qquad Q_5 = N_5 g_s \alpha', \tag{14.23}$$

$$K = 1 + \frac{Q_K}{r^2} \qquad \text{with} \qquad Q_1 = \frac{N_K g_s^2 \alpha'^4}{R_1^2 V}, \tag{14.24}$$

where $V = R_2 R_3 R_4 R_5$ and the charges Q_i have been calculated from the fact that if $H = 1 + Q^{(d)}/r^{d-3}$, then

$$Q^{(d)} = \frac{16\pi G_N^{(d)} m}{(d-3)\omega_{d-2}}, \tag{14.25}$$

where $G_N^{(d)}$ is the Newton constant in d-dimensions and ω_{d-2} is the volume of the $d-2$-dimensional unit sphere; see Skenderis "Black Holes and Branes in String Theory (hep-th/9901050)".

We would now like to reduce this solution to a five-dimensional solution, where the internal compact space will be $T^4 \times S^1$. From the rules for dimensional reduction[‡] over the periodic coordinates $x^1, ..., x^5$, we obtain a five-dimensional metric, and a few scalars corresponding to the nontrivial diagonal metric factors as well as the dilaton. Finally there are three nontrivial gauge fields, whose charges will be associated to Q_1, Q_5 and Q_K. One of them will come from the off-diagonal component of the metric due to the left-moving wave while the other two descend from the $R-R$ two-form. One of the two $R-R$ vectors comes from the C_{05} component of the two-form. The other is the five-dimensional dual of the C_{ij} components of the two-form.

The five-dimensional Einstein metric is given, using the dimension reduction rules, by

$$ds_{E,5}^2 = \lambda^{-2/3} dt^2 + \lambda^{1/3}\left(dr^2 + r^2 d\Omega_3^2\right), \tag{14.27}$$

where

$$\lambda = H_1 H_5 K = \left(1 + \frac{Q_1}{r^2}\right)\left(1 + \frac{Q_5}{r^2}\right)\left(1 + \frac{Q_K}{r^2}\right). \tag{14.28}$$

This is the metric of an extremal three-charged five-dimensional black hole. The horizon is located at $r = 0$, while the area of the horizon

$$A_5 = \omega_3 \left(r^2 \lambda^{1/3}\right)^{3/2}\Big|_{r=0} = \sqrt{Q_1 Q_5 Q_K (2\pi^2)}, \tag{14.29}$$

and the five-dimensional Newton constant is calculated from the ten-dimensional one via

$$G_N^{(5)} = \frac{G_N^{(10)}}{(2\pi)^5 R_1 V}. \tag{14.30}$$

Using these results we see that the entropy for the five-dimensional supersymmetric extremal black hole is given by

$$S = \frac{A_5}{4G_N^{(5)}} = 2\pi\sqrt{N_1 N_5 N_k}. \tag{14.31}$$

[‡]The dimensional reduction rules are

$$ds_{E,d}^2 = e^{-\frac{4}{d-2}\phi_d} ds_S^2 \qquad \text{and} \qquad e^{-2\phi_d} = e^{-2\phi}\sqrt{Det(g_{int})}, \tag{14.26}$$

where ds_E is the Einstein frame metric, ds_S is the string frame metric and g_{int} is the component of the metric in the directions that we reduce over.

Note that for weak coupling, i.e. $g_s N << 1$, the same system above has a description in terms of the worldvolume gauge theories living on the branes. In this type of analysis the black hole entropy becomes a field theory question. The answer to this question agrees exactly with the above black hole entropy. Also, one should note that all of these arguments hold only for supersymmetric theories.

14.3 Holographic Principle

In this section we will touch on one of the most interesting ideas to come out of the study of quantum gravity and string theory in particular: the holographic principle. This is an idea closely related to entropy, so we present it here after we have completed our discussion of black holes and entropy in the last couple of sections. The holographic principle appears to be a quite general feature of quantum gravity, but we discuss it in the context of string theory. So, without further ado:

Conjecture 14.1 *Any theory of gravity in $(d+1)$ dimensions should have a description in terms of a quantum field theory living in a flat (i.e. without gravitational interactions) d-dimensional spacetime.*

After stating the holographic principle we will look at its most successful realization, the Anti-de Sitter/Conformal Field Theory correspondence, or AdS/CFT correspondence for short.

14.3.1 The AdS/CFT Correspondence

The framework of the holographic principle which comes out of string/M-theory is known as the AdS/CFT correspondence. We can quantitatively describe the spacetime using AdS space. The $(d+1)$-dimensional AdS model has a boundary with d dimensions that look like flat space with $d-1$ spatial directions and one time dimension. The AdS/CFT correspondence involves a duality, something were already familiar with from our studies of superstring theories. This duality is between two types of theories:

- theory of gravity living in $d+1$-dimensions,

- and a super Yang-Mills theory defined on the d-dimensional boundary of the spacetime where the gravitational theory is defined.

By super Yang-Mills theory we mean the theory of particle interactions with supersymmetry. The holographic principle comes out of the correspondence between these two theories because Yang-Mills theory, which is happening on the boundary, is equivalent to the gravitational physics happening in the $(d+1)$-dimensional AdS geometry. So, the Yang-Mills theory can be colloquially thought of as a hologram on the boundary of

the real $(d+1)$-dimensional space where the $(d+1)$-dimensional gravitational physics is taking place. Let us now make this discussion more quantitative.

To begin, consider N coincident Dp-branes. At weak coupling they have a description as hypersurfaces where strings can end. There is worldvolume theory describing the collective coordinates of the brane. The worldvolume fields interact among themselves and with the bulk fields. We would like to consider a limit which decouples the bulk gravity but still leaves non-trivial dynamics on the worldvolume. In low energies gravity decouples. So, we consider the limit $\alpha' \to 0$, which implies that the gravitation coupling constant, i.e. Newtons constant, $G_N \sim \alpha' 4$, also goes to zero. We also want to keep the worldvolume degrees of freedom and their interactions. And so, since the worldvolume dynamics are governed by open strings ending on the D-branes, we keep fixed the masses of any string stretched between D-branes as we take the limit $\alpha' \to 0$. In addition, we keep fixed the coupling constant of the worldvolume theory, so all the worldvolume interactions remain present.

Now, for N coincident D-branes, the worldvolume theory is an $SU(N)$ super Yang-Mills theory (we ignore the center of mass part). The Yang-Mills (YM) coupling constant is equal (up to numerical constants) to

$$g_{YM}^2 \sim g_s(\alpha')^{(p-3)/2}. \tag{14.32}$$

Thus, we obtain the following limit,

$$\alpha' \to 0, \qquad U = \frac{r}{\alpha'} = constant, \qquad g_{YM}^2 = fixed, \tag{14.33}$$

yeilds a decoupled theory on the worldvolume.

At strong coupling the Dp-branes are described by the so-called "black" p-brane spacetimes,

$$ds^2 = H_i^{-1/2} ds^2(\mathbb{E}^{(p,1)}) + H_i^{1/2} ds^2(\mathbb{E}^{(9-p)})$$

$$e^{-2\phi} = H_i^{\frac{p-3}{2}} \tag{14.34}$$

$$A_{01\cdots p}^{(p+1)} = H_i^{-1} - 1, \qquad \text{or} \qquad F_{8-p} = \star dH_i,$$

see 13.3.1. In the limit defined by (14.33) the harmonic function H_i becomes

$$H_i \mapsto g_{YM}^2 N(\alpha')^{-2} U^{p-7}, \tag{14.35}$$

see Skenderis hep-th/9901050. Also note that by inserting the new harmonic function, (14.35), back into the metric the spacetime becomes conformally equivalent to $AdS_{p+2} \times S^{8-p}$, where AdS_{p+2} is the $(p+2)$-dimensional anti-de Sitter spacetime and S^{8-p} is the unit $(8-p)$-sphere.

Before we explicitely work out the details for the case of N coincident $D3$-branes, let us first review anti-de Sitter spacetimes, namely 5-dim AdS_5.

In a nutshell, AdS_5 is a four-dimensional spatial ball and an infinite time axis, see figure 13. The radius of the ball is given by $0 \leq r \leq 1$. The radius of curvature is denoted by R, and we lump the remaining spatial dimensions together into a unit three-sphere denoted by Ω_3. The metric which describes the AdS_5 is then written as

Figure 13: AdS_5 spacetime.

$$ds^2 = \frac{R^2}{(1-r^2)^2}\Big(\big(1+r^2\big)dt^2 - 4dr^2 - 4r^2 d\Omega_5^2\Big),$$

or in different coordinates and new parameters

$$ds^2 = \alpha'\left(\frac{R^2}{U^2}dU^2 + \frac{U}{R^2}\big(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\big)\right). \qquad (14.36)$$

To be a bit more precise, AdS_5 is a maximally symmetric[‡] solution to the Einstein equations with a negative cosmological constant. AdS_5 has a conformal boundary (see the figure), i.e. only light-rays can reach the boundary of the spacetime.

Now, let us work out the details for the case of N coincident $D3$-branes. So, to begin, recall that the solution describing a $D3$-brane is given by

$$ds^2 = H^{-1/2}\Big(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\Big) + H^{1/2}\Big(dr^2 + r^2 d\Omega_5^2\Big), \qquad (14.37)$$

along with

$$H = 1 + \frac{g_s N \alpha'^2}{r^4}, \qquad (14.38)$$

for the harmonic function, H. Let us see how this solution behaves in the limit defined by (14.33). In this limit we have that the harmonic function becomes

$$H \mapsto H' = 1 + \frac{g_s N \alpha'^2}{U^4 \alpha'^4}$$

$$= 1 + \frac{g_s N}{U^2 \alpha'^2}$$

$$\approx \frac{g_s N}{U^4 \alpha'^2}. \qquad (14.39)$$

[‡]Maximally symmetric implies that

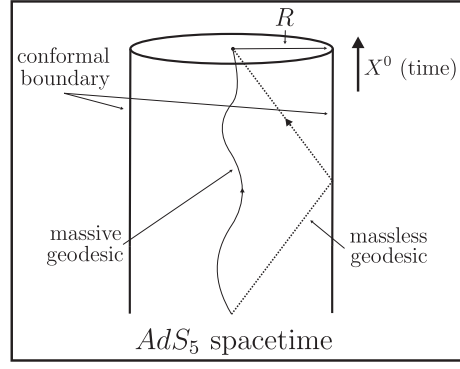$$R_{\mu\nu\rho\sigma} = \frac{1}{l^2}\big(g_{\mu\sigma}g_{\nu\rho} - g_{\nu\sigma}g_{\mu\rho}\big).$$

Pluggin this expression for H back into the metric, (14.37), gives

$$ds^2 = \frac{\alpha' U^2}{(g_s N)^{1/2}}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{(g_s N)^{1/2}}{\alpha' U^2}\left(\alpha'^2 dU^2 + \alpha'^2 U d\Omega_5^2\right)$$

$$= \alpha'\left[\frac{R^2}{U^2}du^2 + \frac{U}{R^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + R^2 d\Omega_5^2\right], \qquad (14.40)$$

where $R = (g_s N)^{1/2}$ is the AdS_5 radius. Note that the first part of the metric, namely

$$\frac{R^2}{U^2}du^2 + \frac{U}{R^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right),$$

is the metric describing an AdS_5 spacetime (see (14.36)) while the second part of the metric, namely

$$R^2 d\Omega_5^2,$$

describes a spacetime with geometry given by a unit 5-sphere. Thus, under the limit described by (14.33) the $D3$-brane solution turns in to a solution which describes a spacetime of the form $AdS_5 \times S^5$.

For the case of N $D3$-branes the worldvolume theory is four-dimensional, $\mathcal{N} = 4$, $SU(N)$ super Yang-Mills (SYM) theory. This is a finite unitary theory for any value of the its coupling constant. On the other hand, this system has a description as a black 3-brane at strong coupling. Also, in order to suppress string loops we need to take N large. For the supergravity description to be valid 't Hoofts coupling constant, $g_{YM}^2 N$, must be large. We therefore get that the strong ('t Hooft) coupling limit of large $d = 4$, $\mathcal{N} = 4$, $SU(N)$ SYM is described by adS supergravity!

$\mathcal{N} = d = 4$ SYM theory is a well-defined unitary finite theory, whereas supergravity is a non-renormalizable theory. It is best to think about it as the low energy effective theory of strings.
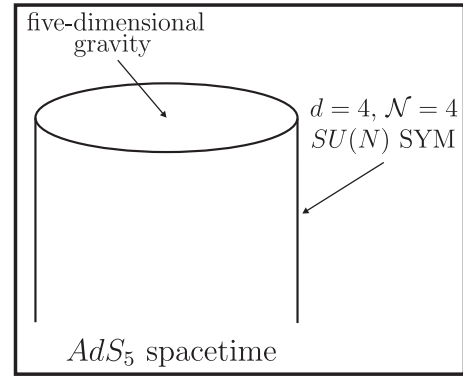
Figure 14: We can think of a four-dimensional, $\mathcal{N} = 4$, $SU(N)$ super Yang-Mills theory living on the conformal boundary of the AdS_5 spacetime.

Therefore, one should really consider strings on $AdS_5 \times S^5$. In this way we reach the celebrated adS/CFT duality (see figure 14):

Four-dimensional, $\mathcal{N} = 4$, $SU(N)$ SYM is dual to string theory on $AdS_5 \times S^5$.

Let us examine again our result. We obtained that five-dimensional AdS gravity is equivalent to $d = 4$, $\mathcal{N} = 4$ SYM theory. In other words, a gravity theory in $d + 1 (= 5)$

dimensions is described in terms of a field theory without gravity in $d(=4)$ dimensions. But this is just the holoraphic principle from before.

Note that if these two theories are really equivalent then there should be a map between the variables. It turns out that this map is given by:

- (Kinematics) For every gauge invariant operator defined on the boundary, there is a corresponding field (bulk field) defined on the interior. For example, we have that

$$\hat{\mathcal{O}} \equiv \hat{F}_{\mu\nu}\hat{F}_{\mu\nu} \qquad \longleftrightarrow \qquad \phi,$$

$$\hat{T}_{\mu\nu} \equiv \hat{F}_{\mu\kappa}\hat{F}^{\kappa}{}_{\nu} + \cdots \qquad \longleftrightarrow \qquad g_{\mu\nu}.$$

- (Dynamics) By the $AdS/$CFT, the path integral

$$Z\big[\Phi_{(0)}\big] \equiv \int_{\Phi=\Phi_{(0)}} \mathcal{D}\Phi \; e^{-S_{grav}}\big[\Phi\big],$$

where $\Phi_{(0)}$ is a bulk field at the boundary (i.e. Φ is a bulk field which we extend to the boundary and at the boundary we denote the value of the bulk field, Φ, by $\Phi_{(0)}$), is equivalent to

$$\left\langle e^{\int \Phi_{(0)}\hat{\mathcal{O}}} \right\rangle.$$

This concludes our discussion of black holes and the AdS/CFT correspondence. For a good overview of these topics see Zwiebach "A First Course in String Theory".

A. Residue Theorem

Let $f(z)$ be a function which is everywhere analytic on the complex plane except at the point w. Then the residue theorem states that the contour integral of a function $f(z)$ around a contour $\mathcal{C}$ enclosing a point w is equal to the residue of $f(z)$ at the point w,

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} dz f(z) = Res(f(z))\Big|_{z=w}. \tag{A.1}$$

In order to calculate the RHS of (A.1) we use the following expression

$$Res(f(z))\Big|_{z=w} = \frac{1}{(n-1)!} \lim_{z \mapsto w} \frac{d^{n-1}}{dz^{n-1}}\Big(f(z)(z-w)^n\Big), \tag{A.2}$$

where we define the residue of a regular function, i.e. function with zeroth order poles, to be zero while the value of n is said to be the order of the pole. Thus, for $f(z) = 0$ we have that its residue is zero by definition. While if $f(z) = 1/(z-w)$ then, since it has a first order pole, by inspection of (A.1) we have that its residue is 1. And, finally, the function $f(z) = 1/(z-w)^2$ has a second order pole at w and using (A.1) we see that its residue vanishes. This implies that

$$\oint_{\mathcal{C}} dz \frac{1}{(z-w)} = 1,$$

$$\oint_{\mathcal{C}} dz \frac{1}{(z-w)^2} = 0,$$

$$\oint_{\mathcal{C}} dz 1 = 0,$$

where $\mathcal{C}$ encloses the point w. See Whittaker "A Course in Modern Analysis".

B. Wick's Theorem

Wick's theorem for bosonic fields says that for the bosonic fields $\{\phi_i\}_{i=1}^k$ and $\{\psi_j\}_{j=1}^l$ their normal ordering is given by

$$:\phi_1\phi_2\cdots\phi_k::\psi_1\psi_2\cdots\psi_l:\;=\;:\phi_1\phi_2\cdots\phi_k\psi_1\psi_2\cdots\psi_l:\;+\;\sum_{\alpha_1,\beta_1}\overparen{\phi_{\alpha_1}\psi_{\beta_1}}\;:\prod_{i\neq\alpha_1}\phi_i\prod_{j\neq\beta_1}\psi_j:$$

$$+\;\sum_{\alpha_1,\beta_1,\alpha_2,\beta_2}\overparen{\phi_{\alpha_1}\psi_{\beta_1}}\,\overparen{\phi_{\alpha_2}\psi_{\beta_2}}\;:\prod_{i\neq\alpha_1,\alpha_2}\phi_i\prod_{j\neq\beta_1,\beta_2}\psi_j:\;+\cdots$$

$$+\;\begin{cases}\displaystyle\sum_{\alpha_1,\beta_1,\cdots\alpha_k,\beta_k}\overparen{\phi_{\alpha_1}\psi_{\beta_1}}\cdots\overparen{\phi_{\alpha_k}\psi_{\beta_k}}\;:\prod_{j\neq\beta_1,\ldots,\beta_k}\psi_j:&(k<l)\\[3em]\displaystyle\sum_{\alpha_1,\beta_1,\cdots\alpha_l,\beta_l}\overparen{\phi_{\alpha_1}\psi_{\beta_1}}\cdots\overparen{\phi_{\alpha_l}\psi_{\beta_l}}\;:\prod_{i\neq\alpha_1,\ldots,\alpha_l}\phi_i:&(k>l)\\[3em]\displaystyle\sum_{\alpha_1,\beta_1,\cdots\alpha_k,\beta_k}\overparen{\phi_{\alpha_1}\psi_{\beta_1}}\cdots\overparen{\phi_{\alpha_k}\psi_{\beta_k}}&(k=l).\end{cases}$$

So, for example,

$$:\phi_1\phi_2::\psi_1\psi_2:\;=\;:\phi_1\phi_2\psi_1\psi_2:\;+\;\overparen{\phi_1\psi_1}:\phi_2\psi_2:\;+\;\overparen{\phi_1\psi_2}:\phi_2\psi_1:\;+\;\overparen{\phi_2\psi_1}:\phi_1\psi_2:\;+\;\overparen{\phi_2\psi_2}:\phi_1\psi_1:$$

$$+\;\overparen{\phi_1\psi_1}\,\overparen{\phi_2\psi_2}\;+\;\overparen{\phi_1\psi_2}\,\overparen{\phi_2\psi_1}\,.$$

Note that the normal ordering of a single field is just the field, i.e. $:\phi_1:\;=\;\phi_1$ and that we only contract with cross terms since by definition the normal ordering removes singularities. Also, for fermionic fields the recipe above is the same except that you have to take into account the anti-commuting property of the fields as you move them through the contractions.

C. Solutions to Exercises

Chapter 2

Solution to Exercise 2.1

a) Under the variation δX^μ, the action,

$$S = -m \int \sqrt{-\dot{X}^2} d\tau + e \int A_\mu(X)\dot{X}^\mu d\tau \tag{C.1}$$

changes by

$$\delta S = m \int \frac{\dot{X}_\mu \delta \dot{X}^\mu}{\sqrt{-\dot{X}^2}} + e \int (\partial_\nu A_\mu(X)\,\delta X^\nu \dot{X}^\mu + A_\mu(X)\delta \dot{X}^\mu)d\tau. \tag{C.2}$$

In the above, note that $A_\mu(X)$ is a function of X^μ and therefore $\delta A_\mu(X) = \partial_\nu A_\mu(X)\delta X^\nu$. Integration by parts yields

$$\delta S = -m \int \delta X^\mu \frac{d}{dt}\left(\frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}}\right) d\tau + e \int (\partial_\nu A_\mu(X)\,\delta X^\nu \dot{X}^\mu - \delta X^\mu \frac{d}{dt}A_\mu(X))d\tau$$

$$= -m \int \delta X^\mu \frac{d}{dt}\left(\frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}}\right) d\tau + e \int (\partial_\nu A_\mu(X)\,\delta X^\nu \dot{X}^\mu - \delta X^\mu \partial_\nu A_\mu(X)\dot{X}^\nu)d\tau$$

$$= \int \left[-m\frac{d}{dt}\left(\frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}}\right) + eF_{\mu\nu}(X)\dot{X}^\nu\right]\delta X^\mu \, d\tau, \tag{C.3}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. For this to hold for arbitrary δX^μ, we have that

$$m\frac{d}{dt}\left(\frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}}\right) = eF_{\mu\nu}(X)\dot{X}^\nu, \tag{C.4}$$

which is the equation of motion. If we choose the parametrization τ to be such that $-\dot{X}^2 = 1$, then this simplifies to [‡]

$$m\ddot{X}_\mu = eF_{\mu\nu}(X)\dot{X}^\nu. \tag{C.5}$$

b) Under the transformation

$$A_\mu(X) \rightarrow A_\mu(X) + \partial_\mu \Lambda(X), \tag{C.6}$$

[‡]One check of this result is that, in the nonrelativistic limit where $X^\mu = (t, \vec{X})$ and $t = \tau$, this reduces to the familiar equation of motion $m\ddot{\vec{X}} = e(\vec{E} + \vec{v} \times \vec{B})$, where $F_{i0} = E_i$ and $F_{ij} = \sum_k \epsilon_{ijk}B_k$ with $i, j, k = 1, 2, 3$.

the action (C.1) changes by

$$\int \partial_\mu \Lambda(X)\, \dot{X}^\mu\, d\tau = \int \frac{d\Lambda(X)}{dX^\mu}\frac{dX^\mu}{d\tau} d\tau = \int \frac{d\Lambda(X(\tau))}{d\tau} d\tau = [\Lambda(X(\tau))]_{\tau=-\infty}^{\tau=\infty}. \qquad \text{(C.7)}$$

If $\Lambda(X)$ vanishes at infinity (namely, $\Lambda(X) \to 0$ as $|X| \to \infty$) and, furthermore, if the particle goes to spatial infinity in the infinite past and future (namely, $|X(\tau)| \to \infty$ as $\tau \to \pm\infty$), this vanishes. Therefore, the action (C.1) is invariant under (C.6).

Solution to Exercise 2.2

a) c is a velocity. Because a velocity is $v = \frac{dX}{dt}$, its dimension is $[c] = [X/t] = [LT^{-1}]$. A relation involving $\hbar$ is $E = \hbar\omega$. (Of course, any other relation involving $\hbar$ will do too. For example, $p = -i\hbar\frac{\partial}{\partial X}$.) . Here, energy E has the same dimension as that of work done by force F, $(work) = (force) \times (distance) = F \times X$. Recalling Newton's second law $F = ma = m\frac{d^2 X}{dt^2}$, the dimension of F is $[F] = [mX/t^2] = [MLT^{-2}]$. So, $[E] = [FX] = [ML^2T^{-2}]$. On the other hand, $[\omega] = [T^{-1}]$. Therefore, $[\hbar] = [E/\omega] = [ML^2T^{-1}]$. Recall Newton's law for gravity, $F = \frac{Gm_1m_2}{r^2}$. This means $[G] = [Fr^2/m_1m_2] = [M^{-1}L^3T^{-2}]$.

In summary,

$$[c] = [LT^{-1}], \qquad [\hbar] = [ML^2T^{-1}], \qquad [G] = [M^{-1}L^3T^{-2}]. \qquad \text{(C.8)}$$

b) Let the Planck length be given by

$$l_p = c^a \hbar^b G^c \qquad \text{(C.9)}$$

with some numbers a, b, c. The dimension of this is, from (C.8),

$$[c^a \hbar^b G^c] = [M^{b-c}L^{a+2b+3c}T^{-a-b-2c}]. \qquad \text{(C.10)}$$

For this to be equal to the dimension of l_p, $[L]$, we need

$$b - c = 0, \qquad a + 2b + 3c = 1, \qquad -a - b - 2c = 0. \qquad \text{(C.11)}$$

Solving this, we get $a = -3/2, b = c = 1/2$. Therefore, the Planck length is

$$l_p = \left(\frac{\hbar G}{c^3}\right)^{1/2}. \qquad \text{(C.12)}$$

Using the numerical values

$$\begin{aligned}
c &= 3 \times 10^8\,\mathrm{m \cdot s^{-1}}, \\
\hbar &= 1.05 \times 10^{-34}\,\mathrm{m^2 \cdot kg \cdot s^{-1}}, \\
G &= 6.67 \times 10^{-11}\,\mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}},
\end{aligned} \qquad \text{(C.13)}$$

we find

$$l_p = 1.6 \times 10^{-35} \, \text{m} = 1.6 \times 10^{-33} \, \text{cm}. \tag{C.14}$$

This is the most natural scale for string theory which purports to be a theory unifying quantum theory and gravity.

c) Just like b), we find that the Planck mass is

$$M_p = \left(\frac{c\hbar}{G} \right)^{1/2} = 2.17 \times 10^{-8} \, \text{kg}. \tag{C.15}$$

In terms of the unit

$$1 \, \text{GeV} = 1.6 \times 10^{-10} \, \text{J}, \qquad 1 \, \text{GeV}/c^2 = 1.8 \times 10^{-27} \, \text{kg}, \tag{C.16}$$

the Planck mass is

$$M_p = 1.2 \times 10^{19} \, \text{GeV}/c^2. \tag{C.17}$$

Solution to Exercise 2.3

First, let us find the transformation rule for $e(\tau)$ so that the action

$$\tilde{S}_0 = \frac{1}{2} \int d\tau (e^{-1}\dot{x}^2 - m^2 e), \tag{C.18}$$

is invariant. Under

$$\tau \to \tau' = f(\tau), \tag{C.19}$$

$d\tau$ and $\dot{X}^\mu$ transform as

$$d\tau \to d\tau' = d\tau \frac{d\tau'}{d\tau} = d\tau \, \dot{f}, \qquad \dot{X}^\mu = \frac{dX^\mu}{d\tau} \to \frac{dX^\mu}{d\tau'} = \frac{d\tau}{d\tau'} \frac{dX^\mu}{d\tau} = \frac{\dot{X}^\mu}{\dot{f}}. \tag{C.20}$$

If e transforms as $e \to e'$, then the transformation of the action (C.18) is

$$\tilde{S}_0 \to \frac{1}{2} \int d\tau \, \dot{f} \left(e'^{-1} \frac{\dot{X}^2}{\dot{f}^2} - m^2 e' \right) = \frac{1}{2} \int d\tau \left((e'\dot{f})^{-1} \dot{X}^2 - m^2 e' \dot{f} \right). \tag{C.21}$$

For this to be equal to (C.18), we need

$$e' = \frac{e}{\dot{f}} = \frac{d\tau}{d\tau'} e. \tag{C.22}$$

This is the transformation rule for e.[‡]

Let us say that we have an arbitrary $e(\tau)$. We would like to find a transformation $e \to e'$ such that we have $e'(\tau') = 1$ in the new coordinate τ'. From (C.22), if $e' = 1$ we have that

$$\dot{f} = e. \tag{C.24}$$

So, we can choose f to be

$$f(\tau) = \int e(\tau)d\tau, \tag{C.25}$$

and thus, given an arbitrary $e(\tau)$, we can always appropriately choose f to go to a gauge where $e' = 1$.[§] This implies that we can always set $e = 1$ by assuming that such a gauge choice has been made.

Solution to Exercise 2.4

Consider the p-brane Polyakov action with the addition of a cosmological constant Λ_p,

$$S_\sigma = -\frac{T_p}{2} \int d^p\tau d\sigma \sqrt{-h}\, h^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X + \Lambda_p \int d^p\tau d\sigma \sqrt{-h}. \tag{C.27}$$

We want to show that, upon solving for the equation of motion for the metric h, this is equivalent to the "Nambu–Goto action"

$$S_{NG} = -T_p \int d^p\tau d\sigma \sqrt{-\det \partial_\alpha X \cdot \partial_\beta X}, \tag{C.28}$$

if we choose the "cosmological constant" Λ_p appropriately.

[‡]Note that this is the same as the transformation law for $\sqrt{-h_{\tau\tau}}$, where $h_{\tau\tau}$ is the "worldline metric". Namely, we can identify $e = \sqrt{-h_{\tau\tau}}$. Because there is only one dimension on the worldline, the "metric" $h_{\alpha\beta}$ has only one component for $\alpha = \beta = \tau$. $\sqrt{-h_{\tau\tau}}$ is the one-dimensional version of $\sqrt{-\det h_{\alpha\beta}}$. With $h_{\tau\tau}$, the action (C.18) can be written as

$$\tilde{S}_0 = -\frac{1}{2} \int d\tau \sqrt{-h_{\tau\tau}}(h^{\tau\tau}(\partial_\tau X)^2 + m^2). \tag{C.23}$$

In this form, it is clear that this is a Polyakov action for a 0-brane with a cosmological constant term.

[§]Note that (C.25) is nothing but the geodesic length along the worldline:

$$f(\tau) = \int \sqrt{-h_{\tau\tau}}\, d\tau. \tag{C.26}$$

Let us consider the variation of the action with respect to $\delta h^{\alpha\beta}$. Remembering the formula $\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h}\,h_{\alpha\beta}\delta h^{\alpha\beta}$, we get

$$\delta S_\sigma = -\frac{1}{2}\int d^p\tau d\sigma\,\sqrt{-h}\left[T_p\left(\partial_\alpha X\cdot\partial_\beta X - \frac{1}{2}h_{\alpha\beta}(\partial X)^2\right) + \Lambda_p h_{\alpha\beta}\right]\delta h^{\alpha\beta}, \quad (C.29)$$

where $(\partial X)^2 \equiv h^{\gamma\delta}\partial_\gamma X\partial_\delta X$. So, the equation of motion is

$$T_p\left(\partial_\alpha X\cdot\partial_\beta X - \frac{1}{2}h_{\alpha\beta}(\partial X)^2\right) + \Lambda_p h_{\alpha\beta} = 0. \quad (C.30)$$

Multiplying this by $h^{\alpha\beta}$ and noting that $h_{\alpha\beta}h^{\alpha\beta} = \delta_\alpha^\alpha = p+1$, we get

$$-\frac{p-1}{2}T_p(\partial X)^2 + (p+1)\Lambda_p = 0. \quad (C.31)$$

For $p = 1$, this means $\Lambda_1 = 0$, which is consistent with the string Polyakov action. For $p \neq 1$, we have

$$(\partial X)^2 = \frac{2(p+1)\Lambda_p}{(p-1)T_p}. \quad (C.32)$$

Substituting this back into (C.30), we can derive

$$h_{\alpha\beta} = \frac{(p-1)T_p}{2\Lambda_p}\,\partial_\alpha X\cdot\partial_\beta X, \quad (C.33)$$

which means that

$$\sqrt{-h} = \left[\frac{(p-1)T_p}{2\Lambda_p}\right]^{\frac{p+1}{2}}\sqrt{-\det\partial_\alpha X\cdot\partial_\beta X}. \quad (C.34)$$

Now, if we plug this back into the Polyakov action (C.27), after some manipulations, we get

$$S_\sigma = -T_p\left[\frac{(p-1)T_p}{2\Lambda_p}\right]^{\frac{p-1}{2}}\int d^{p+1}\sigma\sqrt{-\det\partial_\alpha X\cdot\partial_\beta X}. \quad (C.35)$$

Therefore, if we take the cosmological constant to be

$$\Lambda_p = \frac{p-1}{2}T_p, \quad (C.36)$$

the Polyakov action, with a cosmological constant term, (C.27) reduces to the Nambu–Goto action (C.28).

Chapter 3

Solution to Exercise 3.1

a) If we plug the ansatz $X^\mu(\tau, \sigma) = g(\tau)f(\sigma)$ into the wave equation (3.47), we obtain

$$\frac{f''(\sigma)}{f(\sigma)} = \frac{g''(\tau)}{g(\tau)}. \tag{C.37}$$

For this equation to hold for arbitrary values of τ, σ, it must be that both sides of this equation are equal to a constant independent of τ, σ:

$$\frac{f''(\sigma)}{f(\sigma)} = \frac{g''(\tau)}{g(\tau)} = c. \tag{C.38}$$

Namely,

$$f''(\sigma) = cf(\sigma), \tag{C.39a}$$
$$g''(\tau) = c\,g(\tau). \tag{C.39b}$$

b) If $c \neq 0$, the linearly independent solutions to (C.39a) are

$$e^{\pm\sqrt{c}\,\sigma}. \tag{C.40}$$

For this to be periodic under $\sigma \to \sigma + \pi$, we need

$$e^{\pm\sqrt{c}(\sigma+\pi)} = e^{\pm\sqrt{c}\,\sigma}. \tag{C.41}$$

For this to be true for any σ, we need

$$e^{\pm\pi\sqrt{c}} = 1, \tag{C.42}$$

namely $\pi\sqrt{c} = 2\pi i m$, $m \in \mathbb{Z}$, $m \neq 0$ (note that this takes care of both signs in (C.42)). In other words,

$$c = -4m^2, \qquad m \in \mathbb{Z}, \quad m \neq 0. \tag{C.43}$$

On the other hand, if $c = 0$, the linearly independent solution to (C.39a) are

$$1, \quad \sigma. \tag{C.44}$$

The second one is not periodic under $\sigma \to \sigma + \pi$. So, only the first one is appropriate.

Combining the $c \neq 0$ and $c = 0$ cases, we have

$$c = -4m^2, \quad m \in \mathbb{Z}; \qquad f(\sigma) = e^{\pm 2im\sigma}. \tag{C.45}$$

c) By solving (C.39b) for the values of m given in b), we obtain

$$c = -4m^2, \quad m \in \mathbb{Z}, \quad m \neq 0 \quad \Longrightarrow \quad g(\tau) = e^{\pm 2im\tau},$$
$$c = 0 \quad\quad\quad\quad\quad\quad\quad\quad \Longrightarrow \quad g(\tau) = 1, \ \tau. \tag{C.46}$$

Multiplying f and g, we conclude that $X^\mu(\tau, \sigma)$ is given by a linear combination of the following functions:

$$e^{-2im(\tau+\sigma)}, \quad e^{-2im(\tau-\sigma)}, \quad 1, \quad \tau, \tag{C.47}$$

with $m \in \mathbb{Z}$, $m \neq 0$. Therefore, the most general solution to the wave equation (3.47) satisfying the periodicity (12.3) is

$$X^\mu(\tau, \sigma) = x^\mu + a^\mu \tau + \sum_{n \neq 0} (b_n^\mu e^{-2in(\tau-\sigma)} + \tilde{b}_n^\mu e^{-2in(\tau+\sigma)}), \tag{C.48}$$

where the coefficients $x^\mu, a^\mu, b_n^\mu, \tilde{b}_n^\mu$ are constants. For X^μ to be real, x^μ and a^μ must be real, while $(b_n^\mu)^* = b_{-n}^\mu$, $(\tilde{b}_n^\mu)^* = \tilde{b}_{-n}^\mu$. We can write this as a sum of right- and left-moving parts as

$$X^\mu = X_R^\mu(\tau - \sigma) + X_L^\mu(\tau + \sigma),$$
$$X_R^\mu = \frac{x^\mu}{2} + a^\mu \frac{\tau - \sigma}{2} + \sum_{n \neq 0} b_n^\mu e^{-2in(\tau-\sigma)},$$
$$X_L^\mu = \frac{x^\mu}{2} + a^\mu \frac{\tau + \sigma}{2} + \sum_{n \neq 0} \tilde{b}_n^\mu e^{-2in(\tau+\sigma)}. \tag{C.49}$$

If we set

$$a^\mu = l_s^2 p^\mu, \quad b_n^\mu = \frac{il_s}{2n} \alpha_n^\mu, \quad \tilde{b}_n^\mu = \frac{il_s}{2n} \tilde{\alpha}_n^\mu, \tag{C.50}$$

then we obtain the desired expansion (3.52). Here, p^μ is real, while $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$, $(\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu$.

d) In the above we found that the solution to (3.53), namely the eigenfunction of (3.55), (3.56), is

$$e^{2im\sigma}, \quad m \in \mathbb{Z}, \tag{C.51}$$

with the eigenvalue $c = -4m^2$. So, let us assume that the orthonormal basis is given by

$$f_m(\sigma) = N_m e^{2im\sigma}, \tag{C.52}$$

where N_m is a constant to be determined. We can see that the inner product is

$$\int_0^\pi d\sigma\, f_m(\sigma)^* f_n(\sigma) = N_m^* N_n \int_0^\pi d\sigma\, e^{2i(-m+n)\sigma},$$

$$= \begin{cases} |N_m|^2 \int_0^\pi d\sigma & (m = n) \\ \frac{N_m^* N_n}{2i(-m+n)} \left[e^{2i(-m+n)\sigma} \right]_0^\pi & (m \neq n), \end{cases}$$

$$= \begin{cases} \pi |N_m|^2 & (m = n) \\ 0 & (m \neq n), \end{cases}$$

$$= \pi |N_m|^2 \delta_{mn}. \tag{C.53}$$

For orthonormality (3.57), we can set $N_m = 1/\sqrt{\pi}$. Namely, the orthonormal eigenfunctions are

$$f_m(\sigma) = \frac{e^{2im\sigma}}{\sqrt{\pi}}. \tag{C.54}$$

If we plug (C.54) into the formula (3.60), we obtain

$$\delta(\sigma - \sigma') = \sum_n f_n(\sigma) f_n(\sigma')^* = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} e^{2in(\sigma - \sigma')}. \tag{C.55}$$

e) From (12.3),

$$X^\mu(\tau, \sigma) = x^\mu + l_s^2 p^\mu \tau + \frac{il_s}{2} \sum_m{}' \left(\frac{\alpha_m^\mu}{m} e^{-2im(\tau - \sigma)} + \frac{\tilde{\alpha}_m^\mu}{m} e^{-2im(\tau + \sigma)} \right), \tag{C.56}$$

where $\sum_m{}'$ means a summation over $m \in \mathbb{Z}$ except for $m = 0$. Using the definition (3.62), we obtain

$$P^\mu(\tau, \sigma) = \frac{\dot{X}^\mu}{\pi l_s^2} = \frac{p^\mu}{\pi} + \frac{1}{\pi l_s} \sum_m{}' \left(\alpha_m^\mu e^{-2im(\tau - \sigma)} + \tilde{\alpha}_m^\mu e^{-2im(\tau + \sigma)} \right). \tag{C.57}$$

f) After some computations, for $\{P, P\}$, we get

$$0 = \{P^\mu(\tau, \sigma), P^\nu(\tau, \sigma')\}, \tag{C.58}$$

$$= \frac{1}{\pi^2}\{p^\mu, p^\nu\} + \frac{1}{\pi^2 l_s}\sideset{}{'}\sum_n \left(\{p^\mu, \alpha_n^\nu\}e^{-2in(\tau-\sigma')} + \{p^\mu, \tilde{\alpha}_n^\nu\}e^{-2in(\tau+\sigma')}\right)$$

$$+ \frac{1}{\pi^2 l_s}\sideset{}{'}\sum_m \left(\{\alpha_m^\mu, p^\nu\}e^{-2im(\tau-\sigma)} + \{\tilde{\alpha}_m^\mu, p^\nu\}e^{-2im(\tau+\sigma)}\right)$$

$$+ \frac{1}{\pi^2 l_s^2}\sideset{}{'}\sum_{m,n} \Bigl(\{\alpha_m^\mu, \alpha_n^\nu\}e^{2i(m\sigma+n\sigma')} + \{\alpha_m^\mu, \tilde{\alpha}_n^\nu\}e^{2i(m\sigma-n\sigma')}$$

$$+ \{\tilde{\alpha}_m^\mu, \alpha_n^\nu\}e^{2i(-m\sigma+n\sigma')} + \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\}e^{2i(-m\sigma-n\sigma')}\Bigr)e^{-2i(m+n)\tau}.$$

$$\text{(C.59)}$$

By equating the coefficients on both sides of (C.59), we obtain

$$0 = \{p^\mu, p^\nu\}, \tag{C.60a}$$

$$0 = \{\alpha_m^\mu, p^\nu\}e^{-2im\tau} + \{\tilde{\alpha}_{-m}^\mu, p^\nu\}e^{2im\tau}, \tag{C.60b}$$

$$0 = \{p^\mu, \alpha_n^\nu\}e^{-2in\tau} + \{p^\mu, \tilde{\alpha}_{-n}^\nu\}e^{2in\tau}, \tag{C.60c}$$

$$0 = \{\alpha_m^\mu, \alpha_n^\nu\}e^{-2i(m+n)\tau} + \{\alpha_m^\mu, \tilde{\alpha}_{-n}^\nu\}e^{-2i(m-n)\tau} +$$

$$+ \{\tilde{\alpha}_{-m}^\mu, \alpha_n^\nu\}e^{2i(m-n)\tau} + \{\tilde{\alpha}_{-m}^\mu, \tilde{\alpha}_{-n}^\nu\}e^{2i(m+n)\tau}, \tag{C.60d}$$

where $m, n \neq 0$. Here, to get (C.60a), we equated the constant terms in (C.59). To get (C.60b), (C.60c), and (C.60d), we equated the coefficients of $e^{im\sigma}$ ($m \neq 0$), $e^{in\sigma'}$ ($n \neq 0$), and $e^{i(m\sigma+n\sigma')}$ ($m, n \neq 0$) in (C.59), respectively. This is possible because $f_n(\sigma) = e^{im\sigma}/\sqrt{\pi}$, $m \in \mathbb{Z}$ form a complete basis for functions defined for $\sigma \in [0, \pi)$. By using this twice, any function $g(\sigma, \sigma')$ defined for $\sigma, \sigma' \in [0, \pi)$ can be expanded in terms of $f_m(\sigma)f_n(\sigma')$ with $m, n \in \mathbb{Z}$.

From $\{X, X\}$, similarly (omitting details), we obtain

$$0 = \{x^\mu, x^\nu\} + l_s^2\tau(\{x^\mu, p^\nu\} + \{p^\mu, x^\nu\}) + l_s^4\tau^2\{p^\mu, p^\nu\}, \tag{C.61a}$$

$$0 = \left(\{\alpha_m^\mu, x^\nu\}e^{-2im\tau} - \{\tilde{\alpha}_{-m}^\mu, x^\nu\}e^{2im\tau}\right) + l_s^2\tau\left(\{\alpha_m^\mu, p^\nu\}e^{-2im\tau} - \{\tilde{\alpha}_{-m}^\mu, p^\nu\}e^{2im\tau}\right), \tag{C.61b}$$

$$0 = \left(\{x^\mu, \alpha_n^\nu\}e^{-2in\tau} - \{x^\mu, \tilde{\alpha}_{-n}^\nu\}e^{2in\tau}\right) + l_s^2\tau\left(\{p^\mu, \alpha_n^\nu\}e^{-2in\tau} - \{p^\mu, \tilde{\alpha}_{-n}^\nu\}e^{2in\tau}\right), \tag{C.61c}$$

$$0 = \{\alpha_m^\mu, \alpha_n^\nu\}e^{-2i(m+n)\tau} - \{\alpha_m^\mu, \tilde{\alpha}_{-n}^\nu\}e^{-2i(m-n)\tau} - \{\tilde{\alpha}_{-m}^\mu, \alpha^\nu\}e^{2i(m-n)\tau} +$$

$$+ \{\tilde{\alpha}_{-m}^\mu, \tilde{\alpha}_{-n}^\nu\}e^{2i(m+n)\tau}. \tag{C.61d}$$

From $\{P, X\}$, we obtain

$$\eta^{\mu\nu} = \{p^\mu, x^\nu\} + l_s^2\tau\{p^\mu, p^\nu\}, \tag{C.62a}$$

$$0 = \left(\{\alpha_m^\mu, x^\nu\}e^{-2im\tau} + \{\tilde{\alpha}_{-m}^\mu, x^\nu\}e^{2im\tau}\right) + l_s^2\tau\{\alpha_m^\mu, p^\nu\}e^{-2im\tau} +$$

$$+ l_s^2\tau\{\tilde{\alpha}_{-m}^\mu, p^\nu\}e^{2im\tau}, \tag{C.62b}$$

$$0 = \{p^\mu, \alpha_n^\nu\} e^{-2in\tau} - \{p^\mu, \tilde{\alpha}_{-n}^\nu\} e^{2in\tau}, \tag{C.62c}$$

$$2im\eta^{\mu\nu}\delta_{m+n,0} = \{\alpha_m^\mu, \alpha_n^\nu\} e^{-2i(m+n)\tau} - \{\alpha_m^\mu, \tilde{\alpha}_{-n}^\nu\} e^{-2i(m-n)\tau}$$
$$+ \{\tilde{\alpha}_{-m}^\mu, \alpha^\nu\} e^{2i(m-n)\tau} - \{\tilde{\alpha}_{-m}^\mu, \tilde{\alpha}_{-n}^\nu\} e^{2i(m+n)\tau}. \tag{C.62d}$$

Here, we used the representation of the delta function (3.61).

g) The equations (C.60)–(C.62) should hold for any τ. So, the coefficients of $1, \tau, \tau^2$ in (C.60a), (C.61a), (C.62a) should separately vanish. So, we obtain

$$\{p^\mu, p^\nu\} = [x^\mu, x^\nu] = 0, \qquad \{p^\mu, x^\nu\} = \eta^{\mu\nu}. \tag{C.63}$$

By requiring the coefficients of $e^{-im\tau}$ in (C.60b), (C.60c), (C.61b), (C.61c), (C.62b), (C.62c) to vanish, we obtain

$$\{p^\mu, \alpha_n^\nu\} = \{p^\mu, \tilde{\alpha}_n^\nu\} = \{x^\mu, \alpha_n^\nu\} = \{x^\mu, \tilde{\alpha}_n^\nu\} = 0. \tag{C.64}$$

OK, now let's go to the Poisson brackets among the oscillators $\alpha, \tilde{\alpha}$. From (C.60d)+(C.61d), we obtain

$$\{\alpha_m^\mu, \alpha_n^\nu\} = -\{\tilde{\alpha}_{-m}^\mu, \tilde{\alpha}_{-n}^\nu\} e^{4i(m+n)\tau}. \tag{C.65}$$

For this to be true for any τ, the Poisson brackets must vanish unless $m + n = 0$. So,

$$\{\alpha_m^\mu, \alpha_n^\nu\} = \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\} = 0, \qquad m + n \neq 0. \tag{C.66}$$

For $m + n = 0$, (C.65) implies

$$\{\alpha_m^\mu, \alpha_{-m}^\nu\} = \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_{-m}^\nu\}. \tag{C.67}$$

On the other hand, from (C.60d)−(C.61d), we obtain

$$\{\alpha_m^\mu, \tilde{\alpha}_{-n}^\nu\} = -\{\tilde{\alpha}_{-m}^\mu, \alpha_n^\nu\} e^{4i(m-n)\tau}. \tag{C.68}$$

For this to be true for any τ, the Poisson brackets must vanish unless $m - n = 0$. Namely,

$$\{\alpha_m^\mu, \tilde{\alpha}_n^\nu\} = 0, \qquad m + n \neq 0. \tag{C.69}$$

So, the only non-vanishing Poisson brackets are $\{\alpha_m^\mu, \alpha_{-m}^\nu\} = \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_{-m}^\nu\}$ and $\{\alpha_m^\mu, \tilde{\alpha}_{-m}^\nu\}$. Consider (C.60d)+(C.62d):

$$im\eta^{\mu\nu}\delta_{m+n,0} = \{\alpha_m^\mu, \alpha_n^\nu\} e^{-2i(m+n)\tau} + \{\tilde{\alpha}_{-m}^\mu, \alpha_n^\nu\} e^{2i(m-n)\tau}. \tag{C.70}$$

For $n = -m$, (C.70) means

$$\{\alpha^\mu_m, \alpha^\nu_{-m}\} = im\eta^{\mu\nu}\delta_{m+n,0}. \tag{C.71}$$

On the other hand, for $m = n$, (C.70) means

$$\{\tilde{\alpha}^\mu_{-m}, \alpha^\nu_m\} = 0. \tag{C.72}$$

Combining all, we obtain

$$\{p^\mu, p^\nu\} = \{x^\mu, x^\nu\} = 0, \qquad \{p^\mu, x^\nu\} = \eta^{\mu\nu}, \tag{C.73}$$

$$\{\alpha^\mu_m, \alpha^\nu_n\} = \{\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n\} = im\eta^{\mu\nu}\delta_{m+n,0}, \qquad \{\alpha^\mu_m, \tilde{\alpha}^\nu_n\} = 0. \tag{C.74}$$

Chapter 4

Solution to Exercise 4.1

a) Using the definition $\sigma^{\pm} = \tau \pm \sigma$, we can derive

$$\partial_\tau = \partial_+ + \partial_-, \qquad \partial_\sigma = \partial_+ - \partial_-, \tag{C.75}$$

$$d^2\sigma = d\tau d\sigma = \left| \frac{\partial(\tau, \sigma)}{\partial(\sigma^+, \sigma^-)} \right| d\sigma^+ d\sigma^- = \frac{1}{2} d\sigma^+ d\sigma^-. \tag{C.76}$$

In the second line, $\partial(\tau, \sigma)/\partial(\sigma^+, \sigma^-)$ is the Jacobian for the change of coordinates $(\tau, \sigma) \to (\sigma^+, \sigma^-)$. Another way to see the factor $1/2$ is

$$d\tau \wedge d\sigma = \frac{d\sigma^+ + d\sigma^-}{2} \wedge \frac{d\sigma^+ - d\sigma^-}{2} = -\frac{1}{4} d\sigma^+ \wedge d\sigma^- + \frac{1}{4} d\sigma^- \wedge d\sigma^+ = -\frac{1}{2} d\sigma^+ \wedge d\sigma^-. \tag{C.77}$$

So, the action can be written in the $\sigma^{\pm}$ coordinates as

$$\begin{aligned}
S &= \frac{T}{2} \int d^2\sigma (\dot{X}^2 - X'^2) \\
&= \frac{T}{2} \int \frac{1}{2} d\sigma^+ d\sigma^- \left[(\partial_+ X + \partial_- X)^2 - (\partial_+ X - \partial_- X)^2 \right] \\
&= T \int d\sigma^+ d\sigma^- \partial_+ X \cdot \partial_- X.
\end{aligned} \tag{C.78}$$

b) Under the variation

$$\delta X^\mu = a_n e^{2in\sigma^-} \partial_- X^\mu \equiv a_n f(\sigma^-) \partial_- X^\mu, \tag{C.79}$$

the variation of the action is

$$\begin{aligned}
\delta S &= T \int d\sigma^+ d\sigma^- (\partial_+ \delta X \cdot \partial_- X + \partial_+ X \cdot \partial_- \delta X) \\
&= T \int d\sigma^+ d\sigma^- [\partial_+ (a_n f(\sigma^-) \partial_- X) \cdot \partial_- X + \partial_+ X \cdot \partial_- (a_n f(\sigma^-) \partial_- X)] \\
&= T \int d\sigma^+ d\sigma^- [a_n f(\sigma^-) \partial_+ \partial_- X \cdot \partial_- X \\
&\qquad\qquad + \partial_- (\partial_+ X \cdot a_n f(\sigma^-) \partial_- X) - \partial_- \partial_+ X \cdot a_n f(\sigma^-) \partial_- X] \\
&= T \int d\sigma^+ d\sigma^- \partial_- (\partial_+ X \cdot a_n f(\sigma^-) \partial_- X).
\end{aligned} \tag{C.80}$$

So, the action is invariant up to a total derivative.

Note that the variation (C.79) corresponds to the coordinate change

$$\sigma^- \to \sigma^- + a_n e^{2in\sigma^-}. \tag{C.81}$$

c) Instead of (C.79), consider the variation with a_n being a function of $\sigma^\pm$:

$$\delta X^\mu = a_n(\sigma^+, \sigma^-) e^{2in\sigma^-} \partial_- X^\mu \equiv a_n(\sigma^+, \sigma^-) f(\sigma^-) \partial_- X^\mu, \tag{C.82}$$

Just by repeating what we did in (C.80),

$$
\begin{aligned}
\delta S &= T \int d\sigma^+ d\sigma^- (\partial_+ \delta X \cdot \partial_- X + \partial_+ X \cdot \partial_- \delta X) \\
&= T \int d\sigma^+ d\sigma^- [\partial_+ (a_n f(\sigma^-) \partial_- X) \cdot \partial_- X + \partial_+ X \cdot \partial_- (a_n(\sigma^+, \sigma^-) f(\sigma^-) \partial_- X)] \\
&= T \int d\sigma^+ d\sigma^- [\partial_+ a_n(\sigma^+, \sigma^-) f(\sigma^-) \partial_- X \cdot \partial_- X + a_n(\sigma^+, \sigma^-) f(\sigma^-) \partial_+ \partial_- X \cdot \partial_- X \\
&\qquad\qquad - \partial_- \partial_+ X \cdot a_n(\sigma^+, \sigma^-) f(\sigma^-) \partial_- X] \\
&= T \int d\sigma^+ d\sigma^- \partial_+ a_n(\sigma^+, \sigma^-) f(\sigma^-) \partial_- X \cdot \partial_- X. \tag{C.83}
\end{aligned}
$$

In the third equality, we dropped a total derivative term which can be set to zero by making a_n to vanish at infinity. Comparing this with the definition of the current,

$$\delta S = \int d\sigma^+ d\sigma^- (\partial_+ a_n \, j^+ + \partial_- a_n \, j^-), \tag{C.84}$$

we find the following expression for the current:

$$j^+ = Tf(\sigma^-) \partial_- X \cdot \partial_- X = Te^{2in\sigma^-} \partial_- X \cdot \partial_- X, \qquad j^- = 0. \tag{C.85}$$

Note that, using the expression for the stress-energy tensor (see BSS (2.37)), this can be written as

$$j^+ = Te^{2in\sigma^+} T_{--}. \tag{C.86}$$

The equation of motion for X^μ is $\partial_+ \partial_- X^\mu = 0$. If this is satisfied,

$$\partial_+ j^+ + \partial_- j^- = \partial_+ [Tf(\sigma^-) \partial_- X \cdot \partial_- X] = 2Tf(\sigma^-) \partial_- X \cdot \partial_+ \partial_- X = 0. \tag{C.87}$$

Namely, the current (C.85) is conserved when the equation of motion holds.

Note: another way to derive the conserved current, which may be more familiar to you, is as follows. The Noether procedure says the following: let the transformation

$\phi_i \rightarrow \phi_i + \delta\phi_i$ be a symmetry of the action, namely let the Lagrangian density L be invariant up to a total derivative: $L \rightarrow L + \partial_\alpha K^\alpha$. Then the conserved current is

$$j^\alpha = J^\alpha - K^\alpha, \qquad J^\alpha = \frac{\partial L}{\partial \partial_\alpha \phi_i} \delta\phi_i. \tag{C.88}$$

In the present case, from (C.79),

$$J^\pm = \frac{\partial L}{\partial \partial_\pm X} \cdot \delta X = T\partial_\mp X \cdot a_n f \partial_- X, \tag{C.89}$$

while, from (C.80),

$$K^+ = 0, \qquad K^- = Ta_n f \partial_+ X \cdot \partial_- X. \tag{C.90}$$

Therefore,

$$j^+ = J^+ - K^+ = a_n T f \partial_- X \cdot \partial_- X, \qquad j^- = J^- - K^- = 0. \tag{C.91}$$

This agrees with (C.85) (up to the irrelevant constant factor a_n). In general, the current j^α obtained this way is the same as the current obtained by making a_n a function of $\sigma^\pm$ and looking at the coefficient multiplying $\partial_\alpha a_n$.

d) Since the current j^α transforms as a vector on the worldsheet,

$$j^0 = j^\tau = \frac{\partial\tau}{\partial\sigma^+}j^+ + \frac{\partial\tau}{\partial\sigma^-}j^- = \frac{1}{2}(j^+ + j^-) = \frac{T}{2}e^{2in\sigma^-}\partial_- X \cdot \partial_- X = \frac{T}{2}e^{2in(\tau-\sigma)}T_{--}. \tag{C.92}$$

e) Recall the mode expansion of T_{--} (BBS (2.73)):

$$T_{--} = \frac{2}{\pi T}\sum_m e^{-2im(\tau-\sigma)}L_m, \qquad l_s = \sqrt{2\alpha'} = \frac{1}{\sqrt{\pi T}}. \tag{C.93}$$

Plugging this into (C.92),

$$j^0 = \frac{1}{\pi}\sum_m e^{2i(n-m)(\tau-\sigma)}L_m. \tag{C.94}$$

Then, the Noether charge associated with the current j^α is

$$Q = \int_0^\pi d\sigma\, j^0 = \frac{1}{\pi}\int_0^\pi d\sigma \sum_m e^{2i(n-m)(\tau-\sigma)}L_m = \sum_m \delta_{nm}L_m = L_n. \tag{C.95}$$

Solution to Exercise 4.2

For simplicity of notation we write X^{25} simply as X.

No matter what the boundary condition is, the equation of motion is given by:

$$(\partial_\tau^2 - \partial_\sigma^2)X(\tau, \sigma) = 0. \tag{C.96}$$

Therefore, we can use the separation of variable we used in Homework 2. Namely, we set $X = g(\tau)f(\sigma)$ and find that

$$\frac{f''(\sigma)}{f(\sigma)} = \frac{g''(\tau)}{g(\tau)} = \text{const} \equiv -k^2. \tag{C.97}$$

At this point, k can be any number; it doesn't even have to be real in principle (although we will find that it should be real in the end). If $k \neq 0$, (C.97) means that $f(\sigma) = e^{\pm ik\sigma}$, $g(\tau) = e^{\pm ik\tau}$. If $k = 0$, we have $f(\sigma) = 1, \sigma$ and $g(\tau) = 1, \tau$.

Taking a linear combination of all possible solutions, we conclude that we can always write $X(\tau, \sigma)$ satisfying (C.96) as

$$X(\tau, \sigma) = a_0 + a_1\sigma + a_2\tau + a_3\sigma\tau + \sum_{k \neq 0}(b_k e^{ik\sigma} + \tilde{b}_k e^{-ik\sigma})e^{-ik\tau}, \tag{C.98}$$

with $a_k, b_k, \tilde{b}_k$ being constants. It has not been determined yet what values k can take except that $k \neq 0$; it depends on the boundary condition we impose.

(i) The boundary condition is

$$X(\tau, 0) = X_0, \qquad X(\tau, \pi) = X_\pi. \tag{C.99}$$

Using (C.98),

$$X(\tau, 0) = a_0 + a_2\tau + \sum_k (b_k + \tilde{b}_k)e^{-ik\tau} \equiv X_0, \tag{C.100}$$

$$X(\tau, \pi) = a_0 + \pi a_1 + a_2\tau + \pi a_3\tau + \sum_k (b_k e^{i\pi k} + \tilde{b}_k e^{-i\pi k})e^{-ik\tau} \equiv X_\pi. \tag{C.101}$$

For this to hold for any τ, we need

$$a_0 = X_0, \qquad a_2 = 0, \qquad b_k + \tilde{b}_k = 0, \tag{C.102}$$

$$a_0 + \pi a_1 = X_\pi, \qquad a_2 + \pi a_3 = 0, \qquad b_k e^{i\pi k} + \tilde{b}_k e^{-i\pi k} = 0. \tag{C.103}$$

One can easily see that

$$a_0 = X_0, \qquad a_1 = \frac{1}{\pi}(X_\pi - X_0), \qquad a_2 = a_3 = 0, \qquad \tilde{b}_k = -b_k, \qquad k \in \mathbb{Z}. \tag{C.104}$$

Therefore,

$$X(\tau, \sigma) = X_0 + (X_\pi - X_0)\frac{\sigma}{\pi} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} b_n (e^{in\sigma} - e^{-in\sigma})e^{-in\tau}. \tag{C.105}$$

For X to be real, we need $b_{-n} = b_n^*$. If we set $b_n = -\frac{il_s}{2n}\alpha_n$, this can be written as

$$X(\tau, \sigma) = X_0 + (X_\pi - X_0)\frac{\sigma}{\pi} + l_s \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{\alpha_n}{n} \sin(n\sigma)\, e^{-in\tau}, \tag{C.106}$$

where the reality of X requires $\alpha_{-n} = \alpha_n^*$.

This solution describes an open string whose endpoint at $\sigma = 0$ is ending on a D-brane sitting at $X = X_0$ and whose endpoint at $\sigma = \pi$ is ending on a D-brane sitting at $X = X_\pi$.

The spacetime momentum current $P_\alpha = T\partial_\alpha X$ (BBS (2.67)) is easily computed from (C.106) as

$$P_\tau = -iTl_s \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \alpha_n \sin(n\sigma)e^{-in\tau}, \qquad P_\sigma = T\frac{X_\pi - X_0}{\pi} + Tl_s \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \alpha_n \cos(n\sigma)e^{-in\tau}.$$
$$\tag{C.107}$$

To make it easier to remember that this is a current, let us write $j_\alpha = P_\alpha$. On the worldsheet, there is a flow of a conserved current j_α, and its associated charge, $Q = \int_0^\pi d\sigma\, j^\tau$, is the spacetime momentum. For this charge (= spacetime momentum) to be conserved (time-independent), there must be no current flowing into / out of the string at the boundary $\sigma = 0, \pi$. In equations, this means that the net flow into the string must vanish:

$$j_\sigma|_{\sigma=0} - j_\sigma|_{\sigma=\pi} = 0. \tag{C.108}$$

Using (C.107), we can explicitly evaluate this:

$$j_\sigma|_{\sigma=0} - j_\sigma|_{\sigma=\pi} = P_\sigma(\tau, 0) - P_\sigma(\tau, \pi) = Tl_s \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} (1 - (-1)^n)\alpha_n e^{-in\tau}$$

$$= 2Tl_s \sum_{m \in \mathbb{Z}} \alpha_{2m+1} e^{-i(2m+1)\tau} \neq 0. \tag{C.109}$$

Because this is nonvanishing, spacetime momentum is flowing into the string at the boundary and is not conserved. This is because the boundary condition (C.99), or the existence of D-branes, breaks translational symmetry. More intuitively, D-branes are always pushing/pulling the string to keep the endpoints on them.

Another way to see this non-conservation is to compute the associated charge

$$Q = \int_0^\pi d\sigma \, j^\tau, \qquad j^\tau = h_{\tau\tau} j_\tau = -j_\tau = iT l_s \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \alpha_n \sin(n\sigma) e^{-in\tau}. \tag{C.110}$$

One computes

$$Q = iT l_s \int_0^\pi d\sigma \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \alpha_n \sin(n\sigma) e^{-in\tau} = 2iT l_s \sum_{m \in \mathbb{Z}} \frac{\alpha_{2m+1}}{2m+1} e^{-i(2m+1)\tau}. \tag{C.111}$$

This is not constant in time, meaning that there is flow into / out of the open string from the D-branes. Comparing (C.109) and (C.111), one can explicitly check that (C.109) is really the net flow of charge Q into the string:

$$\frac{dQ}{dt} = j_\sigma|_{\sigma=0} - j_\sigma|_{\sigma=\pi}. \tag{C.112}$$

(ii) Now the boundary condition is

$$X(\tau, 0) = X_0, \qquad \partial_\sigma X(\tau, \pi) = 0. \tag{C.113}$$

Using (C.98),

$$X(\tau, 0) = a_0 + a_2 \tau + \sum_k (b_k + \tilde{b}_k) e^{-ik\tau} \equiv X_0, \tag{C.114}$$

$$\partial_\sigma X(\tau, \pi) = a_1 + a_3 \tau + \sum_k ik(b_k e^{i\pi k} - \tilde{b}_k e^{-i\pi k}) e^{-ik\tau} \equiv 0. \tag{C.115}$$

For these to hold for any τ, we need

$$a_0 = X_0, \qquad a_2 = 0, \qquad b_k + \tilde{b}_k = 0, \tag{C.116}$$
$$a_1 = 0, \qquad a_3 = 0, \qquad b_k e^{i\pi k} - \tilde{b}_k e^{-i\pi k} = 0. \tag{C.117}$$

This means that

$$a_0 = X_0, \qquad a_1 = a_2 = a_3 = 0, \qquad \tilde{b}_k = -b_k, \qquad k \in \mathbb{Z} + \frac{1}{2}. \tag{C.118}$$

Setting $b_k = \frac{il_s}{2k} \alpha_k$, we can write $X(\tau, \sigma)$ as

$$X(\tau, \sigma) = X_0 + l_s \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{\alpha_n}{n} \sin(n\sigma) e^{-in\tau}. \tag{C.119}$$

This solution describes to an open string whose endpoint at $\sigma = 0$ is ending on a D-brane sitting at $X = X_0$ and whose endpoint at $\sigma = \pi$ is free.

Solution to Exercise 4.3

1) From BBS (2.40), (2.41),

$$\partial_- X^\mu = l_s \sum_n \alpha_n^\mu e^{-2in(\tau-\sigma)}, \qquad \partial_+ X^\mu = l_s \sum_n \tilde\alpha_n^\mu e^{-2in(\tau+\sigma)}. \tag{C.120}$$

By plugging this into the expression for stress energy tensor in BBS (2.36), (2.37),

$$T_{--} = \partial_- X \partial_- X = l_s^2 \sum_{k,n} \alpha_k^\mu \alpha_{\mu n} e^{-2i(k+n)(\tau-\sigma)}. \tag{C.121}$$

By setting $k+n \equiv m$, this can be written as

$$
\begin{aligned}
T_{--} &= l_s^2 \sum_{m,n} \alpha_{m-n}^\mu \alpha_{\mu n} e^{-2im(\tau-\sigma)} \\
&= 2l_s^2 \sum_m \left(\frac{1}{2} \sum_n \alpha_{m-n}^\mu \alpha_{\mu n} \right) e^{-2im(\tau-\sigma)} = 2l_s^2 \sum_m L_m e^{-2im(\tau-\sigma)},
\end{aligned} \tag{C.122}
$$

$$L_m \equiv \frac{1}{2} \sum_n \alpha_{m-n}^\mu \alpha_{\mu n}. \tag{C.123}$$

T_{++} and $\tilde L_m$ go exactly the same way.

2) Using the identities

$$[AB, C] = A[B, C] + [A, C]B, \qquad [A, BC] = B[A, C] + [A, B]C, \tag{C.124}$$

repeatedly, one can show

$$[AB, CD] = AC[B, D] + A[B, C]D + C[A, D]B + [A, C]DB. \tag{C.125}$$

Now, using BBS (2.40),

$$[\alpha_m, \alpha_n] = im\eta^{\mu\nu}\delta_{m+n,0}, \tag{C.126}$$

the Poisson bracket for

$$L_m = \frac{1}{2} \sum_n \alpha_{m-n}^\mu \alpha_{\mu n}, \tag{C.127}$$

is computed as

$$
\begin{aligned}
[L_m, L_n] &= \frac{1}{4} \sum_{k,l} [\alpha^{\mu}_{m-k} \alpha_{\mu k}, \alpha^{\nu}_{n-l} \alpha_{\nu l}] \\
&= \frac{1}{4} \sum_{k,l} (\alpha^{\mu}_{m-k} \alpha^{\nu}_{n-l} [\alpha_{\mu k}, \alpha_{\nu l}] + \alpha^{\mu}_{m-k} [\alpha_{\mu k}, \alpha^{\nu}_{n-l}] \alpha_{\nu l} + \alpha^{\nu}_{n-l} [\alpha^{\mu}_{m-k}, \alpha_{\nu l}] \alpha_{\mu k} [\alpha^{\mu}_{m-k}, \alpha^{\nu}_{n-l}] \alpha_{\nu l} \alpha_{\mu k}) \\
&= \frac{i}{4} \sum_{k,l} \Big[k \eta_{\mu\nu} \delta_{k+l,0} \alpha^{\mu}_{m-k} \alpha^{\nu}_{n-l} + k \delta^{\nu}_{\mu} \delta_{k+n-l,0} \alpha^{\mu}_{m-k} \alpha_{\nu l} \\
&\qquad + (m-k) \delta^{\mu}_{\nu} \delta_{m-k+l,0} \alpha^{\nu}_{n-l} \alpha_{\mu k} + (m-k) \eta^{\mu\nu} \delta_{m-k+n-l} \alpha_{\nu l} \alpha_{\mu k} \Big] \\
&= \frac{i}{4} \sum_{k,l} \Big[k (\delta_{k+l,0} \alpha_{m-k} \cdot \alpha_{n-l} + \delta_{k+n-l,0} \alpha_{m-k} \cdot \alpha_{l}) \\
&\qquad + (m-k)(\delta_{m-k+l,0} \alpha_{n-l} \cdot \alpha_{k} + \delta_{m-k+n-l} \alpha_{l} \cdot \alpha_{k}) \Big] \\
&= \frac{i}{2} \sum_{k} [k \alpha_{m-k} \cdot \alpha_{k+n} + (m-k) \alpha_{m+n-k} \cdot \alpha_{k}]. \tag{C.128}
\end{aligned}
$$

In the second equality we used (C.125). By setting $k \to k - n$ in the first term of (C.128),

$$
\begin{aligned}
[L_m, L_n] &= \frac{i}{2} \sum_{k} [(k-n) \alpha_{m+n-k} \cdot \alpha_{k} + (m-k) \alpha_{m+n-k} \cdot \alpha_{k}] \\
&= \frac{i}{2} (m-n) \sum_{k} \alpha_{m+n-k} \cdot \alpha_{k} \\
&= i(m-n) L_{m+n}. \tag{C.129}
\end{aligned}
$$

Chapter 5

Solution to Exercise 5.1

i) We want to compute inner products between the states

$$|a\rangle = L_{-2}|\phi\rangle, \qquad |b\rangle = L_{-1}^2|\phi\rangle, \tag{C.130}$$

where $|\phi\rangle$ satisfies

$$L_0|\phi\rangle = h|\phi\rangle, \qquad L_{n>0}|\phi\rangle = 0, \qquad \langle\phi|\phi\rangle \neq 0. \tag{C.131}$$

For example, let's compute

$$\langle a|a\rangle = \langle\phi|L_{-2}^\dagger L_{-2}|\phi\rangle = \langle\phi|L_2 L_{-2}|\phi\rangle. \tag{C.132}$$

Using the Virasoro algebra,

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{6}m(m^2 - 1)\delta_{m+n,0}, \tag{C.133}$$

we find

$$L_2 L_{-2} = [L_2, L_{-2}] + L_{-2}L_2 = 4L_0 + \frac{c}{2} + L_{-2}L_2. \tag{C.134}$$

Sandwiching (C.134) with $\langle\phi|$ and $|\phi\rangle$, we find

$$\langle a|a\rangle = \langle\phi|(4L_0 + \frac{c}{2} + L_{-2}L_2)|\phi\rangle = \langle\phi|(4h + \frac{c}{2} + 0)|\phi\rangle = (4h + \frac{c}{2})\langle\phi|\phi\rangle, \tag{C.135}$$

where we used (C.131). So, in some sense, this is a generalization of the annihilation (a) / creation ($a^\dagger$) operator algebra of a quantum harmonic oscillator which you are familiar with. Similarly, we find (omitting details of the computation!)

$$\langle a|b\rangle = \langle b|a\rangle = 6h\langle\phi|\phi\rangle, \qquad \langle b|b\rangle = (8h^2 + 4h)\langle\phi|\phi\rangle. \tag{C.136}$$

Therefore, the determinant is

$$\Delta = \langle\phi|\phi\rangle^2 \det \begin{pmatrix} 4h + c/2 & 6h \\ 6h & 8h^2 + 4h \end{pmatrix} = 2h(16h^2 - 10h + 2ch + c)\langle\phi|\phi\rangle^2. \tag{C.137}$$

ii) If we set $h = -1$, then

$$\Delta = 0 \quad \Rightarrow \quad c = 26. \tag{C.138}$$

Vanishing of Δ means that there is a linear combination of $|a\rangle$ and $|b\rangle$ which has zero norm. Now, note that $h = -1$ means that

$$(L_0 - 1)|a\rangle = (L_0 - 1)|b\rangle = 0, \tag{C.139}$$

because $|\phi\rangle$ has $h = -1$ and L_{-n} raises the level (L_0 eigenvalue) by n. Comparing this with the physical state (mass shell) condition

$$(L_0 - a)|\psi\rangle = 0, \tag{C.140}$$

we see that $|a\rangle$ and $|b\rangle$ are physical states for $a = 1$. (Don't confuse the state $|a\rangle$ and the number a appearing in the mass shell condition (C.140).) So, this means that, for $c = 26$, there appears a physical ($h = 1$) state with zero norm. Appearance of additional zero-norm states is important for the absence of negative norm states and it requires the critical dimension of bosonic string, $D = 26^{\ddagger}$.

iii) If we set $c = 1/2$, then

$$\Delta = 0 \quad \Rightarrow \quad h = 0, \frac{1}{16}, \frac{1}{2}. \tag{C.141}$$

Solution to Exercise 5.2

(i) From the open string mass shell condition (5.22):

$$\alpha' M^2 = N - 1, \tag{C.142}$$

we obtain

$$
\begin{aligned}
|\phi_1\rangle &= \alpha^i_{-1}|0; k^\mu\rangle & : \quad & N = 1, & \alpha' M^2 &= 0, \\
|\phi_2\rangle &= \alpha^i_{-1}\alpha^j_{-1}|0; k^\mu\rangle & : \quad & N = 2, & \alpha' M^2 &= 1, \\
|\phi_3\rangle &= \alpha^i_{-3}|0; k^\mu\rangle & : \quad & N = 3, & \alpha' M^2 &= 2, \\
|\phi_4\rangle &= \alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-2}|0; k^\mu\rangle & : \quad & N = 4, & \alpha' M^2 &= 3.
\end{aligned} \tag{C.143}
$$

---

‡If the level-2 Kac determinant,

$$det_2(c, h) = \begin{pmatrix} \langle a|a\rangle & \langle a|b\rangle \\ \langle b|a\rangle & \langle b|b\rangle \end{pmatrix},$$

is equal to zero then the norms $\langle a|a\rangle$, $\langle a|b\rangle$, $\langle b|a\rangle$ and $\langle b|b\rangle$ are linearly independent and so there exists a combination of states $|a\rangle$ and $|b\rangle$ such that the norm of this combination is zero. Thus we have proven that for $h = -1$ and $c = 26$ there exists zero norm states.

$|\phi_1\rangle$ is a massless vector which must be a gauge boson=photon. There are 24 states corresponding to the vector representation of the little group $SO(24)$ for a massless particle. $|\phi_2\rangle$ together with $\alpha^i_{-2}|0; k^\mu\rangle$ makes up $\frac{24\cdot25}{2} + 24 = 324 = \frac{25\cdot26}{2} - 1$ states, corresponding to the traceless symmetric second-rank tensor of the little group $SO(25)$ for a massive particle. Likewise for higher level states.

(ii) From the closed string mass shell condition (5.25):

$$\alpha' M^2 = 4(N - 1) = 4(\tilde{N} - 1), \tag{C.144}$$

we obtain

$$\begin{aligned}
|\phi_1\rangle &= \alpha^i_{-1}\tilde{\alpha}^j_{-1}|0; k^\mu\rangle &&: & N = \tilde{N} = 1, && \alpha' M^2 = 0, \\
|\phi_2\rangle &= \alpha^i_{-1}\alpha^j_{-1}\tilde{\alpha}^k_{-2}|0; k^\mu\rangle &&: & N = \tilde{N} = 2, && \alpha' M^2 = 4.
\end{aligned} \tag{C.145}$$

$|\phi_1\rangle$ is massless and has $24^2 = 576 = 299 + 276 + 1$ states. This splits into the symmetric traceless representation of $SO(24)$ (graviton, $\frac{24\cdot25}{2} - 1 = 299$ states), the antisymmetric representation (Kalb-Ramond B-field, $\frac{24\cdot23}{2} = 276$ states), and the trace part (dilaton, 1 state).

(iii) The state

$$|\phi_3\rangle = \alpha^i_{-1}\tilde{\alpha}^j_{-2}|0; k^\mu\rangle, \tag{C.146}$$

has $N = 1$, $\tilde{N} = 2$ and does not satisfy the level-matching condition $N = \tilde{N}$. So, this is not a physical state.

Solution to Exercise 5.3

The mode expansions for X^-, X^i are

$$X^- = x^- + l_s^2 p^- \tau + i l_s {\sum}' \frac{1}{n} \alpha^-_n e^{-in\tau} \cos n\sigma, \tag{C.147}$$

$$X^i = x^i + l_s^2 p^i \tau + i l_s {\sum}' \frac{1}{n} \alpha^-_i e^{-in\tau} \cos n\sigma, \tag{C.148}$$

where $\sum'$ is summation over $n \in \mathbb{Z}$ with $n \neq 0$. One readily computes

$$\dot{X}^- = l_s^2 p^- + l_s {\sum}' \alpha^-_n e^{-in\tau} \cos n\sigma = l_s \sum_{n\in\mathbb{Z}} \alpha^-_n e^{-in\tau} \cos n\sigma, \tag{C.149}$$

$$X^{-\prime} = -i l_s {\sum}' \alpha^-_n e^{-in\tau} \sin n\sigma = -i l_s \sum_{n\in\mathbb{Z}} \alpha^-_n e^{-in\tau} \sin n\sigma, \tag{C.150}$$

where $\alpha_0^- = l_s p^-$. Therefore

$$\dot{X}^- \pm X^{-\prime} = l_s \sum_{n \in \mathbb{Z}} \alpha_n^- e^{-in(\tau \pm \sigma)}, \tag{C.151}$$

and similarly

$$\dot{X}^i \pm X^{i\prime} = l_s \sum_{n \in \mathbb{Z}} \alpha_n^i e^{-in(\tau \pm \sigma)}, \tag{C.152}$$

where $\alpha_0^i = l_s p^i$.

Therefore, the constraint (5.21)

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{2p^+ l_s^2}(\dot{X}^i \pm X^{i\prime})^2, \tag{C.153}$$

becomes, after substituting (C.151) and (C.152) in,

$$l_s \sum_{n \in \mathbb{Z}} \alpha_n^- e^{-in(\tau \pm \sigma)} = \frac{1}{2p^+} \sum_{k,m \in \mathbb{Z}} \alpha_k^i \alpha_m^i e^{-i(k+m)(\tau \pm \sigma)} = \frac{1}{2p^+} \sum_{n,m \in \mathbb{Z}} \alpha_{n-m}^i \alpha_m^i e^{-in(\tau \pm \sigma)}, \tag{C.154}$$

where in the second equality we set $k + m = n$. Equating the coefficients of $e^{-in(\tau \pm \sigma)}$ on both sides, we obtain

$$\alpha_n^- = \frac{1}{2p^+ l_s} \sum_m \alpha_{n-m}^i \alpha_m^i. \tag{C.155}$$

Actually, the $n = 0$ case is ambiguous because of the ordering issue of operators and we should introduce a constant a, which is undetermined at this level, to write the above as

$$\alpha_n^- = \frac{1}{2p^+ l_s} \left(\sum_m : \alpha_{n-m}^i \alpha_m^i : -a\delta_{n,0} \right). \tag{C.156}$$

Chapter 6

Solution to Exercise 6.1

By acting on

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}, \tag{C.157}$$

with $\partial_\rho \partial_\sigma$, we get

$$\partial_\rho \partial_\sigma (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}\eta_{\mu\nu}\partial_\rho \partial_\sigma (\partial \cdot \epsilon). \tag{C.158}$$

a) By setting $\rho = \mu, \sigma = \nu$ in (C.158), we get

$$2\Box(\partial \cdot \epsilon) = \frac{2}{d}\Box(\partial \cdot \epsilon). \tag{C.159}$$

If $d \neq 1$, this means

$$\Box(\partial \cdot \epsilon) = 0. \tag{C.160}$$

b) By setting $\sigma = \mu$ in (C.158),

$$\Box \partial_\rho \epsilon_\mu + \partial_\rho \partial_\mu (\partial \cdot \epsilon) = \frac{2}{d}\partial_\rho \partial_\mu (\partial \cdot \epsilon). \tag{C.161}$$

Namely,

$$(\frac{2}{d} - 1)\partial_\rho \partial_\mu (\partial \cdot \epsilon) = \Box \partial_\rho \epsilon_\mu. \tag{C.162}$$

Taking the symmetric and antisymmetric parts of this equation under $\rho \leftrightarrow \mu$, we obtain

$$(\frac{2}{d} - 1)\partial_\rho \partial_\mu (\partial \cdot \epsilon) = \frac{1}{2}\Box(\partial_\rho \epsilon_\mu + \partial_\mu \epsilon_\rho), \tag{C.163}$$

$$0 = \Box(\partial_\rho \epsilon_\mu - \partial_\mu \epsilon_\rho). \tag{C.164}$$

Using (1) on the right hand side of (C.163),

$$(\frac{2}{d} - 1)\partial_\rho \partial_\mu (\partial \cdot \epsilon) = \frac{1}{2}\Box \left(\frac{2}{d}\eta_{\rho\mu}(\partial \cdot \epsilon) \right)$$

$$= \frac{\eta_{\rho\mu}}{d}\Box(\partial \cdot \epsilon) = 0, \tag{C.165}$$

where in the last equality we used (C.160). So, unless $d = 2$, we have

$$\partial_\rho \partial_\mu (\partial \cdot \epsilon) = 0. \tag{C.166}$$

c) (C.166) means that $\partial \cdot \epsilon$ contains only a constant term and linear terms in x^μ. By choosing coefficients appropriately, we can write $\partial \cdot \epsilon$ as

$$\partial \cdot \epsilon = d(\lambda - 2b_\alpha x^\alpha). \tag{C.167}$$

d) Using (C.167), now (2) is

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2\eta_{\mu\nu}(\lambda - 2b_\alpha x^\alpha). \tag{C.168}$$

Acting ∂_α on this, we get

$$\partial_\alpha \partial_\mu \epsilon_\nu + \partial_\alpha \partial_\nu \epsilon_\mu = -4\eta_{\mu\nu}b_\alpha. \tag{C.169}$$

If we set $\alpha \leftrightarrow \nu$,

$$\partial_\nu \partial_\mu \epsilon_\alpha + \partial_\nu \partial_\alpha \epsilon_\mu = -4\eta_{\mu\alpha}b_\nu. \tag{C.170}$$

Subtracting (C.170) from (C.169),

$$\partial_\mu(\partial_\alpha \epsilon_\nu - \partial_\nu \epsilon_\alpha) = 4(\eta_{\mu\alpha}b_\nu - \eta_{\mu\nu}b_\alpha). \tag{C.171}$$

e) (C.171) means that $\partial_\alpha \epsilon_\nu - \partial_\nu \epsilon_\alpha$ must contain a constant term and linear terms in x^μ. By setting the constant term to be $2\omega_{\alpha\nu}$, we obtain

$$\begin{aligned} \partial_\alpha \epsilon_\nu - \partial_\nu \epsilon_\alpha &= 2\omega_{\alpha\nu} + 4(\eta_{\mu\alpha}b_\nu - \eta_{\mu\nu}b_\alpha)x^\mu \\ &= 2\omega_{\alpha\nu} + 4(x_\alpha b_\nu - x_\nu b_\alpha). \end{aligned} \tag{C.172}$$

Note that $\omega_{\alpha\nu}$ is antisymmetric under $\alpha \leftrightarrow \nu$, because $\partial_\alpha \epsilon_\nu - \partial_\nu \epsilon_\alpha$ is antisymmetric.

f) Let's write (C.168) again, but with $\mu \to \alpha$:

$$\partial_\alpha \epsilon_\nu + \partial_\nu \epsilon_\alpha = 2(\lambda - 2(b \cdot x))\eta_{\alpha\nu}. \tag{C.173}$$

Summing (C.172) and (C.173) and dividing the result by two,

$$\begin{aligned} \partial_\alpha \epsilon_\nu &= \omega_{\alpha\nu} + (\lambda - 2(b \cdot x))\eta_{\alpha\nu} + 2(x_\alpha b_\nu - x_\nu b_\alpha) \\ &= (\omega_{\alpha\nu} + \lambda\eta_{\alpha\nu}) + 2(-b_\rho\eta_{\alpha\nu} + b_\nu\eta_{\alpha\rho} - b_\alpha\eta_{\nu\rho})x^\rho. \end{aligned} \tag{C.174}$$

This means that ϵ_ν includes terms only up to $\mathcal{O}(x^2)$. Let us set

$$\epsilon_\nu = a_\nu + e_{\nu\rho}x^\rho + f_{\nu\rho\sigma}x^\rho x^\sigma, \qquad f_{\nu\rho\sigma} = f_{\nu\sigma\rho}. \tag{C.175}$$

Acting on this by ∂_α gives

$$\partial_\alpha \epsilon_\nu = e_{\nu\alpha} + 2f_{\nu\rho\alpha}x^\rho. \tag{C.176}$$

Comparing this with (C.174), we see that

$$e_{\nu\alpha} = \omega_{\alpha\nu} + \lambda\eta_{\alpha\nu}, \qquad f_{\nu\rho\alpha} = -b_\rho\eta_{\alpha\nu} + b_\nu\eta_{\alpha\rho} - b_\alpha\eta_{\nu\rho}. \tag{C.177}$$

Plugging this back into (C.175) yields

$$\begin{aligned}
\epsilon_\nu &= a_\nu + (\omega_{\rho\nu} + \lambda\eta_{\rho\nu})x^\rho + (-b_\rho\eta_{\sigma\nu} + b_\nu\eta_{\sigma\rho} - b_\sigma\eta_{\nu\rho})x^\rho x^\sigma \\
&= a_\nu + \lambda x_\nu + \omega_{\rho\nu}x^\rho + b_\nu x^2 - 2(b\cdot x)x_\nu,
\end{aligned} \tag{C.178}$$

or

$$\epsilon_\mu = a_\mu + \lambda x_\mu + \omega_{\nu\mu}x^\nu + b_\mu x^2 - 2(b\cdot x)x_\mu. \tag{C.179}$$

Chapter 7

Solution to Exercise 7.1

(a) The expansion on the Lorentzian cylinder is

$$X(\tau, \sigma) = x + 4p\tau + i \sum_{n \neq 0} \frac{1}{n} (\alpha_n e^{2in\sigma} + \tilde{\alpha}_n e^{-2in\sigma}) e^{-2in\tau}. \tag{C.180}$$

After performing a Wick rotation $\tau \to -i\tau$ and setting $\zeta = 2(\tau - i\sigma)$, this becomes

$$X = x - 4ip\tau + i \sum_{n \neq 0} \frac{1}{n} (\alpha_n e^{-2n(\tau - i\sigma)} + \tilde{\alpha}_n e^{-2n(\tau + i\sigma)})$$

$$= x - ip(\zeta + \bar{\zeta}) + i \sum_{n \neq 0} \frac{1}{n} (\alpha_n e^{-n\zeta} + \tilde{\alpha}_n e^{-n\tilde{\zeta}}). \tag{C.181}$$

(b) In terms of $z = e^{\zeta}$, $\bar{z} = e^{\bar{\zeta}}$, the above becomes

$$X(z, \bar{z}) = x - ip \log|z|^2 + i \sum_{n \neq 0} \frac{1}{n} (\alpha_n z^{-n} + \tilde{\alpha}_n \bar{z}^{-n}). \tag{C.182}$$

(c) In the expression

$$:X(z, \bar{z}) X(w, \bar{w}): = \quad :\left(x - ip \log|z|^2 + i \sum_{m \neq 0} \frac{1}{m} (\alpha_m z^{-m} + \tilde{\alpha}_m \bar{z}^{-m}) \right)$$

$$\left(x - ip \log|w|^2 + i \sum_{n \neq 0} \frac{1}{n} (\alpha_n w^{-n} + \tilde{\alpha}_n \bar{w}^{-n}) \right): ,$$

$$\tag{C.183}$$

the terms that get affected by the creation-annihilation normal ordering are

$$:(-ip \log|z|^2)x: , \tag{C.184}$$

$$:\left(i \sum_{m > 0} \frac{1}{m} \alpha_m z^{-m} \right) \left(i \sum_{n < 0} \frac{1}{n} \alpha_n w^{-n} \right): , \tag{C.185}$$

$$:\left(i \sum_{m > 0} \frac{1}{m} \tilde{\alpha}_m \bar{z}^{-m} \right) \left(i \sum_{n < 0} \frac{1}{n} \tilde{\alpha}_n \bar{w}^{-n} \right): . \tag{C.186}$$

$$- 287 -$$

For example, (C.184) is

$$:(-ip\log|z|^2)x: = -ixp\log|z|^2 = -i([x,p]+px)\log|z|^2$$
$$= \log|z|^2 - ipx\log|z|^2. \tag{C.187}$$

In the last expression, the second term is the same as the one that appears in XX, but the first term is extra. Namely, this extra term contributes to $:XX:-XX$.

Next, (C.185) is

$$:\left(i\sum_{m>0}\frac{1}{m}\alpha_m z^{-m}\right)\left(i\sum_{n<0}\frac{1}{n}\alpha_n w^{-n}\right): = -\sum_{m>0,n<0}\frac{1}{mn}:\alpha_m\alpha_n:z^{-m}w^{-n}$$

$$= \sum_{m,n>0}\frac{1}{mn}:\alpha_m\alpha_{-n}:z^{-m}w^{n} = \sum_{m,n>0}\frac{1}{mn}\alpha_{-n}\alpha_m z^{-m}w^{n}$$

$$= \sum_{m,n>0}\frac{1}{mn}\left([\alpha_{-n},\alpha_m]+\alpha_m\alpha_{-n}\right)z^{-m}w^{n} = \sum_{m,n>0}\frac{1}{mn}\left(-n\delta_{m,n}+\alpha_m\alpha_{-n}\right)z^{-m}w^{n}$$

$$= -\sum_{n>0}\frac{1}{n}\left(\frac{w}{z}\right)^n + \sum_{m,n>0}\frac{1}{mn}\alpha_m\alpha_{-n}z^{-m}w^{n}. \tag{C.188}$$

Again, the first term in the last expression is extra. In exactly the same way, (C.186) is

$$:\left(i\sum_{m>0}\frac{1}{m}\tilde\alpha_m\overline{z}^{-m}\right)\left(i\sum_{n<0}\frac{1}{n}\tilde\alpha_n\overline{w}^{-n}\right): = -\sum_{n>0}\frac{1}{n}\left(\frac{\overline{w}}{\overline{z}}\right)^n + \sum_{m,n>0}\frac{1}{mn}\tilde\alpha_m\tilde\alpha_{-n}\overline{z}^{-m}\overline{w}^{n},$$

$$\tag{C.189}$$

and the first term in the last expression is extra.

Collecting all the extra terms, we find

$$:X(z,\overline{z})X(w,\overline{w}): = X(z,\overline{z})X(w,\overline{w}) + \log|z|^2 - \sum_{n>0}\frac{1}{n}\left[\left(\frac{w}{z}\right)^n + \left(\frac{\overline{w}}{\overline{z}}\right)^n\right]$$

$$= X(z,\overline{z})X(w,\overline{w}) + \log|z|^2 + \log\left(1-\frac{w}{z}\right) + \log\left(1-\frac{\overline{w}}{\overline{z}}\right)$$

$$= X(z,\overline{z})X(w,\overline{w}) + \log|z-w|^2. \tag{C.190}$$

In the second equality, we used

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = -\log(1-x), \qquad |x| < 1. \tag{C.191}$$

Therefore, (C.190) is valid only for $|z| > |w|$. This is because, in the radial quantization, the operator product $X(z,\overline{z})X(w,\overline{w})$ makes sense only if $X(z,\overline{z})$ is "after" $X(w,\overline{w})$ in the radial time.

Solution to Exercise 7.2

(a) The transformation of the 2-point function is

$$\delta_\epsilon G(z_1, z_2) = \delta_\epsilon \langle \Phi_1(z_1)\Phi_2(z_2) \rangle$$
$$= \langle \delta_\epsilon \Phi_1(z_1) \, \Phi_2(z_2) \rangle + \langle \Phi_1(z_1) \, \delta_\epsilon \Phi_2(z_2) \rangle$$
$$= \langle [\epsilon(z_1)\partial_1 + h_1 \partial\epsilon(z_1)]\Phi_1(z_1) \, \Phi_2(z_2) \rangle$$
$$\qquad + \langle \Phi_1(z_1) \, [\epsilon(z_2)\partial_2 + h_2 \partial\epsilon(z_2)]\Phi_2(z_2) \rangle$$
$$= [\epsilon(z_1)\partial_1 + h_1 \partial\epsilon(z_1) + \epsilon(z_2)\partial_2 + h_2 \partial\epsilon(z_2)]\langle \Phi_1(z_1)\Phi_2(z_2) \rangle. \quad \text{(C.192)}$$

For the 2-point function to be invariant, this must vanish. Namely,

$$[\epsilon(z_1)\partial_1 + h_1 \partial\epsilon(z_1) + \epsilon(z_2)\partial_2 + h_2 \partial\epsilon(z_2)]G(z_1, z_2) = 0. \quad \text{(C.193)}$$

(b) By setting $\epsilon(z) = 1$ in (C.193), we have

$$(\partial_1 + \partial_2)G(z_1, z_2) = 0. \quad \text{(C.194)}$$

If we set $x = z_1 - z_2$, $y = z_1 + z_2$, this means

$$\partial_y G(x, y) = 0. \quad \text{(C.195)}$$

Therefore, G is a function of $x = z_1 - z_2$ only, i.e.,

$$G(z_1, z_2) = G(z_1 - z_2). \quad \text{(C.196)}$$

(c) By setting $\epsilon(z) = z$ in (C.193), we have

$$(z_1\partial_1 + h_1 + z_2\partial_2 + h_2)G = 0. \quad \text{(C.197)}$$

Note that

$$\partial_{z_1} G(z_1 - z_2) = \partial_x G(x), \qquad \partial_{z_2} G(z_1 - z_2) = -\partial_x G(x). \quad \text{(C.198)}$$

Therefore, (C.197) becomes

$$(x\partial_x + h_1 + h_2)G(x) = 0, \quad \text{(C.199)}$$

namely

$$\frac{dG}{G} = -(h_1 + h_2)\frac{dx}{x}. \quad \text{(C.200)}$$

Integrating this, we get

$$G(x) = \frac{C}{x^{h_1+h_2}}, \quad \text{(C.201)}$$

with C a constant of integration.

(d) By setting $\epsilon(z) = z^2$ in (C.193), we get

$$0 = [z_1^2 \partial_1 + 2h_1 z_1 + z_2^2 \partial_2 + 2h_2 z_2] \frac{C}{x^{h_1+h_2}}$$

$$= [(z_1^2 - z_2^2) \partial_x + 2(h_1 z_1 + h_2 z_2)] \frac{C}{x^{h_1+h_2}}$$

$$= [-(h_1 + h_2)(z_1 + z_2) + 2(h_1 z_1 + h_2 z_2)] \frac{C}{x^{h_1+h_2}}$$

$$= (h_1 - h_2)(z_1 - z_2) \frac{C}{x^{h_1+h_2}}. \tag{C.202}$$

In the second equality we used (C.198), and in the third equality we used that $\partial_x x^{-(h_1+h_2)} = -(h_1 + h_2)x^{-(h_1+h_2+1)} = -\frac{h_1+h_2}{z_1-z_2}x^{-(h_1+h_2)}$. For (C.202) to vanish, we need $h_1 = h_2 = h$ or $C = 0$. Namely, we have shown

$$G(z_1, z_2) = \langle \Phi_1(z_1)\Phi_2(z_2) \rangle = \begin{cases} \dfrac{C}{(z_1 - z_2)^{2h}} & (h_1 = h_2 = h), \\ 0 & (h_1 \neq h_2). \end{cases} \tag{C.203}$$

Chapter 8

Solution to Exercise 8.1

The stress-energy tensor and the XX OPE are

$$T(z) = -\frac{1}{2} : \partial X \partial X(z) :, \tag{C.204}$$

$$X(z,\bar{z})X(w,\bar{w}) \sim -\ln|z-w|^2. \tag{C.205}$$

Recall also that

$$:\mathcal{F}::\mathcal{G}: = :\mathcal{FG}: + (\text{cross-contractions}) \sim (\text{cross-contractions}). \tag{C.206}$$

Namely, to compute the singular part of the OPE, we only need the cross-contractions between $:\mathcal{F}:$ and $:\mathcal{G}:$.

(i) In this problem, you are asked to compute the OPE of $T, \bar{T}$ with $X, \partial X, \bar{\partial}X, \partial^2 X, :e^{i\sqrt{2}X}:$.

The TX OPE can be computed from the cross-contractions as

$$T(z)X(w,\bar{w}) = -\frac{1}{2}:\partial X \partial X(z): X(w,\bar{w}) \sim -\frac{1}{2} \cdot 2\overline{\partial X \partial X(z)X}(w,\bar{w})$$

$$= -\partial_z(-\ln|z-w|^2)\partial X(z) \sim \frac{1}{z-w}\partial X(z) \sim \frac{1}{z-w}\partial X(w). \tag{C.207}$$

In the last equality, we Taylor expanded $\partial X(z)$ around w:

$$\partial X(z) = \partial X(w) + (z-w)\partial^2 X(w) + \cdots .$$

By switching barred and unbarred quantities, the $\bar{T}X$ OPE is computed as

$$\bar{T}(\bar{z})X(w,\bar{w}) \sim \frac{1}{\bar{z}-\bar{w}}\bar{\partial}X(\bar{w}). \tag{C.208}$$

For computing $T\partial X$, $T\partial^2 X$ OPEs once and for all, let's compute $T\partial^n X$:

$$T(z)\partial^n X(w) = -\frac{1}{2} : \partial X \partial X(z) : \partial^n X(w) \sim -\frac{1}{2} \cdot 2\overline{\partial X \partial X(z)\partial^n X}(w)$$

$$= -\partial_z\partial_w^n(-\ln|z-w|^2)\partial X(z) = (-1)^n\partial_z^{n+1}(\ln|z-w|^2)\partial X(z)$$

$$= \frac{n!}{(z-w)^{n+1}}\partial X(z) \sim \sum_{m=0}^{n} \frac{n!}{m!(z-w)^{n-m+1}}\partial^{m+1}X(w). \tag{C.209}$$

In the last equality, we Taylor expanded $\partial^n X(z)$ around w,

$$\partial^n X(z) = \sum_{m=0}^{\infty} \frac{(z-w)^m}{m!}\partial^{m+n}X(w)$$

, and dropped nonsingular terms.

On the other hand, $\overline{T}\,\partial^n X$ OPEs are[1]

$$\overline{T}(\overline{z})\partial^n X(w) = -\frac{1}{2} : \overline{\partial}X\overline{\partial}X(\overline{z}) : \partial^n X(w) \sim -\frac{1}{2}\cdot 2\overline{\partial X\overline{\partial}X(\overline{z})\partial^n X}(w)$$

$$= -\partial_{\overline{z}}\partial_w^n(-\ln|z-w|^2)\overline{\partial}X(\overline{z}) = \partial_w^n\left(\frac{1}{\overline{z}-\overline{w}}\right)\overline{\partial}X(\overline{z})$$

$$\sim \begin{cases} 0 & n \geq 1, \\ \frac{1}{\overline{z}-\overline{w}}\overline{\partial}X(\overline{w}) & n = 0. \end{cases} \tag{C.210}$$

Now let us read off the OPE. For $n = 1$, (C.209) and (C.210) give

$$T(z)\partial X(w) \sim \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w}, \qquad \overline{T}(\overline{z})\partial X(w) \sim 0. \tag{C.211}$$

For $n = 2$, (C.209) and (C.210) gives

$$T(z)\partial^2 X(w) \sim \frac{2\partial^2 X(w)}{(z-w)^3} + \frac{2\partial^2 X(w)}{(z-w)^2} + \frac{\partial^3 X(w)}{z-w}, \qquad \overline{T}(\overline{z})\partial^2 X(w) \sim 0. \tag{C.212}$$

The OPE for $\overline{\partial}X$ is simply obtained by switching barred and unbarred quantities in (C.211):

$$T(z)\overline{\partial}X(w) \sim 0, \qquad \overline{T}(\overline{z})\overline{\partial}X(w) \sim \frac{\overline{\partial}X(w)}{(\overline{z}-\overline{w})^2} + \frac{\overline{\partial}^2 X(\overline{w})}{\overline{z}-\overline{w}}. \tag{C.213}$$

Rather than computing the OPE just for $: e^{i\sqrt{2}X} :$, let us compute the OPE for $: e^{iaX} :$ with a a general real number.

$$T(z) \, : e^{iaX}(w,\overline{w}): \, = -\frac{1}{2} : \partial X \partial X(z): \, : e^{iaX}(w,\overline{w}):$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{n!}(ia)^n : \partial X \partial X(z): \, : X^n(w,\overline{w}): \, . \tag{C.214}$$

---

[1]Actually, there are contributions proportional to (derivatives of) the delta function at $z = w$, because $\partial\overline{\partial}\log|z|^2 = \partial(\frac{1}{\overline{z}}) = \overline{\partial}(\frac{1}{z}) = \pi\delta^{(2)}(z)$, where $\delta^{(2)}(z) = \delta(\operatorname{Re}z)\delta(\operatorname{Im}z)$. But we can ignore this subtlety if we consider z very close to, but not equal to, w.

When evaluating this OPE, one can contract both of the two ∂Xs with two of the n Xs, or one of the ∂Xs with one of the n Xs. Taking into account the combinatoric factors for these contractions,

$$:\partial X \partial X(z): :X^n(w,\overline{w}): \ \sim n(n-1)(\overline{\partial X(z)X}(w,\overline{w}))^2 :X^{n-2}(w,\overline{w}):$$
$$+ 2n\overline{\partial X(z)X}(w,\overline{w}) :\partial X(z)X^{n-1}(w,\overline{w}):. \quad \text{(C.215)}$$

Plugging this back into (C.214), we find

$$T(z) \ :e^{iaX}(w,\overline{w}): \ \sim -\frac{1}{2}(\overline{\partial X(z)\,iaX}(w,\overline{w}))^2 :e^{iaX}(w,\overline{w}):$$
$$- \overline{\partial X(z)\,iaX}(w,\overline{w}) :\partial X(z)e^{iaX}(w,\overline{w}):$$
$$= \frac{a^2/2}{(z-w)^2} :e^{iaX}(w,\overline{w}): +\frac{ia}{z-w} :\partial X(z)e^{iaX}(w,\overline{w}):$$
$$\sim \frac{a^2/2}{(z-w)^2} :e^{iaX}(w,\overline{w}): +\frac{ia}{z-w} :\partial X e^{iaX}(w,\overline{w}):$$
$$= \frac{a^2/2}{(z-w)^2} :e^{iaX}(w,\overline{w}): +\frac{1}{z-w}\partial[:e^{iaX}(w,\overline{w}):]. \quad \text{(C.216)}$$

One can remember the above result (the first "$\sim$" in (C.216)) as a rule for contractions against $:e^{iaX}:$. Namely, a contraction of X against $:e^{iaX}:$ brings down an iaX, but $:e^{iaX}:$ remains. Just by replacing barred and unbarred quantities, we obtain

$$\overline{T}(\overline{z}) \ :e^{iaX}(w,\overline{w}): \ \sim \frac{a^2/2}{(\overline{z}-\overline{w})^2} :e^{iaX}(w,\overline{w}): +\frac{1}{\overline{z}-\overline{w}}\overline{\partial}[:e^{iaX}(w,\overline{w}):]. \quad \text{(C.217)}$$

For $a = \sqrt{2}$, (C.216) and (C.217) give

$$T(z) \ :e^{i\sqrt{2}X}(w,\overline{w}): \ \sim \frac{1}{(z-w)^2} :e^{i\sqrt{2}X}(w,\overline{w}): +\frac{1}{z-w}\partial[:e^{i\sqrt{2}X}(w,\overline{w}):],$$
$$\text{(C.218)}$$

$$\overline{T}(\overline{z}) \ :e^{i\sqrt{2}X}(w,\overline{w}): \ \sim \frac{1}{(\overline{z}-\overline{w})^2} :e^{i\sqrt{2}X}(w,\overline{w}): +\frac{1}{\overline{z}-\overline{w}}\overline{\partial}[:e^{i\sqrt{2}X}(w,\overline{w}):].$$
$$\text{(C.219)}$$

(ii) The OPE between $T,\overline{T}$ and a conformal primary $\mathcal{O}$ of conformal dimensions $(h,\overline{h})$ is

$$T(z)\mathcal{O}(w,\overline{w}) \sim \frac{h\mathcal{O}(w,\overline{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\overline{w})}{z-w}, \quad \overline{T}(\overline{z})\mathcal{O}(w,\overline{w}) \sim \frac{\overline{h}\mathcal{O}(w,\overline{w})}{(\overline{z}-\overline{w})^2} + \frac{\overline{\partial}\mathcal{O}(w,\overline{w})}{\overline{z}-\overline{w}}.$$
$$\text{(C.220)}$$

Comparing this with the results in part (i), we find the following dimensions of the operators:

operator	$(h, \bar{h})$
X	$(0,0)$
∂X	$(1,0)$
$\bar{\partial} X$	$(0,1)$
$\partial^2 X$	$(2,0)$
$:e^{i\sqrt{2}X}:$	$(1,1)$
$:e^{iaX}:$	$(\frac{a^2}{2}, \frac{a^2}{2})$

$$\text{(C.221)}$$

Note that $\partial^2 X$ is not a primary field because the $T\partial^2 X$ OPE starts from a $(z-w)^{-3}$ term as in (C.212).

More generally, an exponential times a general product of derivatives,

$$: \left(\prod_i \partial^{m_i} X\right) \left(\prod_j \bar{\partial}^{n_j} X\right) e^{iaX}:, \qquad m_i, n_j \geq 1, \tag{C.222}$$

has dimension

$$\left(\frac{a^2}{2} + \sum_i m_i, \frac{a^2}{2} + \sum_j n_j\right). \tag{C.223}$$

Among the operators (C.222), conformal primary operators are only ∂X, $\bar{\partial} X$ and $:e^{iaX}:$. Other operators have higher poles. (This is the case if we have only one scalar X. In string theory where we have multiple X^μs, we can have more primaries.)

Solution to Exercise 8.2

(i) Let's assume that

$$\overbrace{A(z) :B^n}(w): = n\overbrace{A(z)B}(w):B^{n-1}(w):, \tag{C.224}$$

holds for n. Now,

$$\overbrace{A(z) :B^{n+1}}(w): = \overbrace{A(z) :(B^n}B)(w): . \tag{C.225}$$

Here, A can be contracted against B^n or B, so

$$\text{(C.225)} = :\overbrace{A(z)B^n}(w)B(w): + :\overbrace{A(z)B^n(w)B}(w):$$

$$= n\overbrace{A(z)B}(w) :B^{n-1}(w)B(w): + :\overbrace{A(z)B}(w) :B^n(w):$$

$$= (n+1)\overbrace{A(z)B}(w) :B^n(w): . \tag{C.226}$$

In the second equality we used (C.224). So, (C.224) holds for $n+1$ too. On the other hand, (C.224) trivially holds for $n = 1$. It is also true for $n = 0$, if we understand that the right hand side of (C.224) is zero. Therefore, (C.224) is true for any $n \geq 0$.

(ii) By Taylor expanding the exponential operator,

$$
\overbrace{A(z) : \exp B(w):}^{} = \sum_{n=0}^{\infty} \frac{1}{n!} \overbrace{A(z) : B^n(w):}^{} = \sum_{n=0}^{\infty} \frac{1}{n!} n \overbrace{A(z)B(w)} : B^{n-1}(w):
$$

$$
= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \overbrace{A(z)B(w)} : B^{n-1}(w): \; = \overbrace{A(z)B(w)} \sum_{m=0}^{\infty} \frac{1}{m!} : B^m(w):
$$

$$
= \overbrace{A(z)B(w)} : \exp B(w): . \tag{C.227}
$$

In the second equality, we used (C.224).

(iii) By Taylor expanding the exponentials,

$$
: \exp \overbrace{A(z):\; :\exp B}(w): = \sum_{m,n=0}^{\infty} \frac{1}{m!\,n!} : \overbrace{A^m :: B^n}: . \tag{C.228}
$$

In $: \overbrace{A^m :: B^n}:$, we can contract k of m As against k of n Bs, for $k = 1, \ldots, \min(m,n)$. In doing this, there are $\binom{m}{k}$ ways to choose k out of m As, and $\binom{n}{k}$ ways to choose k out of n Bs. Furthermore, there are $k!$ ways to contract k As and k Bs. Therefore,

$$
(\text{C.228}) = \sum_{m,n=0}^{\infty} \sum_{k=1}^{\min(m,n)} \frac{k!}{m!\,n!} \binom{m}{k}\binom{n}{k} \left(\overbrace{A(z)B(w)} \right)^k : A^{m-k}(z)B^{n-k}(w):
$$

$$
= \sum_{m,n=0}^{\infty} \left[\sum_{k=0}^{\min(m,n)} \frac{k!}{m!\,n!} \binom{m}{k}\binom{n}{k} \left(\overbrace{A(z)B(w)} \right)^k : A^{m-k}(z)B^{n-k}(w): \right.
$$

$$
\left. - \frac{1}{m!\,n!} : A^m(z)B^n(w): \right] \tag{C.229}
$$

$$
= \sum_{m,n=0}^{\infty} \sum_{k=0}^{\min(m,n)} \frac{1}{k!(m-k)!(n-k)!} \left(\overbrace{A(z)B(w)} \right)^k : A^{m-k}(z)B^{n-k}(w):
$$

$$
- : \exp A(z)\exp B(w): \tag{C.230}
$$

$$
= \sum_{m',n',k=0}^{\infty} \frac{1}{k!\,m'!\,n'!} \left(\overbrace{A(z)B(w)} \right)^k : A^{m'}(z)B^{n'}(w): - : \exp A(z)\exp B(w):
$$

$$
\tag{C.231}
$$

$$= \exp\left(\overbracket{A(z)B(w)}\right) :\exp A(z)\exp B(w): \ - \ :\exp A(z)\exp B(w): . \qquad \text{(C.232)}$$

In (C.229), we added and subtracted the $k = 0$ term to make the k sum to include $k = 0$. In going from (C.230) to (C.231), we set $m' = m - k, n' = n - k$. It is not immediately obvious that $\sum_{m,n=0}^{\infty}\sum_{k=0}^{\min(m,n)} = \sum_{m',n',k=0}^{\infty}$. A shortcut way to see this is that, because $p! = \Gamma(p + 1)$ is infinite for p a negative integer, we can actually replace $\sum_{k=0}^{\min(m,n)}$ with $\sum_{k=0}^{\infty}$ in (C.230). Then we can freely interchange the order of summation. Or, one can carefully manipulate the sums to show that indeed $\sum_{m,n=0}^{\infty}\sum_{k=0}^{\min(m,n)} = \sum_{m',n',k=0}^{\infty}$.

(iv) By setting $A = iaX, B = -iaX$ in part (iii), we obtain

$$:\exp(ia\overbracket{X})(z): \ :\exp(-ia\overbracket{X})(w):$$
$$= \left[\exp\left(a^2 \overbracket{X(z)X}(w)\right) - 1\right]:\exp(iaX)(z)\exp(-iaX)(w):$$
$$= \left[\exp\left(-a^2 \log|z - w|^2\right) - 1\right]:\exp(iaX)(z)\exp(-iaX)(w):$$
$$= \left(\frac{1}{|z - w|^{a^2}} - 1\right):\exp(iaX)(z)\exp(-iaX)(w):. \qquad \text{(C.233)}$$

This means that

$$:\exp(iaX)(z): \ :\exp(-iaX)(w):$$
$$= \ :\exp(iaX)(z)\exp(-iaX)(w): \ + \ :\exp(ia\overbracket{X})(z): \ :\exp(-ia\overbracket{X})(w):$$
$$= \frac{1}{|z - w|^{a^2}}:\exp(iaX)(z)\exp(-iaX)(w):. \qquad \text{(C.234)}$$

By expanding the right hand side around $z = w$,

$$:\exp(iaX)(z): \ :\exp(-iaX)(w):$$
$$= \frac{1}{|z - w|^{a^2}}\Big[1 + ia(z - w)\partial X(w) + ia(\bar{z} - \bar{w})\bar{\partial}X(\bar{w})+$$
$$-\frac{a^2}{2}:(\partial X)^2(w): \ - \ \frac{a^2}{2}:(\bar{\partial}X)^2(\bar{w}): \ + \cdots\Big]. \qquad \text{(C.235)}$$

If we sandwich this with $\langle 0|$ and $|0\rangle$, the terms in [ ] except "1" give zero. This is because those terms are normal ordered and killed either by $\langle 0|$ or $|0\rangle$. Therefore,

$$\langle:\exp(iaX)(z): \ :\exp(-iaX)(w):\rangle = \frac{1}{|z - w|^{a^2}}. \qquad \text{(C.236)}$$

This means that the conformal dimension of the operator $:\exp(iaX)(z):$ is $a^2/2$. This agrees with the result we obtained in Problem 8.1.

Solution to Exercise 8.3

By definition 8.41,

$$
\begin{aligned}
G &\equiv \langle \phi_1(w_1) \ldots \phi_n(w_n)(\hat{L}_{-k}\phi)(z) \rangle \\
&= \frac{1}{2\pi i} \oint_z dw (w - z)^{-k+1} \langle T(w)\phi_1(w_1) \ldots \phi_n(w_n)\phi(z) \rangle.
\end{aligned}
\tag{C.237}
$$

where $\oint_z dw$ denotes a contour integration around $w = z$. Deforming the contour on the w-plane and picking up possible contributions from residues at $w = w_j$,

$$
\oint_z dw = -\sum_j \oint_{w_j} dw.
\tag{C.238}
$$

Therefore, (C.237) becomes

$$
G = -\frac{1}{2\pi i} \sum_j \oint_{w_j} dw (w - z)^{-k+1} \langle \phi_1(w_1) \ldots (T(w)\phi_j(w_j)) \ldots \phi_n(w_n)\phi(z) \rangle.
\tag{C.239}
$$

Because ϕ_j are primary fields, we have the following $T\phi$ OPE around w_j:

$$
T(w)\phi_j(w_j) = \frac{h_j \phi_j(w_j)}{(w - w_j)^2} + \frac{\partial \phi_j(w_j)}{w - w_j} + \ldots.
\tag{C.240}
$$

Let us look at the term obtained by substituting the first term of (C.240) into (C.239):

$$
-\frac{h_j}{2\pi i} \oint_{w_j} (w - z)^{-k+1}(w - w_j)^{-2} \langle \phi_1(w_1) \ldots \phi_j(w_j) \ldots \phi_n(w_n)\phi(z) \rangle,
\tag{C.241}
$$

where we are focusing on the term with a particular j. Note that nothing other than $(w - z)^{-k+1}(w - w_j)^{-2}$ depends on w. Therefore, the contour integral is obtained by Laurent expanding this quantity around $w = w_j$ and taking the coefficient of the $(w - w_j)^{-1}$ term. Well,

$$
\begin{aligned}
&(w - z)^{-k+1}(w - w_j)^{-2} \\
&= (w_j - z)^{-k+1}(w - w_j)^{-2} + (1 - k)(w_j - z)^{-k}(w - w_j)^{-1} + \mathcal{O}(w - w_j)^0.
\end{aligned}
\tag{C.242}
$$

Therefore, the integral (C.241) is

$$
-h_j(1 - k)(w_j - z)^{-k} \langle \phi_1(w_1) \ldots \phi_j(w_j) \ldots \phi_n(w_n)\phi(z) \rangle.
\tag{C.243}
$$

On the other hand, by substituting the second term of (C.240) into (C.239), we get

$$
-\frac{1}{2\pi i} \oint_{w_j} (w - z)^{-k+1}(w - w_j)^{-1} \langle \phi_1(w_1) \ldots \partial \phi_j(w_j) \ldots \phi_n(w_n)\phi(z) \rangle.
\tag{C.244}
$$

Here,

$$(w-z)^{-k+1}(w-w_j)^{-1} = (w_j-z)^{-k+1}(w-w_j)^{-1} + \mathcal{O}(w-w_j)^0. \qquad \text{(C.245)}$$

Therefore, (C.244) is

$$-(w_j-z)^{-k+1}\langle \phi_1(w_1)\ldots\partial\phi_j(w_j)\ldots\phi_n(w_n)\phi(z)\rangle. \qquad \text{(C.246)}$$

It is clear that "..." in (C.240) does not lead to extra contributions, because the expansion of the terms in "..." do not have a $(w-w_j)^{-1}$ term.

Therefore,

$$
\begin{aligned}
G &= -\sum_j \Big[h_j(1-k)(w_j-z)^{-k}\langle \phi_1(w_1)\ldots\phi_j(w_j)\ldots\phi_n(w_n)\phi(z)\rangle \\
&\qquad\qquad + (w_j-z)^{-k+1}\langle \phi_1(w_1)\ldots\partial\phi_j(w_j)\ldots\phi_n(w_n)\phi(z)\rangle \Big] \\
&= \mathcal{L}_{-k}\langle \phi_1(w_1)\ldots\phi_n(w_n)\phi(z)\rangle, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{(C.247)}
\end{aligned}
$$

where

$$\mathcal{L}_{-k} \equiv -\sum_{j=1}^{n} \left[h_j(1-k)(w_j-z)^{-k} + (w_j-z)^{-k+1}\partial_j \right]. \qquad \text{(C.248)}$$

Chapter 9

Solution to Exercise 9.1

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Solution to Exercise 9.2

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Solution to Exercise 9.3

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Solution to Exercise 9.4

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Chapter 10

Solution to Exercise 10.1

a) Because

$$\delta_{\tau_1}\delta_{\tau_2}X(\tau) = \delta(\tau - \tau_1)\partial_\tau[\delta(\tau - \tau_2)\dot{X}(\tau)]$$
$$= \delta(\tau - \tau_1)\partial_\tau\delta(\tau - \tau_2)\dot{X}(\tau) + \delta(\tau - \tau_1)\delta(\tau - \tau_2)\ddot{X}(\tau), \quad \text{(C.249)}$$

we find

$$[\delta_{\tau_1}, \delta_{\tau_2}]X(\tau) = [\delta(\tau - \tau_1)\partial_\tau\delta(\tau - \tau_2) - \delta(\tau - \tau_2)\partial_\tau\delta(\tau - \tau_1)]\dot{X}(\tau). \quad \text{(C.250)}$$

By the definition of the structure constant $f^{\tau_3}_{\tau_1\tau_2}$ (Eq. (9.71)), this should be equal to

$$f^{\tau_3}_{\tau_1\tau_2}\delta_{\tau_3}X(\tau) = \int d\tau_3\, f^{\tau_3}_{\tau_1\tau_2}\delta(\tau - \tau_3)\dot{X}(\tau) = f^{\tau}_{\tau_1\tau_2}\dot{X}(\tau). \quad \text{(C.251)}$$

By comparing this with (C.250), we find

$$f^{\tau_3}_{\tau_1\tau_2} = \delta(\tau_3 - \tau_1)\partial_{\tau_3}\delta(\tau_3 - \tau_2) - \delta(\tau_3 - \tau_2)\partial_{\tau_3}\delta(\tau - \tau_1). \quad \text{(C.252)}$$

(i) The BRST transformation of X is, using the definition (9.75) and the expression for δ_{τ_1} in (9.70),

$$\delta_B X(\tau) = -i\kappa c^{\tau_1}\delta_{\tau_1}X(\tau) = -i\kappa \int d\tau_1 c(\tau_1)\delta(\tau - \tau_1)\dot{X}(\tau) = -\kappa c(\tau)\dot{X}(\tau). \quad \text{(C.253)}$$

To derive the BRST transformation of $e(\tau)$, we need to know the expression for $\delta_{\tau_1}e(\tau)$. We can do this just like the way we derived (9.70), as follows. From (9.74),

$$\delta e(\tau) = \partial_\tau(\xi(\tau)e(\tau)) = \dot{\xi}(\tau)e(\tau) + \xi(\tau)\dot{e}(\tau)$$
$$= \int d\tau_1\, \delta(\tau - \tau_1)[\dot{\xi}(\tau)e(\tau) + \xi(\tau)\dot{e}(\tau)]. \quad \text{(C.254)}$$

With $\delta(\tau - \tau_1)$, we can replace τ with τ_1 in the integrand. Therefore[2]

$$\delta e(\tau) = \int d\tau_1\, \delta(\tau - \tau_1)[\dot{\xi}(\tau_1)e(\tau) + \xi(\tau_1)\dot{e}(\tau)]$$
$$= \int d\tau_1\, [-\partial_{\tau_1}\delta(\tau - \tau_1)\, \xi(\tau_1)e(\tau) + \delta(\tau - \tau_1)\xi(\tau_1)\dot{e}(\tau)]$$

_____

[2]There is some ambiguity about which τ in (C.254) is replaced by τ_1; for example we could have replaced $e(\tau), \dot{e}(\tau)$ in (C.254) by $e(\tau_1), \dot{e}(\tau_1)$ as well. Such different ways to replace τ by τ_1 lead to expressions for $\delta_{\tau_1}e(\tau)$ which are somewhat different from (C.256). However, such ambiguity corresponds to total derivatives which makes no difference to the final result (C.257).

$$= \int d\tau_1 \left[\partial_\tau \delta(\tau - \tau_1) \, \xi(\tau_1) e(\tau) + \delta(\tau - \tau_1) \xi(\tau_1) \dot{e}(\tau) \right]$$

$$= \int d\tau_1 \, \xi(\tau_1) \partial_\tau [\delta(\tau - \tau_1) e(\tau)]. \tag{C.255}$$

By definition, this must be equal to $\xi^{\tau_1} \delta_{\tau_1} e(\tau) = \int d\tau_1 \, \xi(\tau_1) \delta_{\tau_1} e(\tau)$. So, we obtain

$$\delta_{\tau_1} e(\tau) = \partial_\tau [\delta(\tau - \tau_1) e(\tau)]. \tag{C.256}$$

Substituting the result (C.256) into (9.75),

$$\delta_B e(\tau) = -i\kappa c^{\tau_1} \delta_{\tau_1} e(\tau) = -i\kappa \int d\tau_1 c(\tau_1) \partial_\tau [\delta(\tau - \tau_1) e(\tau)]$$

$$= -i\kappa \partial_\tau \int d\tau_1 c(\tau_1) \delta(\tau - \tau_1) e(\tau) = -\kappa \partial_\tau [c(\tau) e(\tau)]. \tag{C.257}$$

Next, the transformation for c is

$$\delta_B c(\tau) = -\frac{i}{2} \kappa \int d\tau_1 d\tau_2 \, f^\tau_{\tau_1 \tau_2} c(\tau_1) c(\tau_2)$$

$$= -\frac{i}{2} \kappa \int d\tau_1 d\tau_2 [\delta(\tau - \tau_1) \partial_\tau \delta(\tau - \tau_2) - (\tau_1 \leftrightarrow \tau_2)] c(\tau_1) c(\tau_2)$$

$$= -i\kappa \int d\tau_1 d\tau_2 \delta(\tau - \tau_1) \partial_\tau \delta(\tau - \tau_2) c(\tau_1) c(\tau_2)$$

$$= -i\kappa \int d\tau_2 \partial_\tau \delta(\tau - \tau_2) c(\tau) c(\tau_2) = i\kappa \int d\tau_2 \partial_{\tau_2} \delta(\tau - \tau_2) c(\tau) c(\tau_2)$$

$$= -i\kappa \int d\tau_2 \delta(\tau - \tau_2) c(\tau) \dot{c}(\tau_2) = -i\kappa c(\tau) \dot{c}(\tau). \tag{C.258}$$

In the third equality we used that $c(\tau_1)$ and $c(\tau_2)$ anticommute. The remaining relations

$$\delta_B b(\tau) = \kappa B(\tau), \qquad \delta_B B(\tau) = 0, \tag{C.259}$$

are trivially read off from (9.75).

So, in summary, the BRST transformation rule for X, e, c, b, B is

$$\delta_B X = -i\kappa c \dot{X}, \quad \delta_B e = -i\kappa \partial_\tau (ce), \quad \delta_B c = -i\kappa c \dot{c}, \quad \delta_B b = \kappa B, \quad \delta_B B = 0. \tag{C.260}$$

Let us show the nilpotence, writing $\delta_B(\tau) = \delta_B, \delta_B(\tau') = \delta'_B$. In doing this, it is important to keep in mind that κ anticommutes with fermionic variables c, b.

First,

$$\delta_B \delta'_B X = \delta_B(-i\kappa' c \dot{X}) = -i\kappa'[(\delta_B c)\dot{X} + c\partial_\tau(\delta_B X)]$$
$$= -i\kappa'[(-i\kappa c\dot{c})\dot{X} + c\partial_\tau(-i\kappa c\dot{X})] = \kappa'\kappa[-c\dot{c}\dot{X} + c\partial_\tau(c\dot{X})]$$
$$= \kappa'\kappa[-c\dot{c}\dot{X} + c\dot{c}\dot{X} + c^2\ddot{X}] = 0. \qquad (C.261)$$

In the forth equality we used that κ, c anticommute, and in the last equality we used $c^2 = 0$. e is similar:

$$\delta_B \delta'_B e = \delta_B[-i\kappa'\partial_\tau(ce)] = -i\kappa'\partial_\tau[(\delta_B c)e + c(\delta_B e)]$$
$$= -i\kappa'\partial_\tau[(-i\kappa c\dot{c})e + c(-i\kappa\partial_\tau(ce))] = \kappa'\kappa\partial_\tau[-c\dot{c}e + c\partial_\tau(ce)]$$
$$= \kappa'\kappa\partial_\tau[-c\dot{c}e + c\dot{c}e + c^2\dot{e}] = 0. \qquad (C.262)$$

For c,

$$\delta_B \delta'_B c = \delta_B[-i\kappa' c\dot{c}] = -i\kappa'[(\delta_B c)\dot{c} + c\partial_\tau(\delta_B c)]$$
$$= -i\kappa'[(-i\kappa c\dot{c})\dot{c} + c\partial_\tau(-i\kappa c\dot{c})] = \kappa'\kappa[-(c\dot{c})\dot{c} + c\partial_\tau(c\dot{c})]$$
$$= \kappa'\kappa[-c\dot{c}^2 + c\dot{c}^2 + c^2\ddot{c}] = 0, \qquad (C.263)$$

where in the last step we used that $c^2 = \dot{c}^2 = 0$. The rest is almost trivial:

$$\delta_B \delta'_B b = \delta_B(\kappa B) = \kappa\delta_B B = 0, \qquad \delta_B \delta'_B B = \delta_B 0 = 0. \qquad (C.264)$$

So, $\delta_B^2 = 0$ on all fields, X, e, c, b, B.

(ii) Using the general formula,

$$S_2 = -iB_\tau F^\tau = -i\int d\tau\, B(\tau)\,[e(\tau) - 1], \qquad (C.265)$$

$$S_3 = b_\tau c^{\tau_1}\delta_{\tau_1}F^\tau = \int d\tau d\tau_1\, b(\tau)c(\tau_1)\delta_{\tau_1}e(\tau) = \int d\tau d\tau_1\, b(\tau)c(\tau_1)\partial_\tau[\delta(\tau - \tau_1)e(\tau)]$$
$$= -\int d\tau d\tau_1\, \dot{b}(\tau)c(\tau_1)\delta(\tau - \tau_1)e(\tau) = -\int d\tau \dot{b}(\tau)c(\tau)e(\tau). \qquad (C.266)$$

So, the total gauge fixed action is

$$S = S_1 + S_2 + S_3 = \int d\tau\left[\frac{\dot{X}^2}{2e} - iB(e - 1) - e\dot{b}c\right]. \qquad (C.267)$$

(iii) Integrating out B gives a functional version of the delta function and sets $e = 1$. So, we end up with the action

$$S = \int d\tau\left(\frac{\dot{X}^2}{2} - \dot{b}c\right) = \int d\tau\left(\frac{\dot{X}^2}{2} + b\dot{c}\right). \qquad (C.268)$$

(iv) The variation the action (C.268) under (9.77) is

$$
\begin{aligned}
\delta_B S &= \int d\tau \left[\dot{X}\partial_\tau(\delta_B X) + (\delta_B b)\dot{c} + b\partial_\tau(\delta_B c) \right] \\
&= \int d\tau \left[\dot{X}\partial_\tau(-i\kappa c\dot{X}) + i\kappa\left(\frac{\dot{X}^2}{2} + \dot{b}c \right)\dot{c} + b\partial_\tau(-i\kappa c\dot{c}) \right] \\
&= i\kappa \int d\tau \left[-\dot{X}\partial_\tau(c\dot{X}) + \left(\frac{\dot{X}^2}{2} + \dot{b}c \right)\dot{c} + b\partial_\tau(c\dot{c}) \right] \\
&= i\kappa \int d\tau \left[-\dot{X}^2\dot{c} - \dot{X}\ddot{X}c + \frac{\dot{X}^2\dot{c}}{2} + \dot{b}c\dot{c} + b\dot{c}^2 + bc\ddot{c} \right] \\
&= i\kappa \int d\tau \left[-\frac{\dot{X}^2\dot{c}}{2} - \dot{X}\ddot{X}c + \dot{b}c\dot{c} + bc\ddot{c} \right] \\
&= i\kappa \int d\tau \frac{d}{d\tau}\left[-\frac{\dot{X}^2 c}{2} + bc\dot{c} \right] = 0.
\end{aligned}
\tag{C.269}
$$

Even after we set $F = e - 1 = 0$, we have to impose the equation of motion for e. The equation of motion for e derived from the original action (C.267) is

$$
0 = \frac{\delta S}{\delta e} = -\frac{\dot{X}^2}{2e^2} - iB - \dot{b}c.
\tag{C.270}
$$

Namely,

$$
B = i\left(\frac{\dot{X}^2}{2e^2} + \dot{b}c \right) = i\left(\frac{\dot{X}^2}{2} + \dot{b}c \right).
\tag{C.271}
$$

If we use this in the BRST transformation rule (C.260), we obtain the BRST transformation (9.77).

Solution to Exercise 10.2

a) First, let us write down relevant formulas. The definition of the ghost part of the stress energy tensor T^{gh} and the ghost number current j^g is

$$
T^{gh} = -(\partial b)c - 2b\partial c, \qquad j^g = -bc.
\tag{C.272}
$$

Here and below, we omit the normal ordering symbol to avoid clutter. For example, $bc(z)$ means $:bc(z):$. Operator products such as $:bc(z)::bc(w):$ is written as $bc(z)bc(w)$.

The OPE between b, c is

$$b(z)c(w) \sim c(z)b(w) \sim \frac{1}{z - w}, \qquad b(z)b(w) \sim c(z)c(w) \sim 0. \qquad \text{(C.273)}$$

Also recall the OPE

$$T^m(z)T^m(z) \sim \frac{c_m}{2(z-w)^4} + \frac{2T^m(w)}{(z-w)^2} + \frac{\partial T^m(w)}{z-w}. \qquad \text{(C.274)}$$

OK. To derive the transformation rule for c, let us first compute the OPE between j_B and c. Using the explicit expression for j_B, namely the second line in (9.78), we compute

$$j_B(z)c(w) = (cT^m + bc\partial c + \frac{3}{2}\partial^2 c)(z)\, c(w) \sim \overbracket{bc\partial c(z)c}(w) = \frac{c\partial c(z)}{z - w} \sim \frac{c\partial c(w)}{z - w}.$$
$$\text{(C.275)}$$

Using (C.275),

$$
\begin{aligned}
\delta_B c(w) &= \{Q_B, c(w)\} \\
&= \frac{1}{2\pi i} \oint_w dz \{j_B(z), c(w)\} \\
&= \frac{1}{2\pi i} \left[\oint_{|z|>|w|} dz\, j_B(z)c(w) + \oint_{|w|>|z|} dz\, c(w)j_B(z) \right] \\
&= \frac{1}{2\pi i} \left[\oint_{|z|>|w|} dz\, j_B(z)c(w) - \oint_{|w|>|z|} dz\, j_B(z)c(w) \right] \\
&= \frac{1}{2\pi i} \oint_w dz\, j_B(z)c(w) \\
&= \frac{1}{2\pi i} \oint_w dz \left[\frac{c\partial c(w)}{z - w} + (\text{regular}) \right] \\
&= c\partial c(w). \qquad \text{(C.276)}
\end{aligned}
$$

It is gratifying to see that this has the same form as the BRST transformation of c in the point-particle version (third equation in (C.260)), if we replace ∂ here with ∂_τ (up to $-i\kappa$ which can be absorbed in the definition).

You may think that the less explicit expression for j_B, the first line of (9.78), is more useful for computing OPEs. But this is not so, because T^{gh} contains both b and c in it and its OPE with other operators are not simple. So, the explicit expression for j_B (the second line of (9.78)) is easier to work with.

b) We have that

$$j_B(z)b(w) = (cT^m + bc\partial c + \tfrac{3}{2}\partial^2 c)(z)\, b(w)$$

$$= \overline{cT^m(z)b}(w) + b\overline{c\partial c(z)b}(w) + bc\partial \overline{c(z)b}(w) + \tfrac{3}{2}\partial^2 \overline{c(z)b}(w). \quad \text{(C.277)}$$

When computing such contractions between fermionic fields, we must be careful about the anticommuting property of them. We can get the correct sign when we bring the contracted pair of fermions side by side. For example, in the second term in (C.277),

$$b\overline{c\partial c(z)b}(w) = -b(\partial c)\overline{c(z)b}(w) = -\frac{b\partial c(z)}{z - w}. \quad \text{(C.278)}$$

Including other terms,

$$
\begin{aligned}
j_B(z)b(w) &\sim \frac{T^m(z)}{z-w} - \frac{b\partial c(z)}{z-w} + bc(z)\partial_z \frac{1}{z-w} + \frac{3}{2}\partial_z^2 \frac{1}{z-w}\\
&\sim \frac{T^m(z)}{z-w} - \frac{b\partial c(z)}{z-w} - \frac{bc(z)}{(z-w)^2} + \frac{3}{(z-w)^3}\\
&\sim \frac{T^m(w) + \cdots}{z-w} - \frac{b\partial c(w) + \cdots}{z-w} - \frac{bc(w) + (z-w)\partial(bc)(w) + \cdots}{(z-w)^2}\\
&\quad + \frac{3}{(z-w)^3}\\
&\sim \frac{3}{(z-w)^3} - \frac{bc(w)}{(z-w)^2} + \frac{T^m(w) - (\partial b)c(w) - 2b\partial c(w)}{z-w}\\
&= \frac{3}{(z-w)^3} + \frac{j^g(w)}{(z-w)^2} + \frac{T(w)}{z-w}, \quad \text{(C.279)}
\end{aligned}
$$

where T is the total stress-energy tensor, $T = T^m + T^{gh}$.

From the $(z-w)^{-1}$ term, we see that the BRST transformation of the antighost b gives the total stress energy tensor T. This means that $\{Q_B, b\} = T$, which is a general feature of the BRST formalism.

c) Using the OPE between T^m and ϕ^h, we can compute

$$
\begin{aligned}
j_B(z)\phi^h(w) &= (cT^m + bc\partial c + \tfrac{3}{2}\partial^2 c)(z)\, \phi^h(w)\\
&\sim c\overline{T^m(z)\phi}^h(w)\\
&\sim c(z)\left[\frac{h\phi^h(w)}{(z-w)^2} + \frac{\partial \phi^h(w)}{z-w}\right]\\
&= [c(w) + (z-w)\partial c(w) + \cdots]\left[\frac{h\phi^h(w)}{(z-w)^2} + \frac{\partial \phi^h(w)}{z-w}\right]
\end{aligned}
$$

– 305 –

$$\sim \frac{hc\phi^h(w)}{(z-w)^2} + \frac{h(\partial c)\phi^h(w) + c\partial\phi^h(w)}{z-w}. \qquad \text{(C.280)}$$

This gives the BRST transformation rule for matter field, $\delta_B\phi^h \propto h(\partial c)\phi^h + c\partial\phi^h$. Note that the transformation rule for the matter fields X, e in the point particle case, (C.260), was precisely of this form, if we replace ∂ here with ∂_τ and assign $h = 0$ for X and $h = 1$ for e.

d) The computation of the OPE

$$T(z)j_B(w) = \left(T^m - (\partial b)c - 2b\partial c\right)(z)\left(cT^m + bc\partial c + \frac{3}{2}\partial^2 c\right)(w), \qquad \text{(C.281)}$$

is a bit laborious but straightforward and there is nothing essentially difficult. We should use the $T^m T^m$ OPE and use relations such as $c^2 = (\partial c)^2 = 0$. At the end of the day, one obtains (9.80).

The result (9.80) means that, only for $c_m = 26$, the BRST current $j_B(w)$ is a primary field of dimension one. Namely, only for $c_m = 26$, j_B transforms as a vector under general conformal transformations. So, for the BRST operator Q_B, which is an integral j_B, to make sense as a charge, $c_m = 26$ is necessary.

Also note that the additional total derivative term $\frac{3}{2}\partial^2 c$ in j_B is important for this result, although it does not contribute to the charge Q_B. If we had instead

$$j_B = cT^m + bc\partial c + k\partial^2 c, \qquad \text{(C.282)}$$

with k a constant, then the OPE would become

$$T(z)j_B(w) \sim \frac{c_m - 8 - 12k}{2(z-w)^4}c(w) + \frac{3 - 2k}{(z-w)^3}\partial c(w) + \frac{1}{(z-w)^2}j_B(w) + \frac{1}{z-w}\partial j_B(w). \qquad \text{(C.283)}$$

The $(z-w)^{-4}$, $(z-w)^{-3}$ terms must vanish for j_B to be a primary field, from which $c_m = 26$ and $k = 3/2$ follow.

Solution to Exercise 10.3

a) This is a bit laborious but straightforward, so we don't write the process down.

b) By the usual technique of deforming contour,

$$\{Q_B, Q_B\} = \frac{1}{(2\pi i)^2} \oint_{w=0} dw \oint_w dz \, j_B(z)j_B(w). \qquad \text{(C.284)}$$

where the w integral is around the origin, while the z integral is around $z = w$. The w integral picks up the $(z - w)^{-1}$ term in the $j_B(z)j_B(w)$ OPE. Therefore, from (9.81),

$$\{Q_B, Q_B\} = -\frac{c_m - 26}{12} \frac{1}{2\pi i} \oint_{w=0} dw \, c\partial^3 c(w). \tag{C.285}$$

The integral is in general nonvanishing. So, for this to vanish, we need $c_m = 26$. Because $Q_B^2 = \frac{1}{2}\{Q_B, Q_B\}$, This means that the BRST operator is nilpotent only for $c_m = 26$.

Chapter 11

Solution to Exercise 11.1

(i) We can compute the OPE of $\psi^\mu, b, c, \beta, \gamma$ with the relevant part of T. Below, we will omit the normal ordering symbol $::$ to avoid clutter. For example, $bc(z)$ means $:bc(z):$.

For the $T\psi$ OPE, by taking $T_\psi = -\frac{1}{2}\psi^\mu\partial\psi_\mu$,

$$
\begin{aligned}
T_\psi(z)\psi^\mu(w) &= -\frac{1}{2}\psi^\nu\partial\psi_\nu(z)\psi^\mu(w) \sim -\frac{1}{2}\overline{\psi^\nu\partial\psi_\nu(z)\psi^\mu}(w) - \frac{1}{2}\psi^\nu\overline{\partial\psi_\nu(z)\psi^\mu}(w) \\
&= \frac{1}{2}\frac{\eta^{\nu\mu}}{z-w}\partial\psi_\nu(z) - \frac{1}{2}\partial_z(\frac{\delta^\mu_\nu}{z-w})\psi^\nu(z) = \frac{1}{2}\frac{\partial\psi^\mu(z)}{z-w} + \frac{1}{2}\frac{\psi^\mu(z)}{(z-w)^2} \\
&\sim \frac{1}{2}\frac{\psi^\mu(w)}{(z-w)^2} + \frac{\partial\psi^\mu(w)}{z-w}. \qquad\qquad\qquad\qquad\qquad\text{(C.286)}
\end{aligned}
$$

In going to the second line, we got (-1) from commuting fermionic fields. In the last line, we Taylor expanded $\psi(z), \partial\psi(z)$ around $z = w$. From the coefficient of the $(z-w)^{-2}$ term, we obtain $h_\psi = 1/2$.

For b, c, we can use $T_{bc} = -2b\partial c + c\partial b$. Using $b(z)c(w) \sim c(z)b(w) \sim \frac{1}{z-w}$, we compute

$$
\begin{aligned}
T(z)b(w) &\sim -2b\partial\overline{c(z)b}(w) + c\partial\overline{b(z)b}(w) = -2b(z)\partial_z(\frac{1}{z-w}) - \partial b(z)\frac{1}{z-w} \\
&= \frac{2b(z)}{(z-w)^2} - \frac{\partial b(z)}{z-w} \sim \frac{2b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w}, \qquad\qquad\text{(C.287)} \\
T(z)c(w) &\sim -2b\partial\overline{c(z)c}(w) + c\partial\overline{b(z)c}(w) = +2\partial c(z)\frac{1}{z-w} + c(z)\partial_z\frac{1}{z-w} \\
&= \frac{2\partial c(z)}{z-w} - \frac{c(z)}{(z-w)^2} \sim -\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w}. \qquad\qquad\text{(C.288)}
\end{aligned}
$$

Therefore, $h_b = 2$, $h_c = -1$. Actually, the fermionic bc CFT that appears as ghosts for the Virasoro constraint is a special case of the more general bc CFT for which

$$
T_{bc} = (\partial b)c - \lambda\partial(bc) = -\lambda b\partial c + (\lambda-1)c\partial b. \qquad\qquad\text{(C.289)}
$$

The above special case corresponds to $\lambda = 2$. In the general case (C.289), $h_b = \lambda, h_c = 1 - \lambda$.

For β, γ, we can use $T_{\beta\gamma} = -\frac{3}{2}\beta\partial\gamma - \frac{1}{2}\gamma\partial\beta$. Using $\beta(z)\gamma(w) \sim -\frac{1}{z-w}$, $\gamma(z)\beta(w) \sim \frac{1}{z-w}$, we compute

$$
\begin{aligned}
T_{\beta\gamma}(z)\beta(w) &\sim -\frac{3}{2}\beta\partial\overline{\gamma(z)\beta}(w) - \frac{1}{2}\gamma\overline{\partial\beta(z)\beta}(w) = -\frac{3}{2}\beta(z)\frac{-1}{(z-w)^2} - \frac{1}{2}\partial\beta(z)\frac{1}{z-w} \\
&\sim \frac{(3/2)\beta(w)}{(z-w)^2} + \frac{\partial\beta(w)}{z-w}, \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\text{(C.290)} \\
T_{\beta\gamma}(z)\gamma(w) &\sim -\frac{3}{2}\beta\overline{\partial\gamma(z)\gamma}(w) - \frac{1}{2}\gamma\overline{\partial\beta(z)\gamma}(w) = \frac{3}{2}\partial\gamma(z)\frac{1}{z-w} - \frac{1}{2}\gamma(z)\frac{1}{(z-w)^2} \\
&\sim -\frac{(1/2)\gamma(w)}{(z-w)^2} + \frac{\partial\gamma(w)}{z-w}. \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\text{(C.291)}
\end{aligned}
$$

Therefore, $h_\beta = 3/2$, $h_\gamma = -1/2$. Actually, the bosonic $\beta\gamma$ CFT that appears as ghosts for the super-Virasoro constraint is a special case of the more general $\beta\gamma$ CFT for which

$$
T_{\beta\gamma} = (\partial\beta)\gamma - \lambda\partial(\beta\gamma) = -\lambda\beta\partial\gamma + (1-\lambda)\gamma\partial\beta. \quad\quad\text{(C.292)}
$$

The above special case corresponds to $\lambda = 3/2$. In the general case (C.292), $h_\beta = \lambda, h_\gamma = 1 - \lambda$.

In summary, we found

$$
h_\psi = \frac{1}{2}, \quad\quad h_b = 2, \quad h_c = -1, \quad\quad h_\beta = \frac{3}{2}, \quad h_\gamma = -\frac{1}{2}. \quad\quad\text{(C.293)}
$$

(ii) We can compute the OPE of the relevant part of T. Only in this problem, we use "$\sim$" to denote equality up to $\mathcal{O}((z-w)^{-3})$ terms.

First, for the ψ part,

$$
\begin{aligned}
T_\psi(z)T_\psi(w) &= \frac{1}{4}\psi^\mu\partial\psi_\mu(z)\psi^\nu\partial\psi_\nu(w) \sim \frac{1}{4}\left(\overline{\psi^\mu\partial\psi_\mu(z)\psi^\nu\partial}\psi_\nu(w) + \overline{\psi^\mu\partial\psi_\mu(z)\psi^\nu\partial\psi_\nu}(w)\right) \\
&= \frac{1}{4}\left(-\frac{\eta^{\mu\nu}}{z-w}\partial_z\partial_w\frac{\eta_{\mu\nu}}{z-w} + \partial_w\frac{\delta^\mu_\nu}{z-w}\partial_z\frac{\delta^\nu_\mu}{z-w}\right) = \frac{1}{4}\frac{2\eta^{\mu\nu}\eta_{\mu\nu} - \delta^\mu_\nu\delta^\nu_\mu}{(z-w)^4} \\
&\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\text{(C.294)}
\end{aligned}
$$

Because $\eta^{\mu\nu}\eta_{\mu\nu} = \delta^\mu_\nu\delta^\nu_\mu = \delta^\mu_\mu = D$, we get

$$
T_\psi(z)T_\psi(w) = \frac{D/4}{(z-w)^4} + \mathcal{O}((z-w)^{-3}). \quad\quad\quad\text{(C.295)}
$$

Comparing this with

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + (\text{regular}), \qquad \text{(C.296)}$$

we obtain $c_\psi = D/2$.

Next, for the bc part,

$$T_{bc}(z)T_{bc}(w) \sim 4\overbracket{b\partial c(z)b}\partial c(w) - 2\overbracket{b\partial c(z)c}\partial b(w) - 2\overbracket{c\partial b(z)b}\partial c(w) + \overbracket{c\partial b(z)c}\partial b(w)$$

$$\sim 4\partial_w \frac{1}{z-w}\partial_z \frac{1}{z-w} + 2\frac{1}{z-w}\partial_z\partial_w\frac{1}{z-w} \times 2 + \partial_w\frac{1}{z-w}\partial_z\frac{1}{z-w}$$

$$= -\frac{13}{(z-w)^4}. \qquad \text{(C.297)}$$

Therefore, $c_{bc} = -26$. For the general bc CFT (C.289), the central charges becomes $c_{bc} = -3(2\lambda - 1)^2 + 1$.

For the $\beta\gamma$ part,

$$T_{\beta\gamma}(z)T_{\beta\gamma}(w) \sim \frac{9}{4}\overbracket{\beta\partial\gamma(z)\beta}\partial\gamma(w) + \frac{3}{4}\overbracket{\beta\partial\gamma(z)\gamma}\partial\beta(w) + \frac{3}{4}\overbracket{\gamma\partial\beta(z)\beta}\partial\gamma(w) + \frac{1}{4}\overbracket{\gamma\partial\beta(z)\gamma}\partial\beta(w)$$

$$\sim \frac{9}{4}\partial_w\frac{-1}{z-w}\partial_z\frac{1}{z-w} + \frac{3}{4}\frac{-1}{z-w}\partial_z\partial_w\frac{1}{z-w} \times 2 + \frac{1}{4}\partial_w\frac{1}{z-w}\partial_z\frac{-1}{z-w}$$

$$= \frac{11/2}{(z-w)^4}. \qquad \text{(C.298)}$$

Therefore, $c_{\beta\gamma} = 11$. For the general bc CFT (C.289), the central charges becomes $c_{\beta\gamma} = 3(2\lambda - 1)^2 - 1$.

In summary,

$$c_\psi = \frac{D}{2}, \qquad c_{bc} = -26, \qquad c_{\beta\gamma} = 11. \qquad \text{(C.299)}$$

(iii) Including the central charge $c_X = D$ coming from the X^μ fields, the total central charge is

$$c_{total} = c_X + c_\psi + c_{bc} + c_{\beta\gamma} = D + \frac{D}{2} - 26 + 11 = \frac{3}{2}D - 15. \qquad \text{(C.300)}$$

By requiring this to vanish, we obtain the famous critical dimension for superstring:

$$D = 10 \qquad \text{(C.301)}$$

Solution to Exercise 11.2

1) Let us first show the first equation of (11.113). Explicitly, the left hand side is

$$(\gamma^{11})^2 = \gamma^0 \gamma^1 \cdots \gamma^9 \, \gamma^0 \gamma^1 \cdots \gamma^9. \tag{C.302}$$

Let us commute the second γ^0 through just to the right of the first γ^0. Because of (11.110), we get a (-1) each time γ^0 goes through each of the seven γs in between. So,

$$(\gamma^{11})^2 = (-1)^9 (\gamma^0)^2 \gamma^1 \cdots \gamma^9 \, \gamma^1 \cdots \gamma^9. \tag{C.303}$$

Now if we commute the second γ^1 through just to the right of the first γ^1, we similarly get

$$(\gamma^{11})^2 = (-1)^{9+8} (\gamma^0)^2 (\gamma^1)^2 \gamma^2 \cdots \gamma^9 \, \gamma^2 \cdots \gamma^9. \tag{C.304}$$

By repeating this procedure, we finally get

$$(\gamma^{11})^2 = (-1)^{9+8+\cdots+1} (\gamma^0)^2 (\gamma^1)^2 \cdots (\gamma^9)^2 = (-1)^{45} (-1) 1^9 = 1. \tag{C.305}$$

In the second equality, we used $(\gamma^\mu)^2 = \frac{1}{2}\{\gamma^\mu, \gamma^\mu\} = \eta^{\mu\mu}$ (no summation over μ) which follows from (11.110).

Next, let us show the second equation of (11.113).

$$\gamma^{11}\gamma^\mu = (\gamma^0 \cdots \gamma^\mu \cdots \gamma^9)\gamma^\mu \qquad \text{(no summation over } \mu). \tag{C.306}$$

If we commute the last γ^μ through just to the right of the first γ^μ, because we have to go through $(9 - \mu)$ γs,

$$\gamma^{11}\gamma^\mu = (-1)^{9-\mu} \gamma^0 \cdots \gamma^\mu \gamma^\mu \cdots \gamma^9. \tag{C.307}$$

Now, if we commute the first γ^μ through all the way to the left, because we have to go through μ γs,

$$\gamma^{11}\gamma^\mu = (-1)^{(9-\mu)+\mu} \gamma^\mu \gamma^1 \cdots \gamma^\mu \cdots \gamma^9 = (-1)^9 \gamma^\mu \gamma^{11} = -\gamma^\mu \gamma^{11}. \tag{C.308}$$

So, we have shown the second equation of (11.113).

Now let us show (11.115). We first need to show that $(\gamma^{11})^\dagger = \gamma^{11}$.

$$(\gamma^{11})^\dagger = (\gamma^0 \cdots \gamma^9)^\dagger = (\gamma^9)^\dagger \cdots (\gamma^0)^\dagger = -\gamma^9 \cdots \gamma^0. \tag{C.309}$$

where we have used (11.111). Interchanging the order of γs so that we have γ^{11} again (here we start by sending the γ^9 on the left to the right),

$$(\gamma^{11})^\dagger = -(-1)^9 \gamma^8 \cdots \gamma^0 \gamma^9 = -(-1)^{9+8} \gamma^7 \cdots \gamma^0 \gamma^8 \gamma^9 = \cdots$$
$$= -(-1)^{9+8+\cdots+1} \gamma^0 \cdots \gamma^8 \gamma^9 = -(-1)^{45} \gamma^{11} = \gamma^{11}. \qquad \text{(C.310)}$$

So, we have shown that $(\gamma^{11})^\dagger = \gamma^{11}$. Now (11.115) is easy to show:

$$\bar{\psi}_\pm \gamma^{11} = \psi_\pm^\dagger i \gamma^0 \gamma^{11} = -\psi_\pm^\dagger \gamma^{11} i \gamma^0 = -(\gamma^{11} \psi_\pm)^\dagger i \gamma^0 = \mp \psi_\pm^\dagger i \gamma^0 = \mp \bar{\psi}_\pm \gamma^{11}. \quad \text{(C.311)}$$

2.i) The number of independent components in a N dimensional symmetric traceless 2-tensor is $\frac{N(N+1)}{2} - 1$. By setting $N = D - 2 = 8$, we find $\frac{8 \cdot 9}{2} - 1 = 35$.

2.ii) From antisymmetry, none of $i_1, \ldots i_d$ can take the same value. Furthermore, the order of $i_1, \ldots i_d$ does not matter. So the number of independent components is $\binom{D-2}{d} = \binom{8}{d}$.

2.iii) The unconstrained ψ_α^i has $8 \cdot 8 = 64$ components, because the vector index takes $D - 2 = 8$ values while the Majorana-Weyl index α contains 8 independent real components. The gamma-trace part (dilatino) $\lambda_\alpha = (\gamma_i \psi^i)_\alpha$ is a spinor and has 8 components, so the gamma-traceless part (gravitino) $\hat{\psi}_\alpha^i$ has $64 - 8 = 54$ independent real components.

3.i) For IIA/IIB, using (11.114),

$$F^{\mu_1 \cdots \mu_{d+1}} = \bar{\psi}_\mp^L \gamma^{\mu_1 \cdots \mu_{d+1}} \psi_+^R = \bar{\psi}_\mp^L \gamma^{\mu_1 \cdots \mu_{d+1}} \gamma^{11} \psi_+^R. \qquad \text{(C.312)}$$

Let us commute γ^{11} through $\gamma^{\mu_1 \cdots \mu_{d+1}}$. Because $\gamma^{\mu_1 \cdots \mu_{d+1}}$ is nothing but (a sum of) a product of $(d+1)$ γs, the commutation relation (11.113) implies

$$F^{\mu_1 \cdots \mu_{d+1}} = (-1)^{d+1} \bar{\psi}_\mp^L \gamma^{11} \gamma^{\mu_1 \cdots \mu_{d+1}} \psi_+^R$$
$$= \pm (-1)^{d+1} \bar{\psi}_\mp^L \gamma^{\mu_1 \cdots \mu_{d+1}} \psi_+^R = \pm (-1)^{d+1} F^{\mu_1 \cdots \mu_{d+1}}. \qquad \text{(C.313)}$$

In the second equality, we used (11.115). This means that, for $F^{\mu_1 \cdots \mu_{d+1}}$ to be nonvanishing, d must be odd (even) for IIA (IIB).

3.ii) From 3(ii), we have field strengths $\mathbf{F}_2$, $\mathbf{F}_4$ for IIA and $\mathbf{F}_1$, $\mathbf{F}_3$, $\mathbf{F}_5$ for IIB. The RR potentials for these are $\mathbf{C}_1$, $\mathbf{C}_3$, for IIA and $\mathbf{C}_0$, $\mathbf{C}_2$, $\mathbf{C}_4$ for IIB. The independent components in these RR potentials are, from 2(ii),

$$\text{IIA:} \quad \binom{8}{1} + \binom{8}{3} = 8 + 56 = 64,$$
$$\text{IIB:} \quad \binom{8}{0} + \binom{8}{2} + \frac{1}{2} \binom{8}{4} = 1 + 28 + 35 = 64. \qquad \text{(C.314)}$$

In the third term in IIB, we divided by 2 to account for the self-duality of $\mathbf{F}_5$.

3.iii) The bosonic fields that we have not discussed explicitly are the Kaub-Ramond field $B_{\mu\nu}$ which is an antisymmetric 2-tensor with $\frac{8 \cdot 7}{2} = 28$ components, and dilaton Φ which is a scalar. So, the bosonic fields are:

$$
\text{IIA:} \quad \overbrace{\underbrace{35}_{G_{\mu\nu}} + \underbrace{28}_{B_{\mu\nu}} + \underbrace{1}_{\Phi}}^{\text{NS-NS}} + \overbrace{\underbrace{8}_{\mathbf{C}_1} + \underbrace{56}_{\mathbf{C}_3}}^{\text{R}_- \text{-R}_+} = 128,
$$

$$
\text{IIB:} \quad \overbrace{\underbrace{35}_{G_{\mu\nu}} + \underbrace{28}_{B_{\mu\nu}} + \underbrace{1}_{\Phi}}^{\text{NS-NS}} + \overbrace{\underbrace{1}_{\mathbf{C}_0} + \underbrace{28}_{\mathbf{C}_2} + \underbrace{35}_{\mathbf{C}_4}}^{\text{R}_+ \text{-R}_+} = 128.
$$

(C.315)

Here, for example, R$_-$-R$_+$ means the Ramond-Ramond sector with the left-movers having $-$ chirality while the right-movers having $+$ chirality.

For the fermionic fields, we have two vector-spinors coming from the NS-R$_+$ and R$_\mp$-NS sectors ($\mp$ for IIA/IIB). Let us denote them by ψ^{iA}, $A = 1, 2$, where $A = 1$ is for the NS-R$_+$ sector and $A = 2$ is for the R$_\mp$-NS sector. Here we are suppressing the spinor index α. As we saw above, we can decompose these into the gamma-trace part (dilatini) $\lambda^A = \gamma_i \psi^{iA}$ and the gamma-traceless part (gravitini) $\hat{\psi}^{iA} = \psi^{iA} - \frac{1}{10}\gamma^i \gamma_j \psi^{jA}$.

In type IIA the two vector-spinors ψ^{iA} ($A = 1, 2$) have different chiralities, while in type IIB they have the same chirality. Namely,

$$
\begin{aligned}
\text{IIA:} \quad & \gamma^{11}\psi^{i,A=1} = +\psi^{i,A=1}, \quad & \gamma^{11}\psi^{i,A=2} = -\psi^{i,A=2}, \\
\text{IIB:} \quad & \gamma^{11}\psi^{i,A=1} = +\psi^{i,A=1}, \quad & \gamma^{11}\psi^{i,A=2} = +\psi^{i,A=2}.
\end{aligned}
$$

(C.316)

The dilatino λ^A has chirality opposite to ψ^{iA}, because it is multiplied by γ_i. For example, if ψ^{iA} has positive chirality, i.e., $\gamma^{11}\psi^{iA} = \psi^{iA}$, then

$$
\gamma^{11}\lambda^A = \gamma^{11}\gamma_i\psi^{iA} = -\gamma_i\gamma^{11}\psi^{iA} = -\gamma_i\psi^{iA} = -\gamma^{11}\lambda^A,
$$

(C.317)

where in the second equality we used the commutation relation (11.110). On the other hand, the gravitino $\hat{\psi}^{iA}$ has the same chirality as ψ^{iA}. For example, if ψ^{iA} has positive chirality, i.e., $\gamma^{11}\psi^{iA} = \psi^{iA}$, then

$$
\begin{aligned}
\gamma^{11}\hat{\psi}^{iA} &= \gamma^{11}(\psi^{iA} - \frac{1}{10}\gamma^i\gamma_j\psi^{jA}) = \gamma^{11}\psi^{iA} - \frac{1}{10}\gamma^i\gamma_j\gamma^{11}\psi^{jA} \\
&= \psi^{iA} - \frac{1}{10}\gamma^i\gamma_j\psi^{jA} = \hat{\psi}^{iA}.
\end{aligned}
$$

(C.318)

By denoting the chirality by subscripts, the fermionic field content is

$$\text{IIA:} \quad \overbrace{\underbrace{8_-}_{\lambda^{A=1}} + \underbrace{56_+}_{\hat{\psi}^{i,A=1}}}^{\text{NS-R}_+} + \overbrace{\underbrace{8_+}_{\lambda^{A=2}} + \underbrace{56_-}_{\hat{\psi}^{i,A=2}}}^{\text{R}_- \text{-NS}} = 128,$$

$$\text{IIB:} \quad \overbrace{\underbrace{8_-}_{\lambda^{A=1}} + \underbrace{56_+}_{\hat{\psi}^{i,A=1}}}^{\text{NS-R}_+} + \overbrace{\underbrace{8_-}_{\lambda^{A=2}} + \underbrace{56_+}_{\hat{\psi}^{i,A=2}}}^{\text{R}_+ \text{-NS}} = 128.$$

(C.319)

Therefore, for both IIA and IIB, the numbers of bosonic and fermionic fields are the same.

Chapter 12

Solution to Exercise 12.1

(1) Recall that, when we vary a matrix $M_{\mu\nu}$ (not necessarily symmetric) by $\delta M_{\mu\nu}$, the change in $M \equiv \det M_{\mu\nu}$ is given by

$$\delta M = M^{\nu\mu} \delta M_{\mu\nu}. \tag{C.320}$$

From this, one can see that the variation of the action S under small change $\delta M_{\mu\nu}$ in the matrix $M_{\mu\nu}$ is given by

$$\delta S = -\frac{T_p}{2} \int d^{p+1}\sigma \sqrt{-M}\, M^{\nu\mu} \delta M_{\mu\nu}. \tag{C.321}$$

Now, let us consider variation in X^P, $X^P \to X^P + \delta X^P$. Under this variation, $M_{\mu\nu}$ changes by $\delta M_{\mu\nu} = (\partial_\mu \delta X^P \partial_\nu X^Q + \partial_\mu X^P \partial_\nu \delta X^Q)\eta_{PQ}$. Plugging this into (C.321),

$$\delta S = -\frac{T_p}{2} \int d^{p+1}\sigma \sqrt{-M}\, M^{\nu\mu}(\partial_\mu \delta X^P \partial_\nu X^Q + \partial_\mu X^P \partial_\nu \delta X^Q)\eta_{PQ}$$

$$= \frac{T_p}{2} \int d^{p+1}\sigma \left[\partial_\mu(\sqrt{-M}\, M^{\nu\mu} \partial_\nu X^Q \eta_{PQ})\delta X^P + \partial_\nu(\sqrt{-M}\, M^{\nu\mu} \partial_\mu X^P \eta_{PQ})\delta X^Q \right]$$

$$= \frac{T_p}{2} \int d^{p+1}\sigma\, \partial_\mu \left[\sqrt{-M}\,(M^{\nu\mu} + M^{\mu\nu})\partial_\nu X^Q \eta_{PQ} \right]\delta X^P$$

$$= T_p \int d^{p+1}\sigma\, \partial_\mu(\sqrt{-M}\, G^{\mu\nu}\partial_\nu X^Q)\eta_{PQ}\delta X^P. \tag{C.322}$$

To go from the second line to the third, we relabeled $\mu \leftrightarrow \nu$, $P \leftrightarrow Q$ in the second term. Therefore, the equation of motion for X^P is given by

$$\partial_\mu(\sqrt{-M}\, G^{\mu\nu}\partial_\nu X^Q) = 0. \tag{C.323}$$

Next, let us consider small variation in A_μ, $A_\mu \to A_\mu + \delta A_\mu$. For this variation, $M_{\mu\nu}$ changes by $\delta M_{\mu\nu} = k\delta F_{\mu\nu} = k(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)$. Plugging this into (C.321), just like (C.322), we obtain

$$\delta S = -\frac{kT_p}{2} \int d^{p+1}\sigma \sqrt{-M}\, M^{\nu\mu}(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)$$

$$= \frac{kT_p}{2} \int d^{p+1}\sigma \left[\partial_\mu(\sqrt{-M}\, M^{\nu\mu})\delta A_\nu - \partial_\nu(\sqrt{-M}\, M^{\nu\mu})\delta A_\mu \right]$$

$$= \frac{kT_p}{2} \int d^{p+1}\sigma \left[\partial_\mu(\sqrt{-M}\, M^{\nu\mu})\delta A_\nu - \partial_\mu(\sqrt{-M}\, M^{\mu\nu})\delta A_\nu \right]$$

$$= -kT_p \int d^{p+1}\sigma\, \partial_\mu(\sqrt{-M}\, \theta^{\mu\nu})\delta A_\nu. \tag{C.324}$$

Therefore, the equation of motion for A_μ is given by

$$\partial_\mu(\sqrt{-M}\,\theta^{\mu\nu}) = 0. \tag{C.325}$$

(2) In the static gauge and for constant X^i,

$$M_{\mu\nu} = \eta_{\mu\nu} + kF_{\mu\nu}. \tag{C.326}$$

For small k, the inverse matrix is $M^{\mu\nu} = \eta^{\mu\nu} - kF^{\mu\nu} + \mathcal{O}(k^2)$ where $F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma}$, and therefore the antisymmetric part is $\theta^{\mu\nu} = -kF^{\mu\nu} + \mathcal{O}(k^2)$. On the other hand, from (C.326), $\sqrt{-M} = \sqrt{-\det\eta_{\mu\nu}} + \mathcal{O}(k) = 1 + \mathcal{O}(k)$. Therefore, for small k, (C.325) becomes

$$0 = \partial_\mu(\sqrt{-M}\,\theta^{\mu\nu}) = \partial_\mu\left[(1 + \mathcal{O}(k))(-kF^{\mu\nu} + \mathcal{O}(k^2))\right] = -k\partial_\mu F^{\mu\nu} + \mathcal{O}(k^2). \tag{C.327}$$

To the leading order, this gives

$$\partial_\mu F^{\mu\nu} = 0. \tag{C.328}$$

This is nothing but the Maxwell equation.

Aside: Therefore, one can think of the DBI action as a nonlinear generalization of the Maxwell action. Historically, the DBI action was first introduced in the 1930's to remove divergences in the Maxwell theory, but later abandoned, being replaced by QED. But in the context of string theory it reappeared as the effective action describing D-branes.

Solution to Exercise 12.2

(1) In the static gauge and for constant X^i,

$$M_{\mu\nu} = \eta_{\mu\nu} + kF_{\mu\nu}. \tag{C.329}$$

In 3D,

$$M = \det M_{\mu\nu} = \det\begin{pmatrix} -1 & kF_{01} & kF_{02} \\ -kF_{01} & 1 & kF_{12} \\ -kF_{02} & -kF_{12} & 1 \end{pmatrix} = -1 - k^2(F_{12}^2 - F_{01}^2 - F_{02}^2)$$

$$= -1 - k^2(F_{12}F^{12} + F_{01}F^{01} + F_{02}F^{02}) = -1 - \frac{k^2}{2}F_{\mu\nu}F^{\mu\nu}. \tag{C.330}$$

Therefore, the DBI action (12.58) becomes

$$S = -T_2\int d^3\sigma\sqrt{1 + \frac{k^2}{2}F_{\mu\nu}F^{\mu\nu}}. \tag{C.331}$$

(2) Because $\sqrt{1+x} = 1 + \frac{1}{2}x + \mathcal{O}(x^2)$, we can expand (C.331) to obtain

$$S = \int d^3\sigma \left[-T_2 - \frac{k^2 T_2}{4} F_{\mu\nu} F^{\mu\nu} \right] + \mathcal{O}(k^4). \tag{C.332}$$

By comparing this with (12.59), we obtain

$$\Lambda = -T_2 = -\frac{1}{(2\pi)^2 \alpha'^{3/2} g_s}, \qquad e^2 = \frac{1}{k^2 T_2} = \frac{g_s}{\alpha'^{1/2}}, \tag{C.333}$$

where we used $T_2 = \frac{1}{(2\pi)^2 \alpha' g_s}$. By inspecting (12.59), we see that the length dimension of Λ and e^2 is

$$[\Lambda] = (\text{length})^{-3}, \qquad [e^2] = (\text{length})^{-1}. \tag{C.334}$$

Considering $[\alpha'] = (\text{length})^2$, the α' dependence of Λ, e^2 in (C.333) is the expected one.

Solution to Exercise 12.3

(1) Because T-duality exchanges Neumann and Dirichlet boundary conditions, it exchanges

$$\text{NN} \leftrightarrow \text{DD}, \qquad \text{ND} \leftrightarrow \text{DN}. \tag{C.335}$$

Therefore, it is clear that the combinations

$$(\#\text{NN} + \#\text{DD}), \qquad (\#\text{ND} + \#\text{DN}), \tag{C.336}$$

are invariant under T-duality.

(2) Without loss of generality, let the D3-brane extend along the $x^0, \ldots, x^3$ directions and denote it by D3_{123}. Any other configuration of the D3-brane can be obtained by relabeling of coordinates. Let the other D-brane be a $\text{D}(p+q)$-brane extending in p ($0 \le p \le 3$) spatial directions parallel to D3_{123} and in q ($0 \le q \le 6$) spatial directions perpendicular to D3_{123}. Without loss of generality, we can assume that this $\text{D}(p+q)$-brane extends in the $x^1, \ldots, x^p$ directions and $x^4, \ldots, x^{3+q}$ directions. Because we are in type IIB, $p + q$ must be odd.

This configuration can be summarized in the diagram below:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	$\times$	$\times$	$\times$	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\text{D}(p+q)$	$\times$	$\times$	$\times$	$\cdot$	$\times$	$\times$	$\times$	$\cdot$	$\cdot$	$\cdot$

$$\underbrace{\qquad}_{p} \underbrace{\qquad}_{3-p} \underbrace{\qquad}_{q} \underbrace{\qquad}_{6-q}$$

Here, a "×" denotes a direction along which the D-brane is extending, while a "·" denotes a direction along which the D-brane is not extending. The particular configuration displayed above corresponds to $p = 2, q = 3$.

For a string stretching from the D3-brane to the $D(p+q)$-brane, the NN directions are $x^1, \ldots, x^p$, the DD directions are $x^{4+q}, \ldots x^9$ directions, the ND directions are $x^{p+1}, \ldots, x^3$, and DN directions are $x^4, \ldots, x^{3+q}$ (here we are ignoring the x^0 direction). So, in order to have $(\#\text{ND} + \#\text{DN}) = 4$,

$$(\#\text{ND} + \#\text{DN}) = (3 - p) + q = 4, \qquad \text{therefore,} \qquad q = p + 1. \qquad \text{(C.337)}$$

Therefore, the following four cases are the only possibilities, up to relabeling of coordinates:

$$\begin{aligned}
p = 0, \ q = 1: &\quad \text{D3}_{123}, \ \text{D1}_4 \\
p = 1, \ q = 2: &\quad \text{D3}_{123}, \ \text{D3}_{145} \\
p = 2, \ q = 3: &\quad \text{D3}_{123}, \ \text{D5}_{12456} \\
p = 3, \ q = 4: &\quad \text{D3}_{123}, \ \text{D7}_{1234567}
\end{aligned} \qquad \text{(C.338)}$$

How can we T-dualize these configurations to a D1-D5 system? Note that the D1_1-D5_{12345} system given in the problem has the following boundary conditions:

$$\text{NN}: \ x^1, \qquad \text{DD}: \ x^6, x^7, x^8, x^9, \qquad \text{ND}: \ \text{none}, \qquad \text{DN}: \ x^2, x^3, x^4, x^5. \qquad \text{(C.339)}$$

So, we want to T-dualize the configurations (C.338) so that we end up with 1 NN, 4 DD, 0 ND, and 4 DN directions (or 1 NN, 4 DD, 4 ND, and 0 DN directions), remembering that T-duality exchanges N↔D.

For example, consider D3_{123}-D1_4 in (C.338). This has

$$\text{NN}: \ \text{none}, \qquad \text{DD}: \ x^5, x^6, x^7, x^8, x^9, \qquad \text{ND}: \ x^1, x^2, x^3, \qquad \text{DN}: \ x^4. \qquad \text{(C.340)}$$

So, we can T-dualize one of the DD directions, say x^4, and the DN direction x^5 to get the same boundary condition (C.339), up to DN↔ND. Including other configurations, the T-duality transformations that bring the configurations (C.338) to the D1-D5 system are:

$$\begin{aligned}
\text{D3}_{123}, \ \text{D1}_4 \quad &\xrightarrow[T_4 T_5]{} \quad \text{D5}_{12345}, \ \text{D1}_5 \\
\text{D3}_{123}, \ \text{D3}_{145} \quad &\xrightarrow[T_2 T_3]{} \quad \text{D1}_1, \ \text{D5}_{12345} \\
\text{D3}_{123}, \ \text{D5}_{12456} \quad &\xrightarrow[T_1 T_3]{} \quad \text{D1}_2, \ \text{D5}_{23456} \\
\text{D3}_{123}, \ \text{D7}_{1234567} \quad &\xrightarrow[T_1 T_2]{} \quad \text{D1}_3, \ \text{D5}_{34567}
\end{aligned} \qquad \text{(C.341)}$$

where T_i means T-dualization along the x^i direction. Of course the T-duality transformations given in (C.341) are not unique. By appropriate relabeling of coordinates, all these are equivalent to the D1$_1$-D5$_{12345}$ system.

Note: In the above, we considered only "normal" D-branes which extends in the time direction x^0; namely, we did not consider configurations involving D(-1)-branes (D-instantons). If we include D(-1)-branes also, then for example D(-1)-D3$_{123}$ is also a configuration with $(\#\text{ND} + \#\text{DN}) = 4$. However, we enconunter a problem when we T-dualize this into a D1-D5 system. A "T-duality transformation" involving x^0 seems to take D(-1)-D3$_{123}$ to a D1-D5 system, but it is not clear if T-duality along the time direction is meaningful, because we cannot compactify the time direction to give an argument for T-duality just like we did in this chapter for compact spatial directions.

Chapter 13

Solution to Exercise 13.1

The only nonvanishing components of $\Gamma^\kappa_{\mu\nu}$ are

$$\Gamma^-_{+i} = \Gamma^-_{i+} = \partial_i H, \qquad \Gamma^-_{++} = \partial_+ H, \qquad \Gamma^i_{++} = -\frac{1}{2}\partial_i H. \qquad (C.342)$$

This, (C.342), can be shown by plugging the explicit metric (13.94) into the formula for $\Gamma^\kappa_{\mu\nu}$ given in (13.97), but there is an easier way. Namely, consider the action for a massless point particle in this background:

$$S_{pp} = \int d\tau \frac{g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}{2e} = \int d\tau \frac{1}{2e}\left[\dot{x}^+\dot{x}^- + (H(x^i, x^+) - 1)(\dot{x}^+)^2 + \dot{x}^i\dot{x}^i\right]. \qquad (C.343)$$

The equation of motion for x^μ derived from this action must be the geodesic equation (in the $e = 1$ gauge)

$$\ddot{x}^\kappa + \Gamma^\kappa_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0, \qquad (C.344)$$

from which we can read off $\Gamma^\kappa_{\mu\nu}$. Explicitly, the equations of motion derived from (C.343) are

$$\ddot{x}^+ = 0, \qquad \ddot{x}^- + \partial_+ H(\dot{x}^+)^2 + 2\partial_i H\,\dot{x}^+\dot{x}^i = 0, \qquad \ddot{x}^i - \frac{1}{2}\partial_i H(\dot{x}^+)^2 = 0. \qquad (C.345)$$

From these equations, it is easy to see that $\Gamma^\kappa_{\mu\nu}$ is given by (C.342).

From the above formulae (13.97), the Ricci tensor is given by

$$R_{\mu\nu} = \partial_\kappa \Gamma^\kappa_{\mu\nu} + \Gamma^\kappa_{\kappa\rho}\Gamma^\rho_{\mu\nu} - \partial_\mu \Gamma^\kappa_{\kappa\nu} - \Gamma^\kappa_{\mu\rho}\Gamma^\rho_{\kappa\nu}. \qquad (C.346)$$

Carefully inspecting which components of Γ are nonvanishing using (C.342), we can see that

$$R_{\mu\nu} = \partial_i \Gamma^i_{\mu\nu} = \begin{cases} -\frac{1}{2}\partial_i^2 H & \mu = \nu = +, \\ 0 & \text{otherwise.} \end{cases} \qquad (C.347)$$

Therefore, the Einstein equation implies (13.96). To derive (C.347), it is useful to note that (C.342) in particular implies that i) $+$ can't be upstairs, ii) $-$ can't be downstairs, and iii) the only letter that can appear twice is $+$, and only downstairs. For the first term in (C.346), i) means that $\kappa \neq +$, and the fact that H is independent of x^- means that $\kappa \neq -$. The second and the third terms are eliminated by iii). For the forth term, i) and ii) imply that $(\kappa, \rho) = (i, j)$, but that's not possible by iii). So, actually $R_{\mu\nu} = \partial_i \Gamma^i_{\mu\nu}$. But (C.342) means that this is only nonvanishing for $\mu = \nu = +$, which is given by (C.347).

Solution to Exercise 13.2

(1) By using $x^{\pm} = x^1 \pm t$ and completing the square, the gravity wave metric (13.94) can be written as

$$ds^2 = -H^{-1}dt^2 + H[dx^1 + (1 - H^{-1})dt]^2 + \sum_{i=2}^{9} dx^i dx^i, \qquad \text{(C.348)}$$

$$\equiv g_{tt}dt^2 + g_{11}[dx^1 + A_0 dt]^2 + g_{ij}dx^i dx^i. \qquad \text{(C.349)}$$

The dilaton and the B-fields are $\phi = 0$, $B_{\mu\nu} = 0$.

Using the T-duality rule, the fields after T-dualization along x^1 are given

$$\tilde{g}_{11} = \frac{1}{g_{11}} = H^{-1}, \qquad \tilde{B}_{10} = A_0 = 1 - H^{-1}, \qquad e^{-2\tilde{\phi}} = g_{11}e^{-2\phi} = H. \quad \text{(C.350)}$$

While the other components of metric are invariant. Noting that

$$\tilde{B}_{01} = -\tilde{B}_{10} = H^{-1} - 1,$$

the fields after T-duality are indeed given by (13.98) (after dropping the tildes $\tilde{\ }$).

Note that, when H is independent of x^+, the original metric (13.94) represents a gravitational wave in the x^1 direction, which carries some momentum number k^1 along x^1. Because T-duality exchanges the momentum number k^1 and the winding number w^1 (= fundamental string charge), the resulting configuration (13.98) describes the fields produced by a fundamental string(s) stretching along x^1 with winding number $\tilde{w}^1 = k^1$.

(2) The rule, (i), that the Einstein frame metric is invariant,

$$g_E = e^{-\phi/2}g_S = \tilde{g}_E = e^{-\tilde{\phi}/2}\tilde{g}_S, \qquad \text{(C.351)}$$

means that the string frame metric transforms under S-duality as

$$\tilde{g}_S = e^{\tilde{\phi}/2}\tilde{g}_E = e^{\tilde{\phi}/2}g_E = e^{\tilde{\phi}/2 - \phi/2}g_S = e^{-\phi}g_S. \qquad \text{(C.352)}$$

In the last equality we used the rule (iv): $\tilde{\phi} = -\phi$. Therefore, applying S-duality transformation to the metric (13.98), we obtain

$$d\tilde{s}^2 = e^{-\phi}ds^2 = H^{1/2}ds^2 = H^{-1/2}(-dt^2 + (dx^1)^2) + H^{1/2}\sum_{i=2}^{9} dx^i dx^i.$$

The other fields can be worked out as follows. From (ii), we obtain

$$\widetilde{C}_{01} = B_{01} = H^{-1} - 1.$$

From (iv), we obtain

$$e^{-2\widetilde{\phi}} = e^{2\phi} = H^{-1}.$$

Therefore, the fields after S-duality are indeed given by (13.99) (after dropping the tildes $\widetilde{}$).

(3) Applying the T-duality rule to (13.99), we obtain

$$\widetilde{g}_{22} = \frac{1}{g_{22}} = H^{-1/2}, \quad \widetilde{C}_{012} = C_{01} = H^{-1} - 1, \quad e^{-2\widetilde{\phi}} = e^{-2\phi}g_{22} = H^{-1}H^{1/2} = H^{-1/2}.$$

$$\text{(C.353)}$$

Therefore, the fields after T-duality are indeed given by (13.100) (after dropping the tildes $\widetilde{}$).

Solution to Exercise 13.3

(1) Recall that the dimensional reduction rule from M-theory to type IIA supergravity is given by

$$ds_{11}^2 = e^{-\frac{2}{3}\phi}g_{\mu\nu}dx^\mu dx^\nu + e^{\frac{4}{3}\phi}(dx_{10}^2 + C_\mu^{(1)}dx^\mu),$$
$$C_{\mu\nu\rho} = A_{\mu\nu\rho}, \qquad B_{\mu\nu} = A_{\mu\nu,10},$$

$$\text{(C.354)}$$

where $\mu, \nu, \rho = 0, \ldots, 9$.

Let us apply the reduction rule (C.354) to the metric (13.101), with the M-circle direction being x^2. First, we have $H^{-2/3} = e^{\frac{4}{3}\phi}$, hence

$$e^{-2\phi} = H.$$ $$\text{(C.355)}$$

Then the 10D metric can be obtained by multiplying the metric, (13.101), without the $(dx^2)^2$ term by $e^{\frac{2}{3}\phi} = H^{-1/3}$. Namely,

$$ds_{10}^2 = H^{-1/3}\left[H^{-2/3}(-dt^2 + (dx^1)^2) + H^{1/3}\sum_{i=3}^{10} dx^i dx^i\right]$$
$$= H^{-1}(-dt^2 + (dx^1)^2) + \sum_{i=3}^{10} dx^i dx^i.$$

$$\text{(C.356)}$$

Also, the B-field in 10D is

$$B_{01} = A_{012} = H^{-1} - 1. \tag{C.357}$$

Up to relabeling of coordinates $(x^3, \ldots, x^{10}) \rightarrow (x^2, \ldots, x^9)$, the configuration given by (C.355), (C.356) and (C.357) gives the fundamental string solution, see (13.98).

(2) Now let us apply the reduction rule, (C.354), to the metric given in (13.101), with the M-circle direction being x^3. First, we have $H^{1/3} = e^{\frac{4}{3}\phi}$, hence

$$e^{-2\phi} = H^{-1/2}. \tag{C.358}$$

Then, once again, the 10D metric can be obtained by multiplying the metric, (13.101), without the dx^3 term by $e^{\frac{2}{3}\phi} = H^{1/6}$. Namely,

$$\begin{aligned}
ds_{10}^2 &= H^{1/6}\left[H^{-2/3}(-dt^2 + (dx^1)^2 + (dx^2)^2) + H^{1/3}\sum_{i=4}^{10} dx^i dx^i\right] \\
&= H^{-1/2}(-dt^2 + (dx^1)^2 + (dx^2)^2) + H^{1/2}\sum_{i=4}^{10} dx^i dx^i.
\end{aligned} \tag{C.359}$$

Also, the RR 3-form potential in 10D is

$$C_{012} = A_{012} = H^{-1} - 1. \tag{C.360}$$

Now, up to relabeling of coordinates $(x^4, \ldots, x^{10}) \rightarrow (x^3, \ldots, x^9)$, the configuration described by (C.358), (C.359) and (C.360) gives the D2-brane solution, see (13.100).