Physics Skills Workshops

Inverse Square Laws

The purpose of this weeks exercise is to take a look at a very common physical situation – the inverse square law – and then solve some simple problems which all involve them to some extent. So in addition to seeing how and why inverse square laws work, you should remember to put into practice what you've learnt so far about solving physics problems: understanding the basic principles; drawing diagrams; deciding on appropriate assumptions; and only finally doing the maths!

Introduction to Inverse Square Laws

There are a huge number of examples of inverse square laws to be found in physics, they are something we recognise and work out without really thinking about it – consider this question:

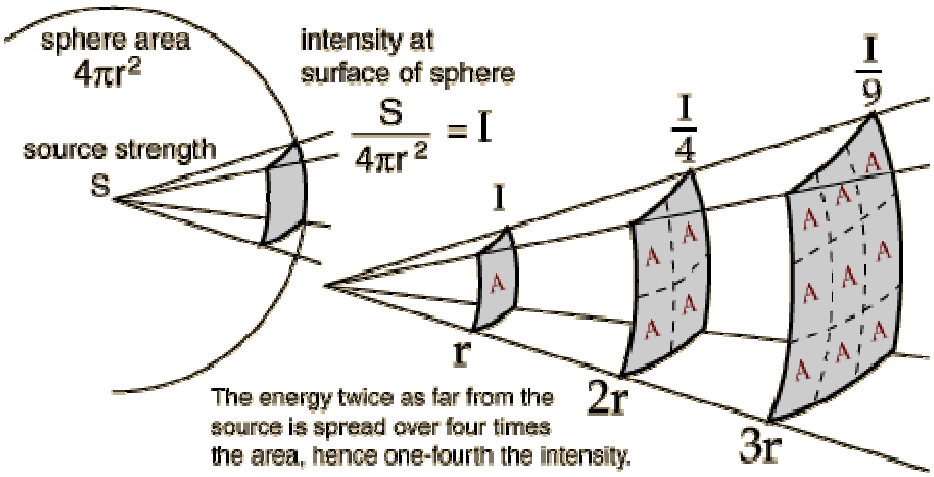
"If you receive a given radiation dose when standing 1m from a radioactive source, how much is the dose reduced if you move to 2m from the source?"

The answer is, of course, that the dose reduces by a factor of 4. Even without knowing the details of the radioactivity, how dose is calculated, or how it's absorbed by the body, you know that moving twice as far away reduces the effect by 4. Why? Because it's an inverse square law…

Important note: If you didn't get that answer, don't panic! All will become clear…

Isotropic Emission, Conservation of Flux, and Inverse Square Laws

Consider a 'source' which emits energy at a rate of S units per second (the type of source, and the units of S are irrelevant for this…). This situation is shown in the diagram below:



Consider a sphere centred on the source, and surrounding it at a radius of r. If we assume the energy flows out isotropically from the source, then the energy received at any point on the sphere should be the same. It's easy to calculate the intensity of

the energy (energy/unit area) as it passes through the sphere, it's just the total energy divided by the surface area of the sphere.

Now extend this idea to spheres at different radii - the surface area of each sphere increases as r^2, so the intensity of the energy (per unit area) must reduce as $1/r^2$, and that's it – the inverse square law.

So, in summary - Isotropic emission from a point generally results in an inverse square law because it mathematically describes the conservation of flux flowing out through concentric shells centred on that point.

That 'flux' can be many things:

- Gravity
- Light intensity
- Electrostatic attraction/repulsion
- Electromagnetic Radiation
- Sound
- Randomly emitted particles (so ionising radiation)

All these situations can be analysed in exactly the same (simple) way.

Example 1 – Gravity in orbit

The International Space Station (ISS) orbits at an altitude of ~350km above the Earth's surface (Earth's radius is 6378.2km). What fraction of the surface gravitational acceleration, g, is felt at this altitude?

So all this talk about feeling weightless in space is a bit misleading! Explain this apparent confusion.

Example 2 – Force between charges

A charge of 8×10^{-18}C is placed at (0,2) on the x,y-plane.

Calculate the force, and its direction, experienced by an electron placed at (4,5).

Where should the electron be moved to so that the force experienced is decreased by a factor of three?

What Examples 1 and 2 show us…

These 2 examples show how simple scaling by an inverse square laws works well in situations where symmetry can be applied, and you are looking for a relative change.

However, when symmetry is not present, or when absolute levels are important, it can be necessary to quantify the flux passing through a specific area at a specific distance from the emitting source.

Solid angle calculations

The solid angle subtended by an area at a distance from a point emitter can be used to calculate the flux received by that area from the emitter.

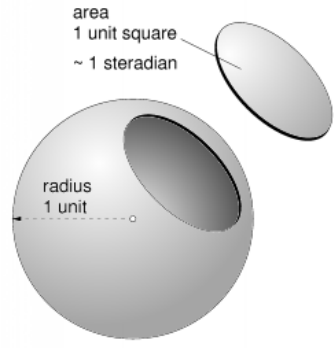
The solid angle subtended by a surface S is defined as the surface area of a unit sphere covered by the surface's projection onto the sphere.

Formally, the solid angle for any shape can be calculated using this formula:

$$\Omega \equiv \iint_S \frac{\hat{\mathbf{n}} \cdot d\mathbf{a}}{r^2},$$

where da is a small area of the shape, and n and r^2 define the unit vector and distance from the origin.

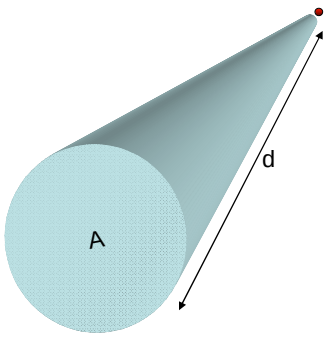
As you can calculate, the whole sky subtends a solid angle of 4π steradians.



There is a much simpler way to calculate solid angles – called the spherical approximation. This assumes that the area A is all equidistant at a distance d from the reference point, and orthogonal to the unit vector from the origin. Under this approximation, we can state that the solid angle

$$\Omega = \frac{A}{d^2}.$$

If you can't remember this formula, just remember the case for the whole sky, where



$A = 4\pi d^2$ and $\Omega = 4\pi$.

Knowing the solid angle subtended by an area allows us to calculate what fraction of the emitted isotropic flux passes through that area.

$$F_{area} = \frac{\Omega}{4\pi} F_{emitted}$$

Where Ω is the solid angle subtended by the area.

Example 3 – Temperature of the planets

Estimating the temperature of the planets is a fairly simple calculation. Getting the right answer is considerably trickier…

Of course, the biggest effect is the amount of energy each planet receives from the Sun, so inverse square laws are fundamental to the calculation.

Step 1 First, calculate the solar energy flux on each planet, in W/m^2, assuming the Sun radiates energy at a rate of 3.8×10^{26} W, and assuming the solar energy is radiated isotropically.

(hint: you need to make a solid angle calculation, with the area of $1m^2$ at the distance of each planet)

Planet	Mean distance from Sun (AU[1])	Solar flux (W/m^2)
Mercury	0.387	
Venus	0.723	
Earth	1	
Mars	1.524	
Jupiter	5.203	
Saturn	9.54	

Step 2 If each planet is in thermal equilibrium, it must be radiating away as much energy as it is receiving.

Use Stefan's law ($F = \sigma T^4$, where $\sigma=5.67 \times 10^{-8} W/m^2$) combined with your values of F calculated above to calculate the expected temperatures of the planets (assuming no atmospheric effects) based on the following data.

Planet	Mean distance from Sun (AU)	Temperature (K)	Actual Temperature
Mercury	0.387		100-700
Venus	0.723		700
Earth	1		250-300
Mars	1.524		120-390
Jupiter	5.203		110-150
Saturn	9.54		95

Compare the calculated values and the actual values on a graph, and comment on the results you have obtained.

Why is this such a poor model…

(i) the albedo of each planet (fraction of the incident solar radiation reflected) is very different – ie 0.06 for Mercury, 0.76 for Venus, 0.4 for Earth

[1] One AU (Astronomical Unit) is the mean Earth-Sun distance, 1.495×10^{11}m

(ii) the presence of an atmosphere dramatically changes the surface temperatures of a planet.

These ‘sub-solar temperatures’ you have calculated are essentially the equilibrium noontime temperatures on slowly rotating planets with no atmospheres.

Example 4 – Sound pressure levels

The engines of a jet taking off emit a noise level of ~140 dB at 1m – which is enough to destroy your hearing very quickly. How far away do you need to stand to reduce the sound level to a more acceptable 85dB, which is about the safe limit for prolonged exposure?

Hint: Although this is a simple invers-square law scaling again, this is a slightly more complex calculation, as you need to know that the dB scale is logarithmic (see box to the right)

First calculate the meaning of the change from 140 dB to 85dB, re-expressing this as a simple ratio of the sound energy levels.

Now calculate the appropriate distance you should move to.

You need to know how the ear perceives loudness. First of all, the ear is very sensitive. The softest audible sound has a power of about 0.000000000001 watt/sq. meter and the threshold of pain is around 1 watt/sq. meter, giving a total range of 120dB. In the second place, our judgment of relative levels of loudness is somewhat logarithmic. If a sound has 10 times the power of a reference (10dB) we hear it as twice as loud. If we merely double the power (3dB), the difference will be just noticeable.

[The calculations for the dB relationships I just gave go like this; for a 10 to one relationship, the log of 10 is 1, and ten times 1 is 10. For the 2 to one relationship, the log of 2 is 0.3, and 10 times that is 3. Incidentally, if the ratio goes the other way, with the measured value less than the reference, we get a negative dB value, because the log of 1/10 is -1.]

Units, Dimensions and Dimensional Analysis

Units

Length can be measured in any number of different units, e.g. meters, cm, km, inches, angstroms, astronomical units, spans, yards.

A measurement of length 2 means nothing unless we state what units we are measuring in. A measurement of 2 furlongs means a length which is 2 times as long as a standard furlong defined somewhere and agreed by everyone.

In general physicists should use the standard international units whose definitions are agreed by an international committee of scientists, http://physics.nist.gov/cuu/Units/units.html, however as long as you define your units carefully it is always possible to convert between different systems.

Dimensions

We implicitly know that it is not possible to say that a length is bigger or smaller than an area or that 1 second is heavier than a kilogram even though we can say that an ounce is smaller than a kilogram.

This is because that separate from the idea of units a measurement has associated with it a quality called dimension.

Although we can't compare areas and lengths we know that they are related by the fact that the area of a square is given by the length of its side squared. We also know that the volume of a cube is given by a length cubed or an area times a length. However no matter what we do with lengths, areas and volumes on their own we cannot make a time.

There is a basic set of dimensions from which we can define the dimension of all other physically measurable quantities these include the most commonly used ones

Length – L, Mass – M, Time –T, Charge – Q.

However there are other, less used quantities, such as fundamental particle type, e.g. quarks, and the equivalent to charge and mass for the other fundamental forces.

Dimensional Analysis

This is just common sense written down so that we can apply it to mathematical equations.

Whilst in maths it is acceptable to write down any set of symbols in equations e.g. A=B in physics only certain equations are acceptable because for instance a length cannot equal a mass. The basic rules are

1) two physical quantities can only be equated if they have the same dimensions
2) two physical quantities can only be added if they have the same dimensions
3) the dimensions of the multiplication of two quantities is given by the multiplication of the dimensions of the two quantities.
4) the dimension of the division of two quantities is given by the division the dimensions of the two quantities.

By analysing an equation in terms of dimensionality we can determine if it could be correct even if we don't know anything else about the physics. This is an extremely powerful technique which has saved many physicists from looking very stupid.

Dimensionless Numbers – not core part of skills session

Often in physics we know the variables which are important to a problem before we understand mechanisms responsible for the behaviour of the system. In that case it is often found that it is possible to produce one or more dimensionless numbers out of the important input parameters and that the behaviour of the system depends on the value of a limited set of these dimensionless numbers independent of the specific values of the much larger set of dimensional input numbers. Famous examples of these dimensionless numbers include the Mach number and Reynolds number.

Examples

A) What are the dimensions of the following common physical quantities? Give two different units for each quantity. e.g. Energy has dimensions ML^2T^{-2} and can be measured in Joules or Calories.

a) Velocity - v,	b) Current - I,	c) Force - $\underline{F}$,
d) Acceleration- $\underline{a}$,	e) Mass Density - ρ,	f) Momentum - p,
g) Pressure - P	h) Magnetic Field - $\underline{B}$,	g) Electric Field $\underline{E}$,
i) Dipole moment -$\underline{d}$,	j) Angular Momentum - $\underline{L}$,	k) Torque - $\underline{\tau}$

B) What are the dimensions of the variable K in the following equations.
You may need the following q is a charge, t is a time, r is distance, En is an energy, f is a frequency.

i) $\underline{F}=K\underline{L}$

ii) $\underline{E} = \dfrac{q\underline{r}}{Kr^2}$

iii) $\underline{\tau} = P\underline{K} \times \underline{r}$

iv) $P = K\rho|\underline{a}|$

v) $En = Kf$

C) Which of the following equations could be true and which cannot? Give a dimensional analysis proof of your answer.
V is a volume, $\underline{\omega}$ is an angular velocity with units radians per sec, r_0 is a distance,

i) $\dfrac{\partial^2 \underline{E}}{\partial x^2} = v^2 \dfrac{\partial^2 \underline{E}}{\partial t^2}$

ii) $PV = \sum \dfrac{1}{2} mv^2 + \dfrac{1}{2} \underline{L}.\underline{\omega}$

iii) $r = r_0 \sin(3t)$