PHYS1022 - Electricity and Magnetism

Integration Reminder

Integration is the reverse of the process of differentiation. In the usual notation

$$\int f'(x)dx = f(x) + \text{constant}$$

The derivative of the RHS gives you $f'(x)$. The examples we have used in the course are (k and c are constants)

$$\int k\ dx = kx + c$$

$$\int kx\ dx = \frac{1}{2}kx^2 + c$$

$$\int \sin\theta\ d\theta = -\cos\theta + c$$

$$\int \cos\theta\ d\theta = \sin\theta + c$$

$$\int \frac{dx}{x} = \ln x + c$$

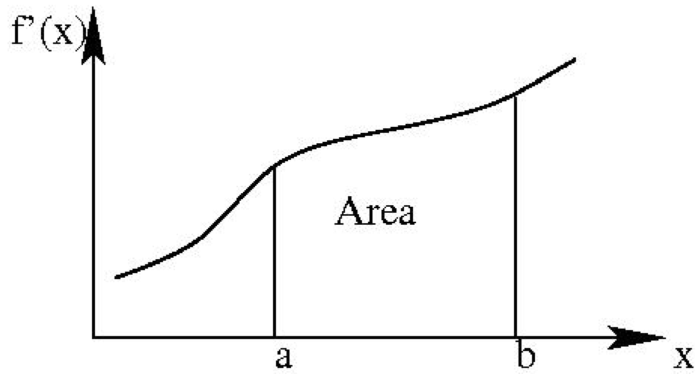
and two trickier ones you might want to check your differentiation on

$$\int \frac{a}{(L^2 + a^2)^{3/2}}\ da = \frac{-1}{\sqrt{L^2 + a^2}} + c$$

$$\int \frac{dy}{\sqrt{x^2 + y^2}} = \ln[2y + 2\sqrt{x^2 + y^2}] + c$$

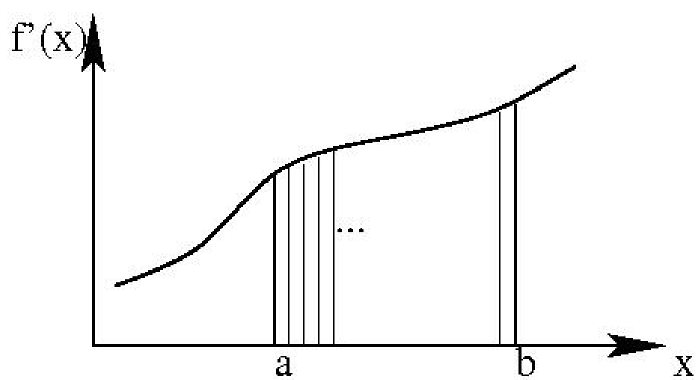
The Link to Area & Adding Up

Newton showed that integrating between limits computes the area under the curve $f'(x)$.



$$\text{Area} = \int_a^b f'(x)dx = [f(x)]_a^b = f(b) - f(a)$$

Since we can also write the area as a sum over the areas of infinitesimal rectangles (here we imagine n of them) we know that



$$\sum_n f'(x_n)dx = \int_a^b f'(x)dx$$

In particular therefore if we ever have a sum of the form shown we can indentify $f'(x)$ and use integration to do the summation.

Integrating Between Limits

If we have an expression such as

$$\frac{dI}{dt} = -\frac{I}{RC}$$

Here we're looking at a current's dependence on time. We can integrate to find I as a function of t

$$\int \frac{dI}{I} = -\frac{1}{RC}\int dt$$

BUT we must fix the limits of integration - if we are integrating to find $I(t = T)$ in terms of $I(t = 0)$ then we do the time integration from $t = 0$ to $t = T$.

The I integration must share the same initial and starting points - so we integrate from $I(t = 0)$ to $I(t = T)$ - typically we might call those values I_0 and just I.

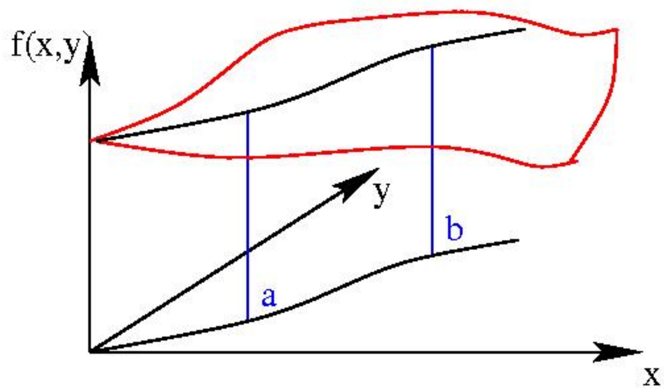
$$\int_{I_0}^{I} \frac{dI}{I} = -\frac{1}{RC}\int_0^T dt$$

$$\ln(I/I_0) = -\frac{1}{RC}T$$

$$I = I_0 e^{-\frac{1}{RC}T}$$

Line Integrals

If we have a function that depends on two variables (x, y) then we can plot like this



The red surface is the value of the function at each point over the $x - y$ plane.

We can do many "line integrals". We could integrate along the x-axis in which case we ignore all the y extent of the diagram and just work in the $f(x, y = 0) - x$ plane and compute the area under the curve. We could do the same but just keeping the $F(x = 0, y)$-y plane.

Or we could pick the black curve in the $x-y$ plane. The integral is done by taking infintesimal lengths of the line, multiplying that length, dl, by the value of $f(x, y)$ above the dl piece. Then we add up the results for all the pieces that make up the line.

$$\text{Line Integral} = \sum_{\text{along line}} F(x, y) dl$$

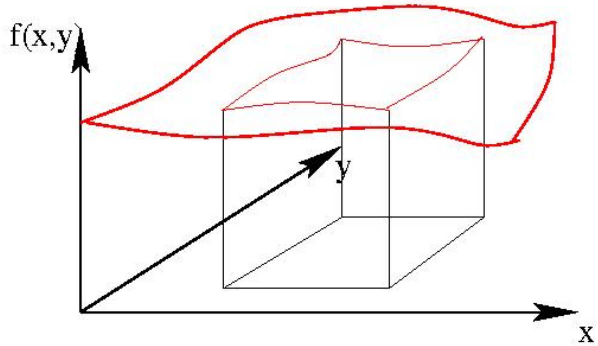
In this course we have only done line integrals along lines where $f(x, y)$ is a constant, f_l. In problems with Gauss' law or Ampere's law we use symmetry to find such lines. The integral is then simply given by

$$\text{Line Integral} = f_l \sum_{\text{along line}} dl$$

which is just $f_l \times$ the length of the line.

Area Integrals

For our function $f(x,y)$ we can also compute the volume between the red surface and some shape in the $x-y$ pla



The only examples I did in the course were where $f(x,y)=1$. The volume is then equal to the area of the shape on the bottom plane.

For example I did the area of a square between $x=0..a$, $y=0..a$

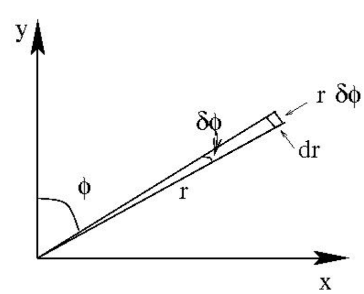
$$Area = \int_0^a \int_0^a dxdy = [x]_0^a [y]_0^a = a^2$$

Note my integration factor is $dxdy$ which is the area of an infintesimal square on the $x-y$ plane. I'm adding them all up to get the total area.

We also did the area of a circle of radius a, centred on the origin. Here we use polar coordinates

$$0 < r < \infty, \qquad 0 < \phi < 2\pi$$

We make an infinitesimal square on the plane by changing $r \rightarrow r+dr$ and $\phi \rightarrow \phi + d\phi$. The latter induces an arc of length $rd\phi$.

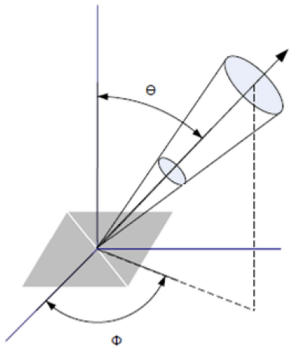


The infinitesimal area element is therefore $rdrd\phi$. Adding them up over a circle gives the total area

$$Area = \int_0^a \int_0^{2\pi} rdrd\phi = \left[\frac{r^2}{2}\right]_0^a [\phi]_0^{2\pi} = \pi a^2$$

Volume Integrals

If I want to compute the surface area or volume of a sphere I need to set up polar coordinates.



$$0 < r < \infty, \qquad 0 < \theta < \pi, \qquad 0 < \phi < 2\pi$$

The range in θ and ϕ that an object subtends is the "solid angle". eg the sun and the moon subtend the same solid angle even though they are different sizes leading to solar eclipses.

To get an infinitesimal cube we vary $r \to r + dr$, $\theta \to \theta + \delta\theta$ and $\phi \to \phi + d\phi$. This induces the sides of a cuboid with lengths dr, $rd\theta$ and $r \sin\theta d\phi$.

The top surface area of the cube is

$$dA = r^2 \sin\theta \; d\theta \; d\phi$$

Summing them up gives the surface area of a sphere of radius a

$$A = a^2 \int_0^{\pi} \int_0^{2\pi} \sin\theta \; d\theta \; d\phi \;\; = \;\; a^2 \;\; [-\cos\theta]_0^{\pi} \;\; [\phi]_0^{2\pi} \;\; = \;\; a^2.2.2\pi \;\; = \;\; 4\pi a^2$$

The volume element is

$$dV = r^2 \sin\theta \; dr \; d\theta \; d\phi$$

If we sum over the volume of a sphere we get

$$V = \int_0^a \int_0^{\pi} \int_0^{2\pi} r^2 \sin\theta \; dr \; d\theta \; d\phi \;\; = \;\; 4\pi \left[\frac{r^3}{3} \right]_0^a \;\; = \;\; \frac{4}{3}\pi a^3$$

here the angular integrals are just the same as in the surface area case.

We can also see that by constructing the sphere's volume from spherical shells of area $4\pi r^2$ and width dr and adding them up

$$V = 4\pi \int_0^a r^2 dr = \frac{4}{3}\pi a^3$$