III Coulomb's Law

The force exerted on charge Q_2 by charge Q_1 is

$$\vec{F}_{1 on 2} = \frac{k Q_1 Q_2}{r^2} \hat{\underline{r}}_{1\to 2}$$

Note the force is a <u>vector</u> – it has a <u>magnitude</u> & a <u>direction</u>

$$k = \frac{1}{4\pi\epsilon_0}$$

ϵ_0 - permittivity of free space
$8 \cdot 85 \times 10^{-12}\ C^2 N^{-1} m^{-2}$

$\hat{\underline{r}}_{1\to 2}$ is a unit vector pointing along the line between two charges towards the one feeling the force

Note: $\hat{\underline{r}} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$

NB/ leave the sign on Q – it sorts out attraction & repulsion

NB/ always check the direction of the force is right!

$|\vec{F}| \propto Q_1 Q_2$

$|\vec{F}| \propto 1/r^2$

Vectors: $\vec{F}_{12}$ is a vector with 3 components giving the magnitude of the force experienced in the x, y & z directions

$$\underline{F} = \vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = F_x \hat{\underline{i}} + F_y \hat{\underline{j}} + F_z \hat{\underline{k}}$$

Multiple Charge Systems

If a charge is being acted on by many other charges the total force is the vector sum

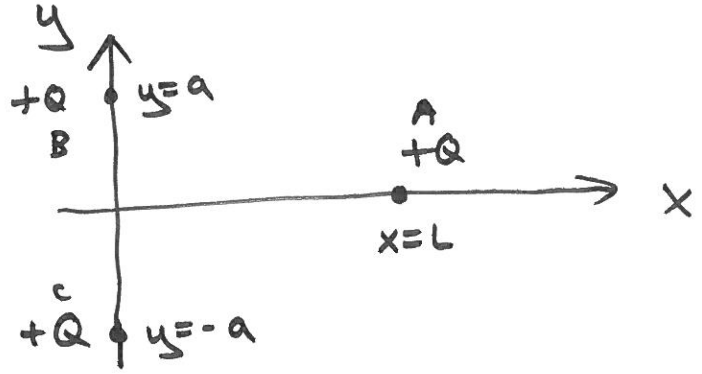
$$\vec{F}_{TOT(2)} = \sum_i \vec{F}_{i2}$$

NB/ $\vec{F}_1 + \vec{F}_2 = (F_x^1 + F_x^2)\hat{\underline{i}} + (F_y^1 + F_y^2)\hat{\underline{j}} + (F_z^1 + F_z^2)\hat{\underline{k}}$

Note a charge never experiences a force due to it's own charge!

Please: mark vectors as vectors!
do not write vector = scalar!
do not divide by a vector!

eg



The force on the charge at $x=L$ is

$$\vec{F}_A = \frac{kQ^2}{r_{BA}^3}\vec{r}_{BA} + \frac{kQ^2}{r_{CA}^3}\vec{r}_{CA}$$

$$\vec{r}_{BA} = \underline{\hat{r}}_{BA}|\vec{r}_{BA}| = \underline{\hat{r}}_{BA}\, r_{BA}$$

Now $\vec{r}_{BA} = \begin{pmatrix} L \\ -a \\ 0 \end{pmatrix}$ $\qquad \vec{r}_{CA} = \begin{pmatrix} L \\ a \\ 0 \end{pmatrix}$

$$r_{BA}^3 = r_{CA}^3 = (L^2+a^2)^{3/2}$$

$$\vec{F}_A = \frac{kQ^2}{(L^2+a^2)^{3/2}}\left[\begin{pmatrix} L \\ -a \\ 0 \end{pmatrix} + \begin{pmatrix} L \\ a \\ 0 \end{pmatrix}\right]$$

$$= \frac{kQ^2}{(L^2+a^2)^{3/2}}\begin{pmatrix} 2L \\ 0 \\ 0 \end{pmatrix}$$

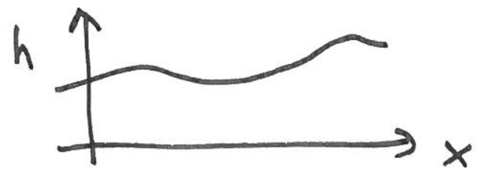
$$= \frac{2kLQ^2}{(L^2+a^2)^{3/2}}\,\underline{\hat{i}}$$

Note both ~~B~~ B & C push A in +ve x-direction.

But one pushes to +ve & one to −ve y-direction – hence $F_y = 0$.

IV ELECTRIC FIELD

Fields are like "hills" - they take a value (the height of the hill) at every point in space. The height of the hill is a <u>scalar</u> field



<u>Vector</u> fields have three numbers at each point in space eg for moving water we could assign the $\overrightarrow{\text{velocity}}$ of the water at each point in space.

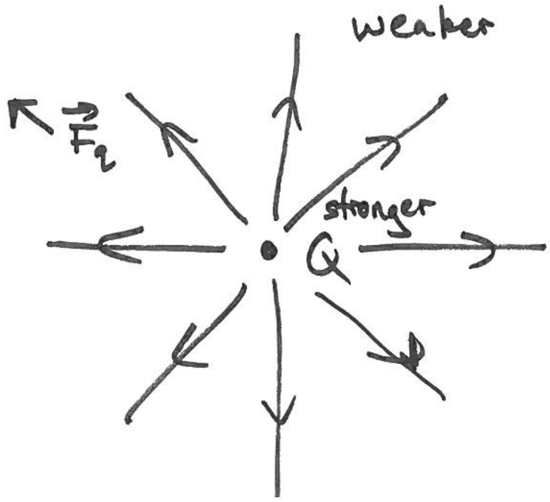
We can invent a concept of a space filling field whose vector value tells you the force a charge of 1C would experience at each point in space

$$\vec{E} = \frac{\vec{F}_q}{q}$$ here the force on q normalized by q

UNITS: NC^{-1}

We can draw $\vec{E}$ with lines whose tangent is in the direction of $\vec{E}$ & whose density are proportional to $|\vec{E}|$

q point charge



Rules for drawing field lines

1/ Begin on +ve charges
End on -ve charges

2/ Uniformily spaced near a charge (rotational sym)

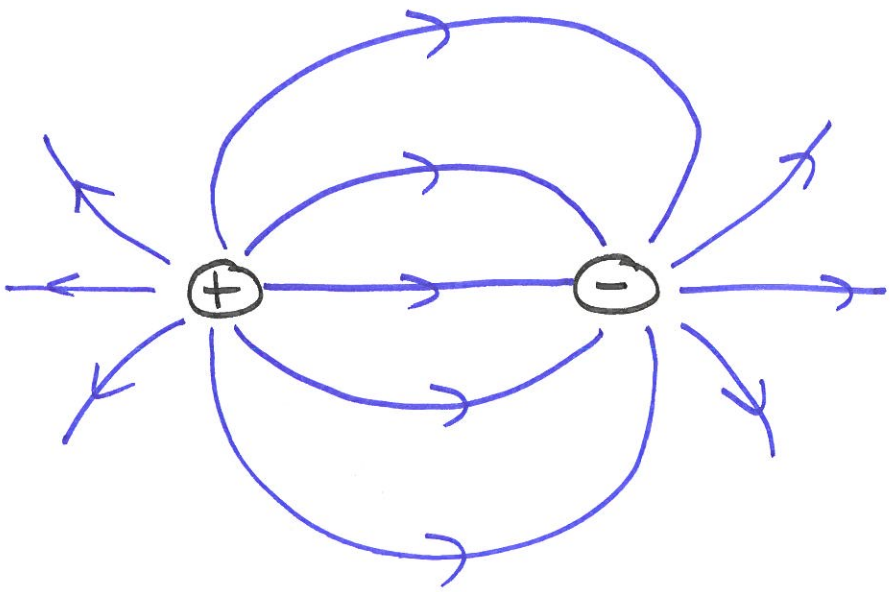
3/ Number of lines entering / leaving a charge is proportional to Q.

4/ Line density $\propto$ magnitude of field

5/ Far away always look like single charge source

6/ Lines don't cross (can't have 2 directions for the force)

eg



NB at large r all the field lines have reconnected & $\vec{E} = 0$ (density of lines $\to 0$)

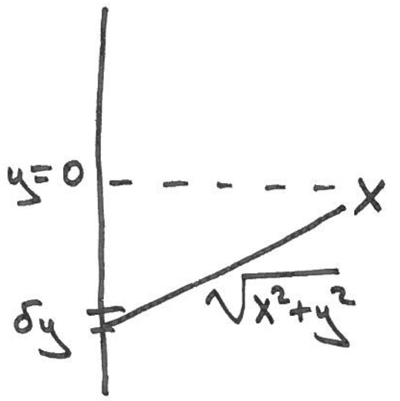
Electric Fields From Multiple Charges

Coulomb forces from many charges vectorially add.
So therefore do $\vec{E}$ contributions

$$\vec{E}_{TOT} = \sum_i \vec{E}_i$$

Note: a charge never experiences it's own field!

eg <u>Infinite Charged Wire</u>, $\lambda\, Cm^{-1}$



What is the $\vec{E}$ field at x?

Erm... we know the $\vec{E}$ field due to a point charge

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{\underline{r}}$$

So... divide wire into point charges & add up contributions

$$\delta\vec{E} = \frac{\lambda\,\delta y}{4\pi\varepsilon_0 (x^2+y^2)^{3/2}}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Now we can sum by $\int_{-\infty}^{\infty} dy \ldots$

<u>Result:</u> $$\int \frac{x}{(x^2+y^2)^{3/2}}\,dy = \frac{y}{x(x^2+y^2)^{1/2}} + c$$

Check: $\frac{d}{dy}\left[\frac{y}{x(x^2+y^2)^{1/2}}+c\right] = \frac{1}{x(x^2+y^2)^{1/2}} - \frac{1/2\, y}{x(x^2+y^2)^{3/2}} \cdot 2y$

$$= \frac{1}{(x^2+y^2)^{3/2}}\left(\frac{x^2+y^2}{x} - \frac{y^2}{x}\right)$$

$$= x/(x^2+y^2)^{3/2} \quad \checkmark$$

Returning to the problem... component by component:

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{x\,dy}{(x^2+y^2)^{3/2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\frac{y}{x(x^2+y^2)^{1/2}}\right]_{-\infty}^{\infty}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\frac{2}{x}$$

$$E_y = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{-y}{(x^2+y^2)^{3/2}}\,dy$$

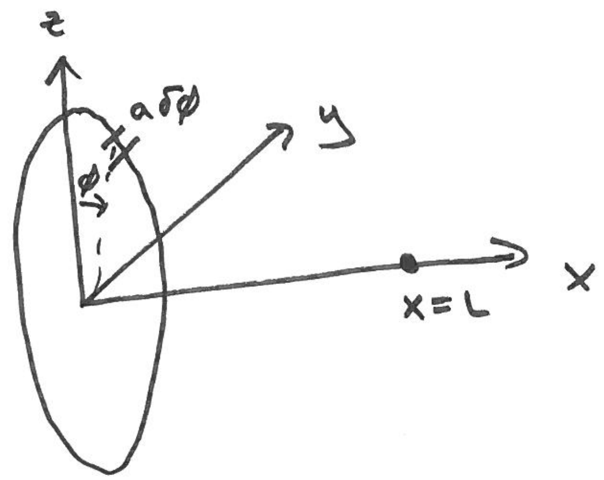
$$= \frac{\lambda}{4\pi\epsilon_0}\left[\frac{-1}{(x^2+y^2)^{1/2}}\right]_{-\infty}^{\infty}$$ using another "standard integral"

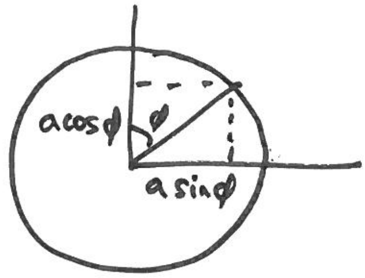
$$= 0$$

Generally we have (by symmetry)

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\,\hat{r}$$

eg charged ring, radius a, charge Q





We will find $\vec{E}$ at $x = L$ on line ⊥ to the ring through it's centre.

Total charge Q $\Rightarrow$ $\lambda = Q/2\pi a$

Consider a single arc of length $a\delta\phi$

$$\delta\vec{E} = \frac{\delta q}{4\pi\varepsilon_0 (L^2+a^2)^{3/2}} \begin{pmatrix} L \\ -a\sin\phi \\ -a\cos\phi \end{pmatrix} \qquad \left(\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^3}\vec{r}\right)$$

$$= \frac{Q/2\pi a \cdot \delta\phi\, a}{4\pi\varepsilon_0 (L^2+a^2)^{3/2}} \begin{pmatrix} L \\ -a\sin\phi \\ -a\cos\phi \end{pmatrix}$$

Now sum over all arcs

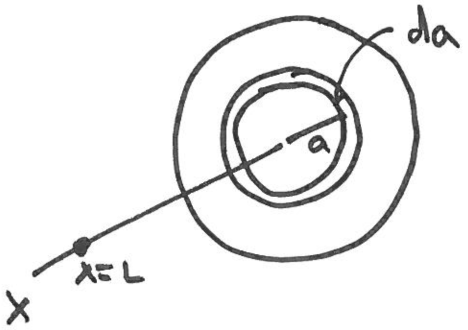
$$\vec{E} = \int_0^{2\pi} \frac{Q/2\pi}{4\pi\varepsilon_0 (L^2+a^2)^{3/2}} \begin{pmatrix} L \\ -a\sin\phi \\ -a\cos\phi \end{pmatrix} d\phi$$

$$= \frac{Q/2\pi}{4\pi\varepsilon_0 (L^2+a^2)^{3/2}} \begin{pmatrix} L \cdot 2\pi \\ a\,[\cos\phi]_0^{2\pi} \\ -a\,[\sin\phi]_0^{2\pi} \end{pmatrix} \begin{matrix} \\ \leftarrow 0 \\ \leftarrow 0 \end{matrix}$$

$$= \frac{QL}{4\pi\varepsilon_0 (L^2+a^2)^{3/2}}\, \underline{\hat{i}}$$

eg charged disc, radius R, charge density σ

We can use the previous result for a ring – now consider building the disc from rings of thickness δa



$$\delta \vec{E}_{\substack{at\\x=L}} = \frac{\overbrace{\sigma \; 2\pi a \, \delta a}^{\delta q}}{4\pi\epsilon_0 \, (L^2+a^2)^{3/2}} \; L \; \hat{\underline{i}}$$

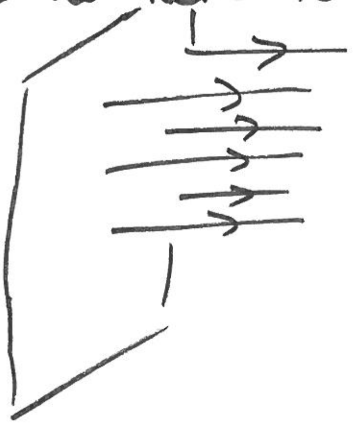
$$\vec{E}_{\substack{at\\x=L}} = \frac{\sigma L}{2\epsilon_0} \underbrace{\int_0^R \frac{a \, da}{(L^2+a^2)^{3/2}}}_{\left[\frac{-1}{(L^2+a^2)^{1/2}}\right]_0^R} \; \hat{\underline{i}}$$

$$\vec{E}_{\substack{at\\x=L}} = \frac{\sigma L}{2\epsilon_0} \left(\frac{1}{L} - \frac{1}{(L^2+R^2)^{1/2}} \right) \hat{\underline{i}}$$

For an infinite plane sheet $R \Rightarrow \infty$

$$\vec{E}_{\substack{\infty\\sheet}} = \frac{\sigma}{2\epsilon_0} \; \hat{\underline{i}}$$

Note the field doesn't fall off – the field lines have nowhere to spread out ... this is an artefact



of assuming an infinite plane ... but true near a large plane ...