VII POTENTIAL

Potential Energy

Consider the energy needed to bring two +ve charges together

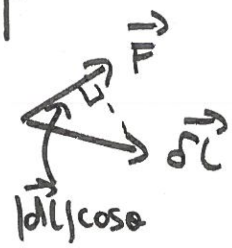
$+Q \qquad \qquad \xleftarrow{+q}$

Work done by field on particle = Force × distance moved in direction of force

$$W = \int \vec{F}.d\vec{l}$$

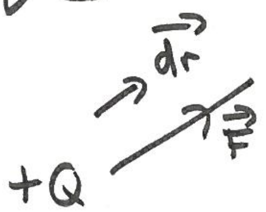
$\vec{\delta l}$ is a short vector along the path

sum over all pieces along the path



We break path up in case $\vec{F}$ varies along path or path is curved.

eg Moving q from ∞ towards Q on radial path



$$\vec{F}.\vec{dr} = |F||\vec{dr}|\underbrace{\cos 0^\circ}_{1}$$

$$W = \int_{\infty}^{r} \frac{Qq}{4\pi\epsilon_0 r^2}\,dr = \frac{-Qq}{4\pi\epsilon_0 r}$$

The minus sign is because the particle has gained potential energy – the field has lost energy. An external agent would give energy to q.

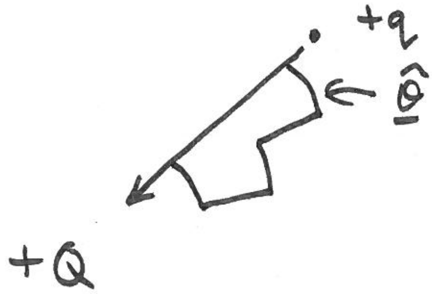
The change in energy of the particle is

$$\Delta U = U_f - U_i = -W$$
$$= -\int_i^f \vec{F} . \vec{dl}$$

The usual convention is that a particle at ∞ has zero energy.

The energy needed to move a particle to some point is <u>independent of the path</u> it takes.

Any path can be split into infintessimal motions in $\hat{\underline{r}}$ or $\hat{\underline{\theta}}$



pieces of motion are perpendicular to $\vec{F}$ & don't contribute to ΔU

The energy change along any $\hat{\underline{r}}$ piece of path is independent of θ

$\Rightarrow$ energy change is same for any paths with same initial & final points.

Potential

we now define a property of a $\vec{E}$ field, potential $\phi(x)$

" the energy an ~~e~~external agent must provide to move a unit charge to the point $\vec{r}$ from ∞ "

(this is − work done by the field)

$$\phi_f = \frac{U_f - U_\infty}{q}$$

$$= -W/q$$

$$\phi(f) = -1/q \int_\infty^f \vec{F} \cdot d\vec{l}$$

$$\phi(f) = -\int_\infty^f \vec{E} \cdot d\vec{l}$$ Unit: JC^{-1}

Note: because the path doesn't matter this a single number at each point f.

Note: we assume bringing in the test charge does not move the other charges & change $\vec{E}$ - it may have to be very small for this to be true.

eg For $\vec{E}$ of a point charge ϕ only depends on radial distance from the source charge

$$\phi(r) = \frac{q}{4\pi\epsilon_0 r}$$

eg if we have multiple charges the net potential is just the sum of the individual contributions:

$$\phi_{TOT} = -\int_{\infty}^{f} \vec{E} \cdot d\vec{l}$$

$$= -\int_{\infty}^{f} \sum_i \vec{E}_i \cdot d\vec{l}$$

$$= -\int_{\infty}^{f} \vec{E}_1 \cdot d\vec{l} - \int_{\infty}^{f} \vec{E}_2 \cdot d\vec{l} - \ldots$$

all independent of path

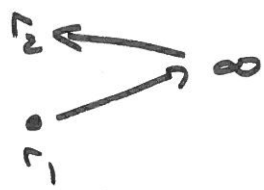
ie $\phi_{TOT} = \sum_{\text{charges}} \dfrac{q_i}{4\pi\epsilon_0 r_i}$

Potential Difference

The potential difference between two points is called the voltage, V

$$V = \Delta\phi = -\int_{\infty}^{r_2} \vec{E} \cdot d\vec{l} + \int_{r_1}^{\infty} \vec{E} \cdot d\vec{l}$$

$$V = -\int_{r_1}^{r_2} \vec{E} \cdot d\vec{l}$$



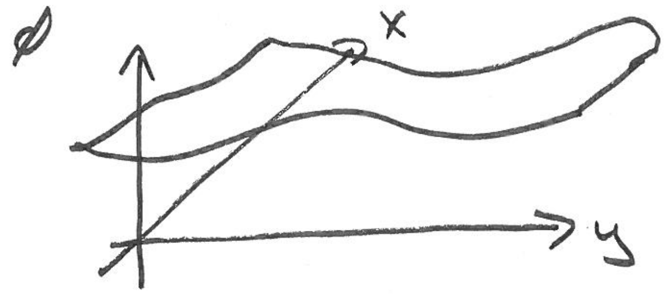
Note if a charge "moves through a potential difference" its change in energy is just qV.

eg if e^- moves through 1V $\quad \Delta U = 1.6 \times 10^{-19}\,J$

"electron volt" $\quad \equiv 1\,eV$

Relation Between Potential & Force

Potential is just a single number at each point in space to be compared with 3 for $\vec{E}$. We can recover $\vec{E}$ from ϕ:



For small local motion

$$\delta\phi = -\vec{E}\cdot\vec{dl}$$

so if we move in $\hat{x}$ by δx $\quad E_x \delta x = -\delta\phi$

$$E_x = -\frac{d\phi}{dx}$$

" " $\hat{y}$ $\quad E_y = -\frac{d\phi}{dy}$

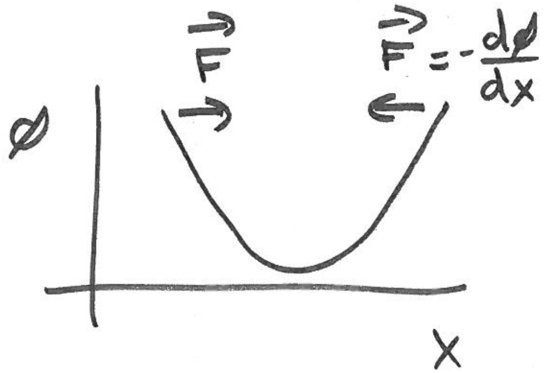
" " $\hat{z}$ $\quad E_z = -d\phi/dz$

We write $\boxed{\vec{E} = -\vec{\nabla}\phi}$

where $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

it's called "grad", "del" or "nabla"

ϕ is also a good concept because +ve charges "roll" "down hill".



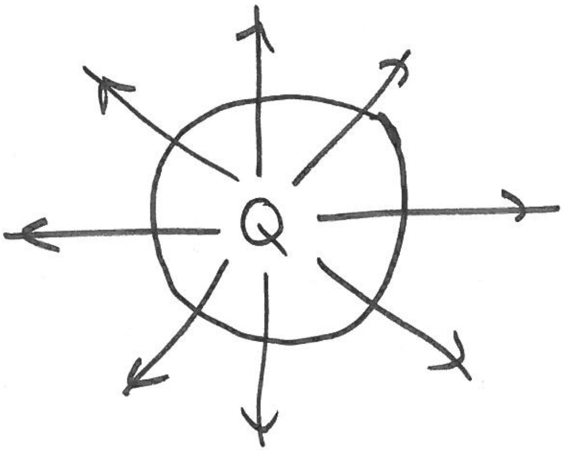
Watch out! -ve charges "roll" "up hill".

Equipotentials

Lines along which ϕ does not change are directions in which there is no force

$$\vec{E} = -\vec{\nabla}\phi = 0$$

eg

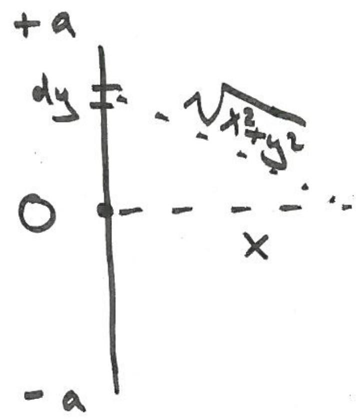


Equipotentials are always at 90° to $\vec{E}$ lines

so $\vec{E} \cdot \vec{\delta l} = 0$

eg the surface of a conductor is an equipotential.

EG POTENTIAL DUE TO A WIRE OF LENGTH $2a$ & CHARGE DENSITY λ



As usual we split wire into δl pieces, treat them as point charges, & sum.

$$\phi = \sum \frac{\delta q}{4\pi\epsilon_0 r}$$

$$= \sum \frac{\lambda \delta y}{4\pi\epsilon_0 \sqrt{x^2+y^2}}$$

$$\phi = \int_{-a}^{a} \frac{\lambda}{4\pi\epsilon_0} \frac{dy}{\sqrt{x^2+y^2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \left[\ln\left(2\left(y + \sqrt{x^2+y^2}\right)\right) \right]_{-a}^{a}$$

differentiate if you want to check!

$$= \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{\sqrt{x^2+a^2} + a}{\sqrt{x^2+a^2} - a} \right]$$

$\ln A - \ln B = \ln A/B$

We can take the $a \to \infty$ limit

$$\phi = \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{2a}{a(1+x^2/a^2)^{1/2} - a} \right]$$

Use the binomial expansion $(1+\delta)^n = 1 + n\delta + \ldots$

$$\phi = \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{2a}{a - a + \frac{1}{2} x^2/a + \ldots} \right]$$

$$\phi = \frac{\lambda}{4\pi\epsilon_0} \ln \frac{4a^2}{x^2}$$

$$= \frac{-\lambda}{4\pi\epsilon_0} \ln x^2 + \text{const}$$

$$= \frac{-\lambda}{2\pi\epsilon_0} \ln x$$

Now $\vec{E} = -\vec{\nabla}\phi = \frac{\lambda}{2\pi\epsilon_0 x} \hat{\underline{x}}$

This matches our previous result for $\vec{E}$ from an ∞ charged wire!

The opposite limit is $x \gg a$ &

$$\phi = \frac{\lambda}{4\pi\epsilon_0} \ln\left[\frac{x+a}{x-a}\right]$$

$$\vec{E} = -\vec{\nabla}\phi = \frac{-\lambda}{4\pi\epsilon_0} \underbrace{\left(\frac{1}{x+a} - \frac{1}{x-a}\right)}_{\frac{x-a-(x+a)}{x^2-a^2}} \hat{\underline{x}}$$

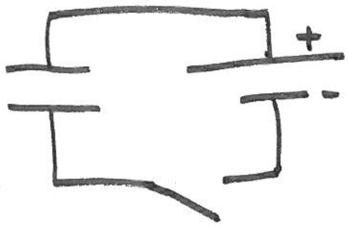
$$= \frac{\lambda}{4\pi\epsilon_0} \frac{2a}{x^2} \hat{\underline{x}}$$

$$= \frac{\overbrace{2\lambda a}^{Q}}{4\pi\epsilon_0} \frac{1}{x^2} \hat{\underline{x}}$$

We recover a point-charge like result!

VIII Capacitors

We can use two parallel plates to store energy by moving charges from one to other

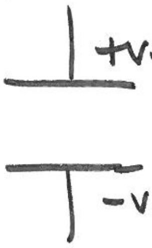


battery generates p.d.

Close switch & +ve charges move from + terminal towards −

Charge accumulates on capacitor plates

Disconnect battery

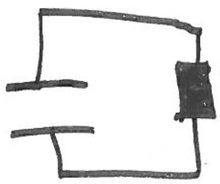


+ve charge

−ve charge

There's energy stored here because charges want to move together.

Reconnect plates & a current flows heating a resistor



A capacitor is a very simple battery!

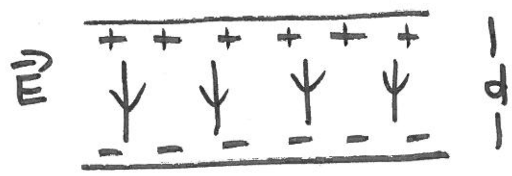
Potential difference (V) between plates $\propto$ Q stored

$$V = \frac{Q}{C} \quad , \quad C = Q/V$$

↑ Capacitance $C^2 J^{-1}$ "Farad"

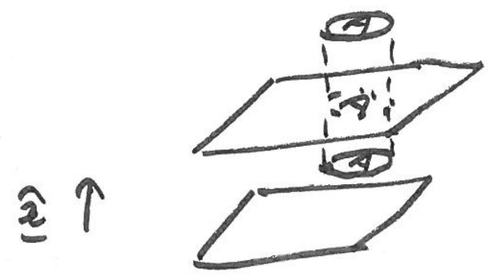
"charge stored / unit p.d."

EG Parallel Plates



the $\vec{E}$ field here is zero... we saw $\vec{E}$ doesn't fall off away from ∞ plate ... so out here sum of + & - plate $\to 0$.

We'll assume the plates are large & ignore end effects.... first compute $\vec{E}$



Symmetry: $\vec{E} = E(z)\,\hat{z}$

Choose Gaussian surface shown

$$\int_{GS} \vec{E} \cdot d\vec{A} = E(z)\,A$$

Gauss' Law: $E(z)\,A = Q/\epsilon_0 = \dfrac{\sigma A}{\epsilon_0}$

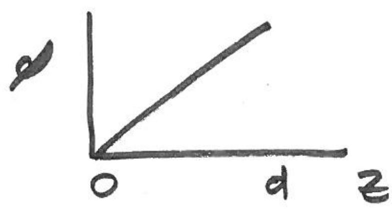
$$\Rightarrow \vec{E} = -\frac{\sigma}{\epsilon_0}\,\hat{z}$$

Next compute potential difference:

$$\Delta\phi = -\int_0^d \vec{E} \cdot d\vec{z}$$

$\vec{E}$ & $d\vec{z}$ opposite
$\cos\theta = -1$

$$= |\vec{E}|\,d$$



$$C = Q/V = \frac{\sigma A_{plate}}{|\vec{E}|\,d}$$

$$= \frac{\sigma A_{plate}}{\sigma/\epsilon_0 \; d}$$

$$= \epsilon_0 A_{plate} / d$$

$$C = \frac{\epsilon_0 A_{plate}}{d}$$

We can store more Q for a given V if A_{plate} is bigger or d is smaller.

It is also beneficial to put a <u>dielectric</u> material between the plates

+Q

− − − −

medium polarizes

+ + + +

−Q

$\vec{E}$ is reduced by polarization

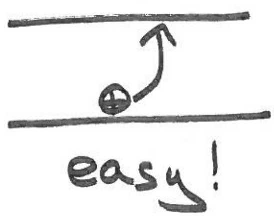
$$|\vec{E}| = \frac{\sigma}{\epsilon_0 \epsilon}$$

relative permittivity

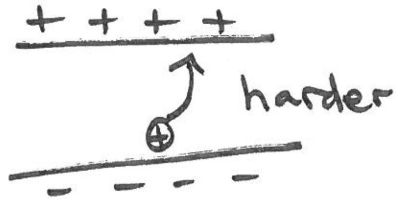
$$C = \frac{\epsilon \epsilon_0 A_{plate}}{d}$$

eg		ϵ
	Air at stp	1.00058
	Polythene	2.3
	Ethanol	26

Charging A Capacitor



... later



energy needed to move a δq $= \delta q \, V(q)$

↑ V grows as move more q!

To get the total energy sum up the bits

$$U = \int_i^f V \, dq$$

$$= \int_0^Q \frac{q}{C} \, dq$$

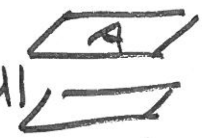
$$= \frac{1}{2} \, Q^2/C$$

$$\left(= \tfrac{1}{2} QV = \tfrac{1}{2} CV^2 \quad \text{using } C = Q/V \right)$$

Where is the energy you put in? Stored in the $\vec{E}$ field...

For // plates $\quad C = \epsilon_0 A/d$

Energy/unit volume $= \tfrac{1}{2} CV^2 / Ad$

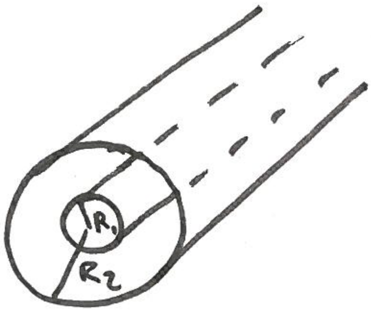


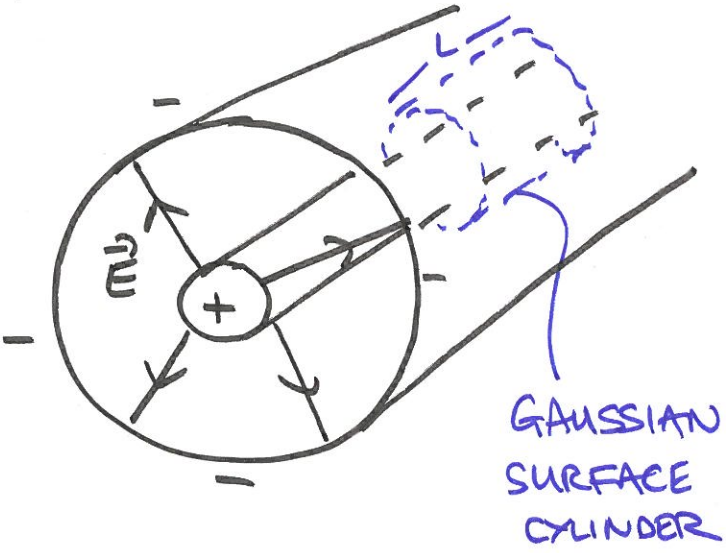
$$= \tfrac{1}{2} \left(\frac{\epsilon_0 A}{d} \right) (Ed)^2 / Ad$$

this is the general expression for energy in $\vec{E}$ field.

$$= \tfrac{1}{2} \epsilon_0 E^2$$

EG Cylinderical Capacitor / Co-axial Cable





By sym: $\vec{E} = E(r)\hat{r}$

- **FIND $\vec{E}$:**

$$\int \vec{E} \cdot d\vec{A} = |\vec{E}|\, 2\pi r l = \frac{q}{\epsilon_0} = \frac{2\pi R_1 L \sigma}{\epsilon_0}$$

$$\vec{E} = \frac{R_1 \sigma}{r \epsilon_0} \hat{r}$$

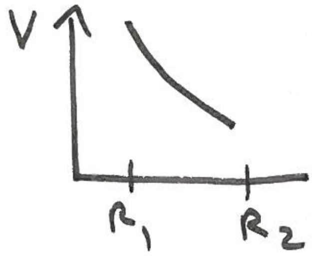
- **FIND V:**

$$\Delta \phi = -\int_{R_1}^{R_2} \vec{E} \cdot d\vec{r}$$

radial path from R_1 to R_2

$$= -\int_{R_1}^{R_2} \frac{R_1 \sigma}{\epsilon_0} \frac{dr}{r}$$

$$= -\frac{R_1 \sigma}{\epsilon_0} \ln R_2/R_1$$



- **FIND C:**

$$C = Q/V = \frac{\sigma\, 2\pi R_1 L_{TOT}}{R_1 \sigma/\epsilon_0 \ln R_2/R_1}$$

$$= \frac{2\pi \epsilon_0 L_{TOT}}{\ln R_2/R_1}$$

Engineers use co-axial cables to send TV signals by sending waves through the $\vec{E}$ field.