XIV MAXWELL'S EQUATIONS

$$\int_{\substack{\text{closed}\\\text{surface}}} \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$ Gauss' Law

$$\int_{\substack{\text{closed}\\\text{surface}}} \vec{B}\cdot d\vec{A} = 0$$ no magnetic charges

$\int \vec{B}.d\vec{A}$

$$\int_{\substack{\text{closed}\\\text{loop}}} \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$ Faraday's law

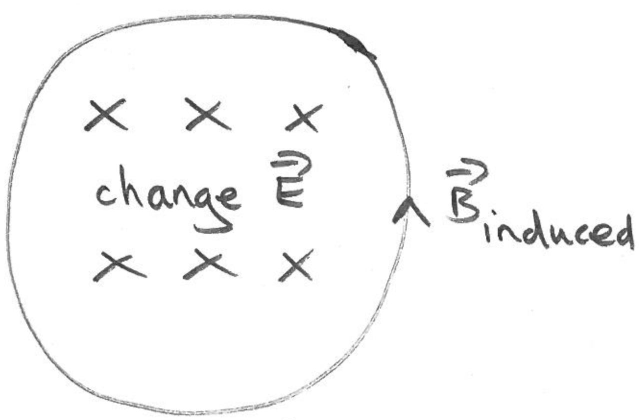
$$\int_{\substack{\text{closed}\\\text{loop}}} \vec{B}\cdot d\vec{l} = \mu_0 I$$ Ampère's Law

There is a lack of symmetry:

(a) no magnetic charges / currents (ho hum!)

(b) $\int \vec{B}.d\vec{l} = \mu_0 I +$ " $\frac{d\Phi_E}{dt}$ " ?

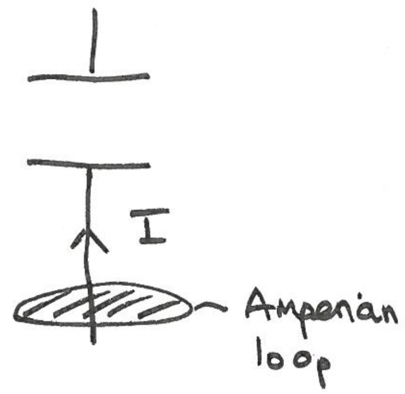
Is it true that a changing $\vec{E}$ field generates a circular $\vec{B}$ field?



?

An argument: Ampère's Law needs work in time dependent problems...

eg <u>charging a capacitor</u>

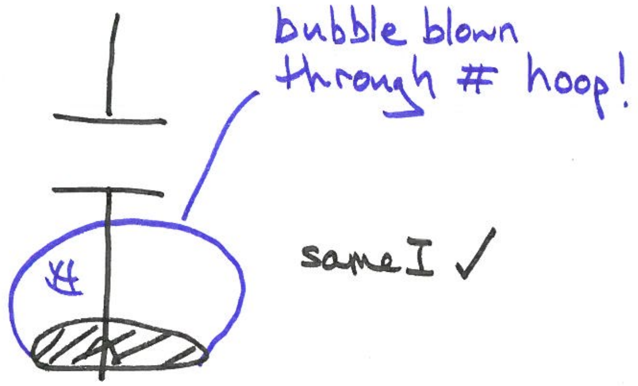


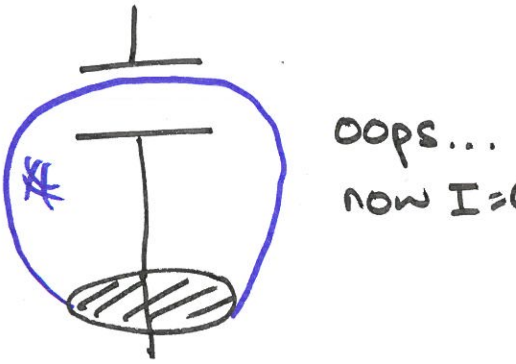
Ampère's law says that

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

what do we mean by I_{enc} for the simple loop shown?

A sensible answer might be $I = \int \vec{J} \cdot d\vec{A}$ where the integral is for any surface bounded by the loop





We're in trouble because I doesn't flow between the plates.

There is something that exists between the plates & equals I.... apply Gauss' Law to the volume in right hand picture above ($\vec{E}$ exists only between the plates)

$$\int_{\substack{\text{closed} \\ \text{surface}}} \vec{E} \cdot d\vec{A} = Q/\epsilon_0$$

take a time derivative

$$\int_{cs} \frac{d\vec{E}}{dt} \cdot d\vec{A} = I/\epsilon_0$$

so we can think of the growing $\vec{E}$ as giving an effective "displacement" current between the plates.

$$I_{dis} = \epsilon_0 \int \frac{d\vec{E}}{dt} \cdot d\vec{A}$$

$$= \epsilon_0 \frac{d\Phi_E}{dt}$$

For this to contribute to the Maxwell Eq'ns we need

$$\int_{\substack{\text{closed}\\\text{loop}}} \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

This is indeed the correct 4TH ME.

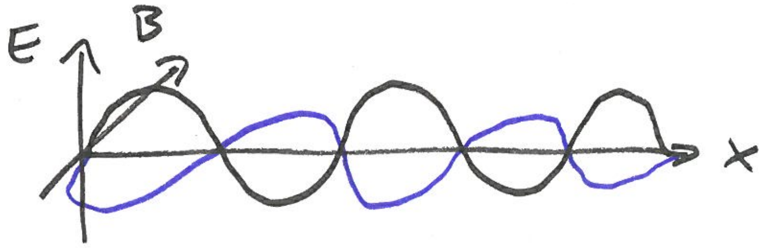
Maxwell's Equations in Vacuum

In the absence of charges we have

$$\int_{cs} \vec{E} \cdot d\vec{A} = 0 \qquad \int_{cs} \vec{B} \cdot d\vec{A} = 0$$

$$\int_{cl} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \qquad \int_{cl} \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

Remarkably these can have non-zero solutions. Waves where a changing $\vec{B}$ generates $\vec{E}$ & vice versa



It turns out these waves move at speed

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \ ms^{-1}$$

This is light! Those continuing will learn more in year 2.

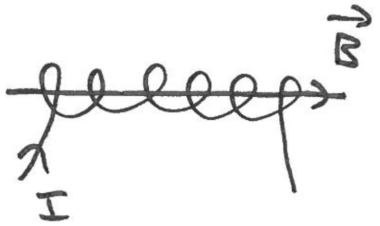
Note: the existence of light in the absence of charges means $\vec{E}$ & $\vec{B}$ are real objects not just a notation to compute forces on charges!

Future: in fact EM contains strong hints of Relativity (3rd year)... they contain a hidden symmetry (gauge invariance) that "explains" why nature chooses forces of this sort....

XV INDUCTORS

SELF INDUCTANCE

A current carrying coil generates $\vec{B}$ $(\propto I)$, the flux of which with itself is non-zero



$$\Phi_B = L I$$

↑ coefficient of self inductance (unit: Henry)

Faraday's law : $\quad V = -\dfrac{d\Phi_B}{dt}$

$$V = -L\, dI/dt$$

If I increases get a "back emf" that opposes change.... if I decreases get emf that supports I.

eg <u>self inductance of solenoid</u>

We've shown $\quad B = \mu_0 n I$

$$\Phi = \mu_0 n I \times \underbrace{\pi r^2}_{\text{x-section area}} \times n l$$

↖ total number of turns

$$\Rightarrow L = \frac{\Phi}{I} = \mu_0 \pi r^2 n^2 l$$

Energy stored in $\vec{B}$

We can use this result to find out how much energy is needed to create $\vec{B}$. While I increases in a circuit from 0 to I the power expended makes $\vec{B}$

$$\text{Power} = VI$$

$$= L\frac{dI}{dt}\cdot I$$

$$\text{Work done} = \int_0^t LI\frac{dI}{dt}\,dt$$

$$= \int_0^I LI\,dI$$

$$= \tfrac{1}{2}LI^2$$

For the solenoid this stored energy $= \frac{1}{2}\mu_0 \pi r^2 n^2 l I^2$

volume

$$\text{energy/volume} = \tfrac{1}{2}\mu_0 n^2 I^2$$

$$= \tfrac{1}{2}B^2/\mu_0$$

This in fact is the general expression for the energy in a $\vec{B}$ field.

MUTUAL INDUCTANCE

Two neighbouring solenoids also generate mutual Φ_B

$$\Phi_2 = M_{12} I_1$$

$$\Phi_1 = M_{21} I_2$$

coeff of mutual inductance

Such systems are used in transformers.
The voltage in the second coil differs from that inducing it in the first by the ratio of the number of turns.

AC CIRCUIT THEORY

Inductors only play a role in AC circuits where the current changes.

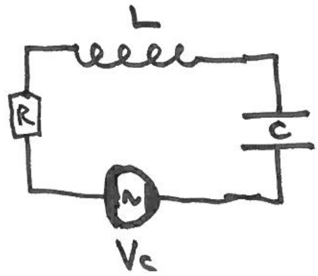
Kirchoff's Laws: (i) charge is conserved

$$\sum I = 0 \quad \text{at any point}$$

(ii) $\int \vec{E}.\vec{dl} = 0$ (assumes no $\frac{d\Phi_B}{dt}$ for circuit)

$$\sum_{\text{loop}} V = 0$$

eg LCR Circuit

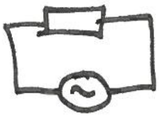


$$V_R + V_L + V_c = V_s$$

$$IR + L\frac{dI}{dt} + Q/C - V_s = 0$$

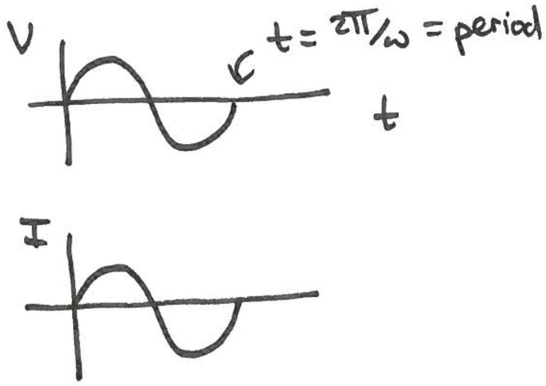
note opposing signs

① Pure R



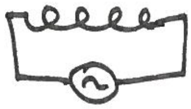
The AC generator imposes $V = V_0 \sin \omega t$

$$I = \frac{V}{R} = \frac{V_0}{R} \sin \omega t$$



V & I are in phase.

② Pure L



$$L \frac{dI}{dt} = V_0 \sin \omega t$$

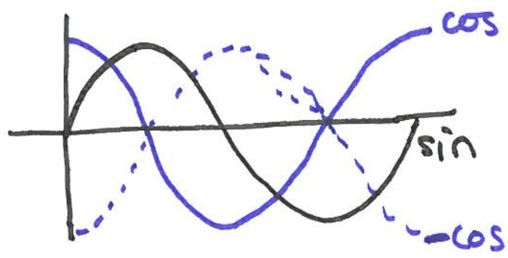
$$\int dI = \int \frac{V_0}{L} \sin \omega t \, dt$$

$$I = -\frac{V_0}{\omega L} \cos \omega t + \text{const}$$

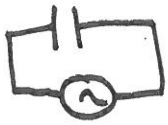
zero since no DC voltage to drive constant current

$$I = \frac{V_0}{\omega L} \sin(\omega t - \pi/2)$$

The current lags by $\pi/2$ behind the voltage.



③ Pure C



$$Q = CV$$

$$I = C \frac{d}{dt} V_0 \sin\omega t$$

$$= C V_0 \omega \cos\omega t$$

$$= C V_0 \omega \sin(\omega t + \pi/2)$$

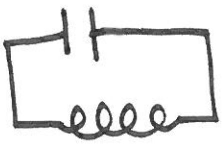
I leads V by $\pi/2$

capacitor: I leads V

Mnemonic: C I V I L

inductor: V leads I

④ LC Circuit



dI/dt

$$L \frac{d^2q}{dt^2} + \frac{q}{c} = 0$$

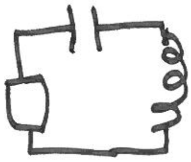
$$\frac{d^2q}{dt^2} = -\frac{1}{LC} q \qquad \text{c.f.} \quad \ddot{x} = -\omega^2 x$$

This is simple Harmonic Motion ... the charge leaves C but L then "reloads" it onto C.

solution: $q = Q_0 \cos(\omega t + \delta)$

with $\omega^2 = 1/LC$.

⑤ LCR discharge

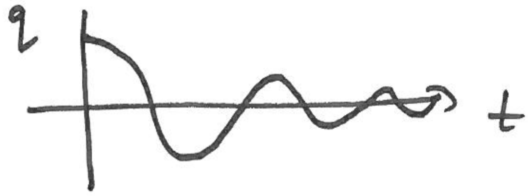


$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

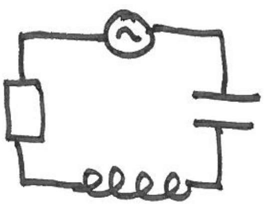
This is equivalent to damped SHM. The solution is

$$q = Q_0 e^{-Rt/2L} \cos \omega t$$

The resistance removes energy from the circuit (as heat) weakening the oscillations



⑥ Driven LCR



The current is the same throughout the circuit

$$I = I_0 \cos \omega t$$

The voltage drops are (CIVIL)

$$V_R = I_0 R \cos \omega t$$

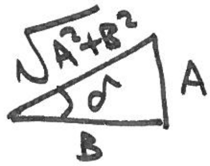
$$V_L = I_0 L \omega \cos(\omega t + \pi/2) = I_0 L \omega \sin \omega t$$

$$V_C = \frac{I_0}{\omega C} \cos(\omega t - \pi/2) = -\frac{I_0}{\omega C} \sin \omega t$$

The sum of dropped voltages equal V_s

$$V_s = \underbrace{I_o R}_{A} \cos\omega t + \underbrace{I_o \left(L\omega - \frac{1}{\omega C}\right)}_{B} \sin\omega t$$

$$= \sqrt{A^2+B^2} \left[\frac{A}{\sqrt{A^2+B^2}} \cos\omega t + \frac{B}{\sqrt{A^2+B^2}} \sin\omega t \right]$$

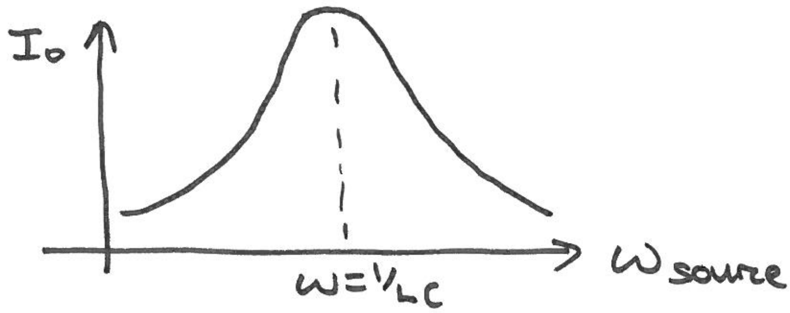
Introduce triangle  $\delta = \text{ArcTan}\, A/B$

$$V_s = \sqrt{A^2+B^2} \left[\sin\delta \cos\omega t + \cos\delta \sin\omega t \right]$$

$$= \underbrace{\sqrt{A^2+B^2}}_{V_o} \sin(\omega t + \delta)$$

$$V_o^2 = (I_o R)^2 + I_o^2 (\omega L - 1/\omega C)^2$$

$$\Rightarrow \quad I_o = \frac{V_o}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$



There is a "resonance" when the driving frequency matches the SHM frequency of the LC circuit.