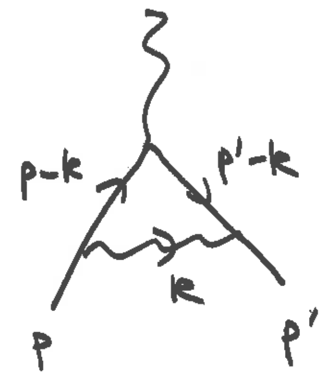
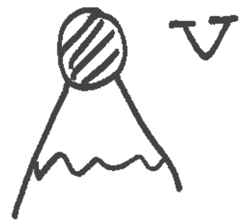
IR DIVERGENCES IN QED

There are some further divergences in the loop diagrams we have studied which come from $k \to 0$ in loops.

eg



or



$$\int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2} (-ie\gamma^\mu) \frac{i}{\not p - \not k - m} \, V \, \frac{i}{\not p' - \not k - m} (-ie\gamma_\mu)$$

In the "eikonal approximation" take $k \to 0$

& use $$\frac{1}{(p-k)^2 - m^2} = \frac{1}{-2p.k}$$ since $k^2 \simeq 0$, $p^2 = p'^2 = m^2$

$$-ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu \not p \, V \, \not p' \gamma_\mu}{k^2 (2p.k)(2p'.k)}$$

The k integral

$$\int_0^\Lambda k^2 dk^2 \frac{1}{k^2} \frac{1}{p.k} \frac{1}{p'.k} \sim \ln \Lambda/0 \to \infty$$

Note the divergence is p & p' dependent so can't be absorbed into parameters.

Let's keep going more rigorously

<u>Numerator</u>: sandwich between the external spinors

$$N = \bar{u}(p)\, \gamma^\mu (\not{p}+m)\, V\, (\not{p}'+m)\, \gamma_\mu\, u(p')$$

We can use $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ & $(\not{p}-m)u = 0$
(Dirac eq'n)

$$\bar{u}(p)\, \gamma^\mu (\not{p}+m) = \bar{u}(p)\left[-\not{p}\gamma^\mu + 2p^\mu + m\right]$$
$$= 2\bar{u}(p)\, p^\mu$$

$$\Rightarrow N = 4\, p\cdot p' \;\; \bar{u}(p)\, V\, u(p')$$

The full diagram is now:

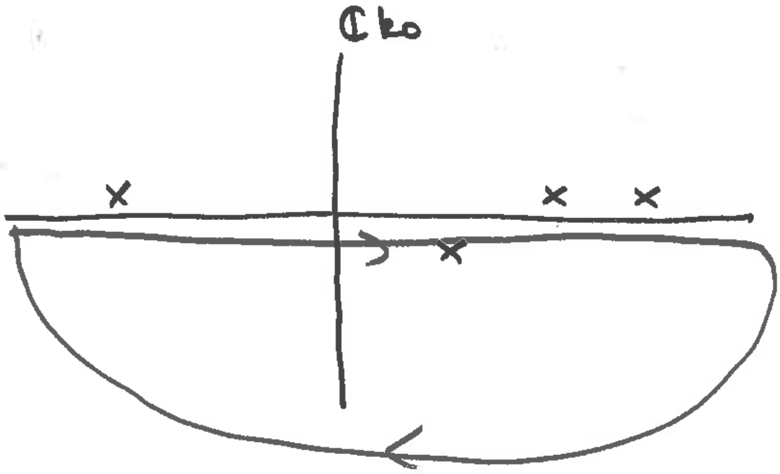
$$-ie^2\, p\cdot p'\, V \int \frac{d^4k}{(2\pi)^4}\; \frac{1}{k^2+i\epsilon}\; \frac{1}{p\cdot k - i\epsilon}\; \frac{1}{p'\cdot k - i\epsilon}$$

We can do the k_0 integral by contour integration

$$-ie\, p\cdot p'\, V \int \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3}\; \frac{1}{(k_0^2 - \mathbf{k}^2 + i\epsilon)\,(p_0 k_0 - \mathbf{p}\cdot\mathbf{k} - i\epsilon)\,(p_0' k_0 - \mathbf{p}'\cdot\mathbf{k} - i\epsilon)}$$

$$\wr$$

$$(k_0 + |\mathbf{k}| - i\epsilon)(k_0 - |\mathbf{k}| + i\epsilon)$$



Choose lower contour

$$2\pi i \; \frac{1}{2|\mathbf{k}|\,(p_0|\mathbf{k}| - \mathbf{p}\cdot\mathbf{k})\,(p_0'|\mathbf{k}| - \mathbf{p}'\cdot\mathbf{k})}$$

$$e^2\, p\cdot p'\, V \int \frac{d^3\underline{k}}{(2\pi)^3}\,\frac{1}{2|\underline{k}|}\,\frac{1}{(p_0|\underline{k}|-\underline{p}\cdot\underline{k})}\,\frac{1}{(p_0'|\underline{k}|-\underline{p}'\cdot\underline{k})}$$

$$= e^2\, p\cdot p'\, V \int_\lambda^E \frac{|\underline{k}|^2\, dk}{(2\pi)^3}\, d\Omega\, \frac{1}{2|\underline{k}|^2}\,\frac{1}{(E-p\cos\theta)}\,\frac{1}{(E'-p'\cos\theta')}$$

$$= e^2\, \frac{p\cdot p'}{2(2\pi)^3}\, V \ln E/\lambda \int d\Omega\, \frac{1}{(E-p\cos\theta)}\,\frac{1}{(E'-p'\cos\theta')}$$

This diverges as $\lambda \to 0$

BLOCH – NORDSIECK THEOREM

This divergence is not present in physical scattering ~~any~~ probabilities. In particular in $|-i\mathcal{M}|^2$ the leading terms are the interference terms between

 &  $\sim e^2$

The claim is this will cancel against an equal divergence from the interference between

 &  $\sim e^2$

As $k \to 0$ these soft photons become undetectable by a detector of given resolution so should be included in the process...

 &  $\sim \frac{e^2}{(2\pi)^3}\, p\cdot p'\, |V|^2 \ln E/\lambda\; I_\Omega$

2 cross terms

So now we look at 

two cross terms

$$\sim 2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2|\underline{k}|} \sum_{\varepsilon} \left[\varepsilon_\mu (-ie\gamma^\mu) \frac{i}{\not p - \not k - m} V \right] \left[\frac{-i}{\not p' - \not k - m} (ie\gamma^\nu) V^\dagger \varepsilon_\nu^* \right]$$

sum over phase space of k

Take eikonal approx $k \to 0$... & use $\sum \varepsilon_\mu \varepsilon_\nu^* = -g_{\mu\nu}$

$$\sim -\int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|}\, e^2\, \frac{\gamma^\mu (\not p + m) V (\not p' + m) \gamma_\mu V^\dagger}{(2\, p\cdot k)\, (2 p'\cdot k)}$$

The numerator is precisely that of before & gives $4\, p\cdot p'\, |V|^2$. Now the photon is on-shell $k_0 = |\underline{k}|$

$$\sim -\int_\lambda^{\Delta E} e^2\, p\cdot p'\, |V|^2\, \frac{|\underline{k}|^2 dk\, d\Omega}{(2\pi)^3 |\underline{k}|^3}\, \frac{1}{(E - p\cos\theta)(E' - p'\cos\theta)}$$

Here ΔE is the detector resolution.

$$\sim -\frac{e^2}{(2\pi)^3}\, p\cdot p'\, |V|^2 \ln \Delta E/\lambda\;\; I_\Omega$$

The two contributions to the probability leave the finite answer

$$-\frac{e^2}{(2\pi)^3}\, p\cdot p'\, |V|^2 \ln \Delta E/E\;\; I_\Omega .$$

DIM REG

You can use dim reg to track IR divergences too

eg $$I = \int \frac{d^dk}{(2\pi)^d} \underset{\alpha}{\frac{1}{k^2}} \underset{\beta}{\frac{1}{k^2 - 2p.k}} \underset{\gamma}{\frac{1}{k^2 - 2p'.k}}$$

Feyn Param:

$$D = k^2 - 2\beta\, p.k - 2\gamma\, p'.k$$

shift $k = k' + \beta p + \gamma p'$

$$D = k'^2 \underbrace{- (\beta+\gamma)^2 m^2 - 2\beta\gamma\, p.p'}_{-A^2}$$

$$I = -i \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+A^2)^3} \, d\beta\, d\gamma$$

Wick rotate

$$= -i \frac{\Gamma(1+\epsilon)}{(4\pi)^{2-\epsilon}\,\Gamma(3)} \int A^{-2-2\epsilon} \, d\beta\, d\gamma$$

$$= \frac{-i}{32\pi^2} \int_0^1 d\beta \int_0^{1-\beta} d\gamma \left((\beta+\gamma)^2 m^2 + 2\beta\gamma\, p.p' \right)^{-1-\epsilon}$$

Change variables $\beta = \rho\bar{\beta}$ $\quad \gamma = \rho(1-\bar{\beta})$

$$\text{Jacobian} = \begin{vmatrix} \rho & -\rho \\ \bar{\beta} & (1-\bar{\beta}) \end{vmatrix} = \rho(1-\bar{\beta}) + \rho\bar{\beta} = \rho$$

$$I = \frac{-i}{32\pi^2} \int_0^1 \rho^{-1-2\epsilon} \, d\rho \int_0^1 d\bar{\beta} \frac{1}{(m^2 + \bar{\beta}(1-\bar{\beta})2p.p')^{1+\epsilon}}$$

$\underset{\sim}{\ } -\frac{1}{2\epsilon}$ which is the divergence.

So interference of  with  &  with  are finite.... likewise

 has an IR divergence

Interference of  with  cancels vs 

In the end this is all related to the fact

 & 

do not have IR divergences – the different processes are different "cuts" of these diagrams.

<u>COLLINEAR DIVERGENCES</u>

The I_Ω can have further divergences

$$I_\Omega \sim \int d\Omega \, \frac{1}{E - p\cos\theta} \, \frac{1}{E' - p'\cos\theta'}$$

If the external states are massless $E = p$ & $E' = p'$ & there are divergences at $\theta = 0$, $\theta' = 0$.

To cancel everything need to compute all possible <u>in</u> & <u>out</u> states.

In dim reg the $\vec{p}$ integral also gives a $\frac{1}{\epsilon}$ pole.