THE CHIRAL ANOMALY

Some global symmetries of classical theories don't survive the quantization process - they are said to be anomalous. The classic example is the chiral flavour anomaly of QCD.

$$\mathcal{L} = i\bar{\psi}_L \not{D} \psi_L + i\bar{\psi}_R \not{D} \psi_R + m\bar{\psi}_L \psi_R$$

For N flavours we expect when $m=0$:

$SU(N_f)_L \otimes SU(N_f)_R$ $\quad \psi_L \to u\psi_L$, $\quad \psi_R \to \hat{u}\psi_R$ $\quad$ } these are OK ✓

$U(1)_L \otimes U(1)_R$ $\quad \psi_L \to e^{i\alpha}\psi_L$, $\quad \psi_R \to e^{i\beta}\psi_R$

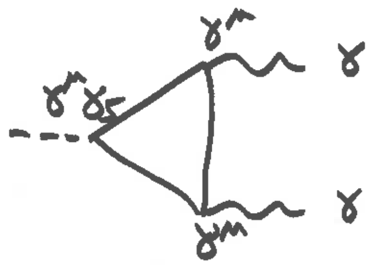
$U(1)_B$ $\quad \psi_L$ & ψ_R transform the same $\alpha = \beta$ ✓

$U(1)_A$ $\quad \alpha = -\beta$ is anomalous.

You can also write it as $\quad \psi \to e^{i\alpha\gamma_5}\psi$

$\gamma_5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ in chiral basis.

The breaking was first seen in the 1-loop diagram



$$\sim \varepsilon_{\sigma\nu\tau\rho} F^{\sigma\nu} F^{\tau\rho} \neq 0$$

If we tried to gauge $U(1)_A$ we'd end up with a "3 photon" veretex forbidden by gauge invariance.

FUJIKAWA'S NON-PERTURBATIVE PROOF

Consider the fermion fields in Z & "Fourier expand"

$$\psi(x) = \sum_n a_n \phi_n(x) \qquad a_n = \int d^4x \, \bar{\phi}_n \psi$$

$$\bar{\psi}(x) = \sum_n \phi_n^\dagger(x) \bar{b}_n \qquad \bar{b}_n = \int d^4x \, \bar{\psi} \phi_n$$

where ϕ_n are eigenvalues of $i\not{D}$

$$i\not{D} \phi_n = \lambda_n \phi_n(x) \qquad \text{ie} \quad \phi_n(x) = u_r e^{-ik.x}$$

$$u_r = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} ..$$

these are orthogonal $\int \bar{\phi}_i \phi_j \, d^4x = \delta_{ij}$

The interesting result comes from looking at the path integral measure which changes under a $U(1)_A$ transform

$$D[\psi] D[\bar{\psi}] \to \prod_n da_n \prod_m d\bar{b}_m$$

Now under $U(1)_A$

$$\psi \to \psi + i\omega \gamma^5 \psi + \ldots$$

$$a_n \to a_n + i\omega \int d^4x \, \bar{\phi}_n \gamma_5 \psi \equiv a_n'$$

$$\bar{b}_n' = \bar{b}_n + i\omega \sum_m \int d^4x \, \bar{\phi}_m \bar{b}_m \gamma^5 \phi_n$$

$\uparrow$ γ_5 & γ_0 anti-commute.

Now for $\prod_n a_n \to \prod_n a_n'$ we get a determinant factor

$$\det \underset{\substack{\text{at small}\\ \omega}}{\simeq} \det \begin{bmatrix} 1+iK_{11} & 0 & \cdots \\ 0 & 1+iK_{22} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

(K_{ij} terms come in squared)

$$K_{ii} = \omega \int \bar{\phi}_n \, \alpha \, \gamma_5 \, \phi_n \, d^4x$$

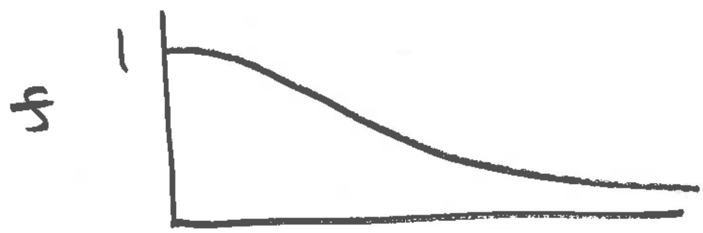
Computing & re-exponentiating we get

$$\det{}^{-1} \simeq e^{-i \sum_n \omega \int \bar{\phi}_n \gamma_5 \phi_n \, d^4x}$$

Since there are an ∞ number of momentum states for n the sum is divergent.

<u>Regulator</u>

We will multiply by a smooth, Lorentz & gauge invariant function of the form $f[(\frac{i}{g}\not{D})^2]$



$f'(\infty) = 0$
$f(\infty) = 0$

Now

$$(\not{D})^2 = \gamma^\mu \gamma^\nu D_\mu D_\nu$$

$$= \tfrac{1}{2} [\gamma^\mu, \gamma^\nu] D_\mu D_\nu + \tfrac{1}{2} \{\gamma^\mu, \gamma^\nu\} D_\mu D_\nu$$

$$= D^2 + \tfrac{1}{2} \gamma^\mu \gamma^\nu [D_\mu, D_\nu]$$

$$= D^2 + \tfrac{i}{2} g F_{\mu\nu} \gamma^\mu \gamma^\nu$$

We can Taylor expand

$$f\left(\left(\frac{i\not{D}}{g}\right)^2\right) = f\left(\frac{-D^2}{g}\right) - f'\left(\frac{-D^2}{g}\right) \frac{i}{2} F_{\mu\nu} \gamma^\mu \gamma^\nu$$

$$+ f''\left(\frac{-D^2}{g}\right) \left(\frac{-1}{8}\right) F_{\mu\nu} \gamma^\mu \gamma^\nu F_{\lambda\sigma} \gamma^\lambda \gamma^\sigma + \dots$$

$$\det{}^{-1} \simeq e^{-i\sum_n \omega \int \bar{\phi}_n^{\dagger}\, \gamma_5\, f\left(\frac{-\not{D}^2}{g}\right) \phi_n \, d^4x}$$

$$\sum_n \uparrow \sum_{r=1}^{4}\int \frac{d^4k}{(2\pi)^4} \qquad \bar{\phi}_n \sim \bar{u}_r e^{-ikx} \qquad \phi_n \sim u_r e^{ikx}$$

Sandwiching the γ-matrices between the u_r projects out the trace

$$\simeq e^{-i\omega \iint \frac{d^4k}{(2\pi)^4} d^4x \ \mathrm{Tr}\,(\gamma_5 f)}$$

But

$$\mathrm{Tr}\,\gamma_5 = 0$$

$$\mathrm{Tr}\,\gamma_5 \gamma^\mu \gamma^\nu = 0$$

$$\mathrm{Tr}\,\gamma_5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma = -4i\,\varepsilon^{\mu\nu\lambda\sigma}$$

$\mathrm{Tr}\,\gamma_5 \cdots = 0$ all higher

So only the third term contributes

$$\det{}^{-1} = e^{-i\omega \int \frac{d^4k}{(2\pi)^4} \left(-\frac{1}{8}\right) F_{\mu\nu} F_{\lambda\sigma} f'' \left(-4i\,\varepsilon^{\mu\nu\lambda\sigma}\right)}$$

Now $\int \frac{d^4k}{(2\pi)^4} f''(-\not{D}^2) \xrightarrow{\text{k shift}} \int \frac{d^4k'}{(2\pi)^4} f''(-k'^2)$

$$\overset{\text{Wick Rotate}}{=} \frac{-i}{16\pi^2} \int k^2\, dk^2\, f''(k^2)$$

$$= \frac{-i}{16\pi^2} \left\{ \left[k^2 f'\right]_0^\infty - \int_0^\infty f'(k^2)\, dk^2 \right\}$$

$$= \frac{-i}{16\pi^2}\left[f(0) - f(\infty)\right] = \frac{-i}{16\pi^2}$$

So when we make the $U(1)_A$ transform the action changes due to the integration measure by

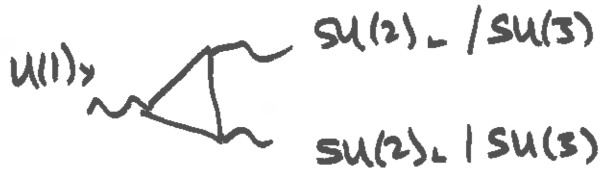
$$\Delta \mathcal{L} = -\frac{\omega}{32\pi^2} F\tilde{F}$$

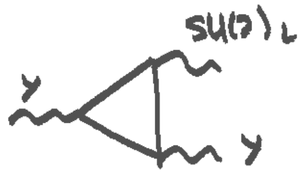
We must also change $D[\bar{\Psi}]$ & so

$$\Delta \mathcal{L} = -\frac{\omega}{16\pi^2} F\tilde{F}$$

<u>CONCLUSION</u>: if the action changes it's not a symmetry.

In the SM hypercharge is chiral & one has to worry about the triangle graphs





These only cancel for the particular choices of Y-charge <u>across</u> a particle generation (u, d, e, ν_e)

See Peskin pg 707.