COMPLEX ANALYSIS

A point on the complex plane is $z = x + iy$

Now imagine some function of z

$$f(z) = u(z) + i v(z)$$

If it is an "analytic function" it has a well defined derivative

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \left\{ \frac{f(z+\Delta z) - f(z)}{\Delta z} \right\}$$

$$= \frac{\partial f}{\partial x} \quad \text{but also} \quad \frac{\partial f}{i\,\partial y}$$

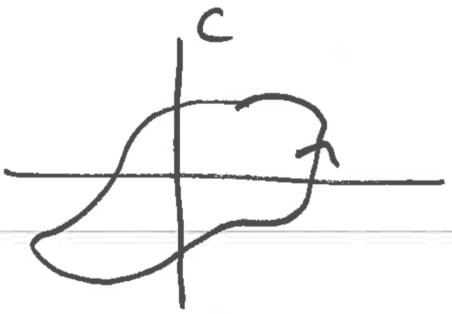
$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \& \quad -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

If this makes sense then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

"Cauchy Riemann eq'ns"

CONTOUR INTEGRATION



Assume f analytic inside contour

$$\oint_c f(z)\,dz = 0$$ Cauchy's Theorem

Proof: $$\oint_c f(z)\,dz = \oint_c (u+iv)(dx+idy)$$

$$= \oint_c \left([u+iv]\,dx - [-iu+v]\,dy\right)$$

$\oint_c \vec{A}\cdot d\vec{l} = \int \vec{\nabla}\times\vec{A}\cdot dS$

$$= \int \left(\frac{\partial}{\partial x}[v-iu] + \frac{\partial}{\partial y}[u+iv] \right) dA$$

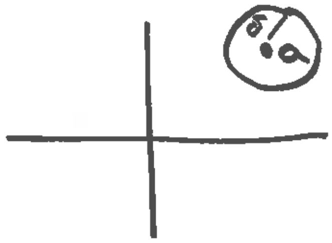
$$= \int (\text{sum of Cauchy-R eq'ns})\, dA$$

$$= 0$$

More general meromorphic fns can be written as (near $z=a$)

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \underbrace{\sum_{n=1}^{\infty} b_n (z-a)^{-n}}_{\uparrow\ \text{"poles"}}$$

Now do a contour integral around $z=a$



$$z = a + \delta e^{i\theta}$$

$$dz = i\delta e^{i\theta}\,\delta\theta$$

$$\oint_c f(z)\,dz = i\delta \int_0^{2\pi} d\theta \left[\sum_{n=0}^{\infty} a_n \delta^n e^{i(n+1)\theta} + \sum_{n=1}^{\infty} b_n \delta^{-n} e^{i(1-n)\theta} \right]$$

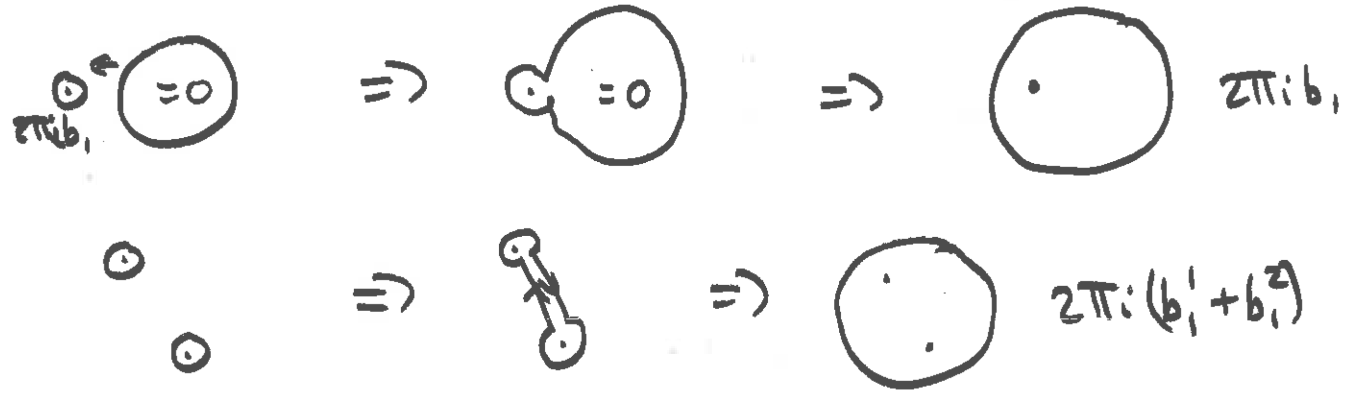
Now $\displaystyle\int_0^{2\pi} e^{im\theta}\, d\theta = 0 \qquad m \neq 0$

$\qquad\qquad\qquad\qquad = 2\pi \qquad m = 0$

$$\Rightarrow \oint_c f(z)\,dz = 2\pi i\, b_1$$

↑ this is the residue of the simple pole $\dfrac{b_1}{z-a}$

Now merge some contours:



Generically

$$\boxed{\oint f(z)\,dz = 2\pi i \sum \text{residues}}$$

$+i\epsilon$ PRESCRIPTION

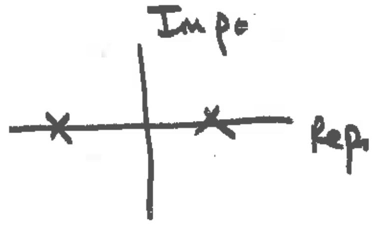
The scalar ~~(gree)~~ Green's function is

$$G_F(x-x') = \frac{1}{(2\pi)^4}\int d^4p \; \frac{1}{p^2-m^2}\, e^{-ip\cdot(x-x')}$$

$$\frac{1}{p^2-m^2} \wr \frac{1}{(p_0+\sqrt{|p|^2+m^2})}\;\frac{1}{(p_0-\sqrt{|p|^2+m^2})}$$

So the p_0 integral is tricky due to the poles

We move to the complex plane

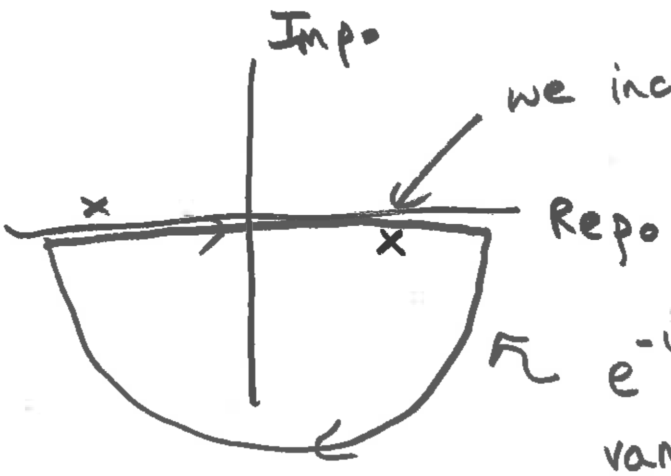


We use the Feynman-Stuckleburg interpretation:

$+E$ states moving forwards in t are pts

$-E$ states " backwards " " $\overline{\text{pts}}$

When $t > t'$ so this is forward propagation in time we must complete the contour in the lower plane



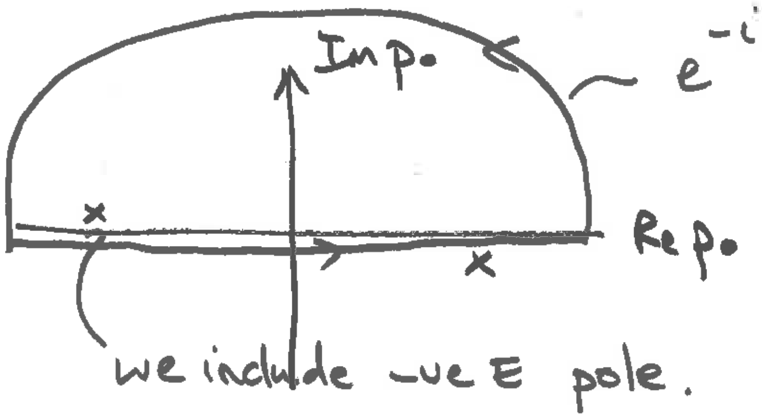
we include the +ve E pole.

$$\approx e^{-ip_0(t-t')} \to e^{-E(t-t')}$$

-ve imag, +ve

which vanishes at $E \to \infty$

When $t < t'$ & we describe propagation back in time we can complete the contour in the top plane & include $-E$ pole



$e^{-ip_0(t-t')} \to e^{+E(t-t')}$ ← -ve which vanishes at $E \Rightarrow \infty$.

Thus

$$G_F(x-x') = \frac{1}{(2\pi)^4} \int d^4p \, \frac{1}{p^2 - m^2 + i\epsilon} \, e^{-ip.(x-x')}$$

WICK ROTATION

The key idea is that one can distort contours as long as no poles enter or leave the contours' interior

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