THE MAGNETIC MOMENT OF THE ELECTRON

The tree level amplitude for e^- A^μ interaction is

$$\mathcal{M} = -i \int J^\mu A_\mu \, d^4x$$

$$e \, \bar{\psi}_f \gamma^\mu \psi_i$$

$$\rightarrow e \, \bar{u}_f \gamma^\mu u_i \, e^{i(p_f - p_i)\cdot x}$$

external lines → spinors ...

There are two sorts of interaction here

$$\bar{u}_f \gamma^\mu u_i = \frac{1}{2m} \bar{u}_f \left[(p_f + p_i)^\mu + i \sigma^{\mu\nu} (p_f - p_i)_\nu \right] u_i$$

$$\sigma^{\mu\nu} = \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$$

anti-commutator

$$-2g^{\mu\nu} + \gamma^\mu \gamma^\nu$$

cancels 1st term

use $\bar{u}_f \not{p}_f = m\bar{u}_f$ or $\not{p}_i u_i = m u_i$

"Gordon Decomposition"

The first term has no spin structure & is like the electric interaction of a scalar $\sim -ie(p_f - p_i)^\mu$

The second term couples different spin components.

Let's look at it in the non-rel limit in a $\vec{B}$ field which is time independent

$$A^\mu \rightarrow (0, \vec{A}(\vec{x}))$$

We can now do the t integral in $\mathcal{M}$

$$\mathcal{M} = -i \, 2\pi \, \delta(E_f - E_i) \int J_\mu A^\mu \, d^3x$$

$\nu = 0$ is zero

only $\vec{A}$ non-zero

$$\int \frac{e}{2m} \bar{u}_f \, i \underbrace{\sigma_{\mu\nu}}_{} (p_f - p_i)^\nu u_i A^\mu \, d^3x$$

$\Rightarrow$ only non-zero when spatial

$$(p_f - p_i)^\nu A^\mu \rightarrow \partial^\nu A^\mu$$

In the non-rel limit spinors become $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

So we're interested in the top components of

$$\frac{1}{2} \gamma^0 [\gamma^i, \gamma^j] \sim \frac{1}{2} \begin{pmatrix} -[\sigma^i, \sigma^j] & 0 \\ \cdots & \cdots \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i \, \epsilon_{ijk} \sigma_k$$

So $i(\sigma_{23}, \sigma_{31}, \sigma_{12}) \longrightarrow -i\vec{\sigma}$

& we arrive at

$$\mathcal{M} \sim \int \underbrace{u_f^\dagger}_{\text{2 spinor}} \left(\frac{e}{2m} \vec{\sigma} \cdot \underbrace{(\vec{\nabla} \times \vec{A})}_{\vec{B}} \right) u_i \, d^3x$$

This is a magnetic moment interaction

$$-\vec{\mu}\cdot\vec{B} \qquad \text{with} \qquad \vec{\mu} = \frac{-e\hbar}{2m}\vec{\sigma}$$

Compare to the orbital spin

$$\vec{\mu}_{orb} = \frac{-e}{2m}\vec{L}$$

So people write $\vec{\mu}_{spin} = -g\dfrac{e}{2m}\vec{S}$ $\quad \leftarrow \frac{\hbar}{2}\vec{\sigma}$

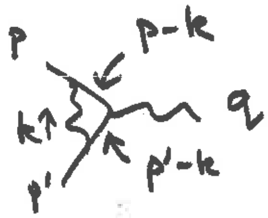
gyromagnetic ratio

The tree level Dirac eq'n predicts $g=2$

Expt $(g-2) = 0.00232\ldots$ so it's pretty good.

<u>ONE LOOP CORRECTION</u>

The deviation from 2 is due to



$$-ie\Gamma^\mu = \mu^{2\epsilon}\int\frac{d^4k}{(2\pi)^4}(-ie\gamma^\nu)\frac{i[(\not p'-\not k)+m]}{(p'-k)^2-m^2}\gamma^\mu\frac{i[(\not p-\not k)+m]}{(p-k)^2-m^2}(-ie\gamma^\rho)\frac{-ig_{\nu\rho}}{k^2}$$

F.P.: $\quad 2\int d\alpha\, d\beta\, d\gamma\; \delta(1-\alpha-\beta-\gamma)\,\dfrac{N}{D^3}$

$$\begin{aligned} D &= \beta[(p'-k)^2-m^2] + \alpha[(p-k)^2-m^2] + \gamma k^2 \\ &= \beta(k^2-2p'\cdot k) + \alpha(k^2-2p\cdot k) + \gamma k^2 \\ &= k^2 - 2(\beta p'\cdot k + \alpha p\cdot k) \end{aligned}$$

$$k^\mu = k'^\mu + p\alpha + p'\beta$$

$$D = k'^2 - (p\alpha + p'\beta)^2$$

$$= k'^2 - m^2\alpha^2 - m^2\beta^2 - 2\alpha\beta\, p.p' \qquad q^2 = (p \mp p')^2 = -2p.p' \pm 2m^2$$

$$= k'^2 - m^2(\alpha+\beta)^2 + \alpha\beta q^2$$

As $q \to 0$ $\quad D = k'^2 - m^2(\alpha+\beta)^2.$

Numerator:

$$N = \gamma^\nu[\gamma.(p' - \underbrace{k}_{k'+p\alpha+p'\beta}) + m]\gamma^\mu[\gamma.(p - \underbrace{k}_{k'+p\alpha+p'\beta}) + m]\gamma_\nu$$

The $\not{k}'\not{k}'$ term we already computed – it's a divergent correction to e_R. Linear terms in k' vanish on integration

$$N_{rest} = \gamma^\nu[\gamma.(p' - \alpha p - \beta p') + m]\gamma^\mu[\gamma.(p - \alpha p - \beta p') + m]\gamma_\nu$$

$$= m^2\gamma^\nu\gamma^\mu\gamma_\nu \to -2m^2\gamma^\mu \quad \text{finite } e_R \text{ renorm.}$$

$$+ \gamma^\nu[\gamma.(p' - \alpha p - \beta p')]\gamma^\mu m\gamma_\nu \to 4m(p' - \alpha p - \beta p')^\mu$$

Use spinor identities

$$+ \gamma^\nu m\gamma^\mu[\gamma.(p - \alpha p - \beta p')]\gamma_\nu \to 4m(p - \alpha p - \beta p')^\mu$$

$$+ \underbrace{\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta\gamma_\nu}_{-2\gamma^\beta\gamma^\mu\gamma^\alpha}\underbrace{(p' - \alpha p - \beta p')_\alpha}_{p' + \frac{1}{2}(\gamma-1)(p+p')}\underbrace{(p - \alpha p - \beta p')_\beta}_{p + \frac{1}{2}(\gamma-1)(p+p')}$$

We can use the symmetry $\alpha \xleftrightarrow{F.P.} \beta$ on the middle two terms

$$\to 4m(p+p')^\mu(\underbrace{1-\alpha-\beta}_{\gamma})$$

$$\alpha + \beta + \gamma = 1$$

Now we sandwich the term between external spinors $\bar{u}(p')\,u(p)$

& use $\not{p}\,u(p) = m\,u(p)$ $\qquad$ $\bar{u}(p')\,\not{p}' = \bar{u}(p')\,m$

For the general term use $p'-p=q$ to write as

$$-2\,\gamma^\beta\gamma^\mu\gamma^\alpha\left[\gamma p + \tfrac{1}{2}(\gamma+1)q\right]_\alpha\left[\gamma p' - \tfrac{1}{2}(1+\gamma)q\right]_\beta$$

p & p' give m's. We have

$-2m^2\gamma^2\gamma^\mu$ $\Leftarrow$ more finite eq renorm.

$$+2\,\tfrac{1}{4}(1+\gamma)^2\,\gamma^\beta\gamma^\mu\gamma^\alpha\,q_\alpha q_\beta$$

$$\gamma.q\,\gamma^\mu\gamma.q = 2q^\mu\not{q} + q^2\gamma^\mu$$

this vanishes between spinors $-m \leftarrow -p \quad p' \rightarrow m$

goes to m

$$p'-p$$

$$+2m\,\underbrace{\gamma^\beta\gamma^\mu}_{-\gamma^\mu\gamma^\beta+2g^{\mu\beta}}\,\tfrac{1}{2}\gamma(\gamma+1)\,q_\beta \;-\; 2m\,\underbrace{\gamma^\mu\gamma^\alpha}_{-\gamma^\alpha\gamma^\mu+2g^{\mu\alpha}}\,\tfrac{1}{2}\gamma(\gamma+1)\,q_\alpha$$

$p'-p$ goes to m

$m\gamma^\mu$ eq renorm

$$\rightarrow -4m\;\tfrac{1}{2}(\gamma+1)\gamma\,(p'+p)^\mu$$

Thus in total $\quad 4m(p+p')^\mu\gamma - 2m\,\gamma(\gamma+1)(p'+p)^\mu$

$$= +2m\,(p+p')^\mu\,\gamma(1-\gamma)$$

Now use Gordon decomposition

$$= 2m\left(2m\gamma^\mu - i\sigma^{\mu\nu}q_\nu\right)\gamma(1-\gamma)$$

Phew! So the term we want is ($q^2 \to 0$):

$$-ie\Gamma^\mu = -e^3 2\int d\alpha\, d\beta\, d\gamma \underbrace{\int \frac{d^4k'}{(2\pi)^4} \frac{\delta(\alpha+\beta+\gamma-1)}{[k'^2 - m^2(\alpha+\beta)^2]^3}} 2m\gamma(1-\gamma) iq_\nu \sigma^{\mu\nu}$$

Wick rotate

$$i\int \frac{(k'^2 + A^2 - A^2)\, dk^4}{16\pi^2} \frac{1}{[k^2+A^2]^3}$$

$$= \frac{i}{16\pi^2}\left[\frac{-1}{[k^2+A^2]} + \frac{1/2\, A^2}{[k^2+A^2]^2}\right]_0^\infty$$

$$= \frac{i}{16\pi^2}\left[\frac{1}{A^2} - \frac{1}{2}\frac{1}{A^2}\right]$$

$$= \frac{i}{32\pi^2} \frac{1}{(\alpha+\beta)^2 m^2} \qquad \alpha+\beta = 1-\gamma$$

$$= \frac{i}{32\pi^2} \frac{1}{m^2} \frac{1}{(1-\gamma)^2}$$

$$-ie\Gamma^\mu = -i\frac{e^3}{16\pi^2}\int d\alpha\, d\beta\, d\gamma\, \delta(\alpha+\beta+\gamma-1) \frac{2m\gamma\cancel{(1-\gamma)}}{m^2(1-\gamma)^{\cancel{2}}} iq_\nu\sigma^{\mu\nu}$$

$$= -ie\frac{\alpha}{\cancel{8}\pi}\frac{1}{2m} \underbrace{\int_0^1 d\gamma \int_0^{1-\gamma} d\beta \frac{\gamma}{1-\gamma}}_{\int_0^1 d\gamma\, \gamma = 1/2} iq_\nu\sigma^{\mu\nu}$$

$$= -ie\frac{1}{2m}\frac{\alpha}{2\pi}(iq_\nu\sigma^{\mu\nu})$$

the correction to $\frac{g-2}{2}$!

PS How did we manage to drag 2 different coeffs from $\bar{\psi} A \psi$ when gauge invariance says just e?

Well we've generated at loop level a contribution to a HDO

$$\frac{1}{m_e} \bar{\psi} D_\mu D_\nu \sigma^{\mu\nu} \psi$$