Q.F.T 3

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NAGT + renormalization + IR divergences

Peskin + Schroeder

Rhyder

COMPLEX ANALYSIS

A point on complex plane $\quad z = x + iy$

Functions: $\quad f(z) = u(z) + i\,v(z)$

"analytic fn" have well defined derivatives

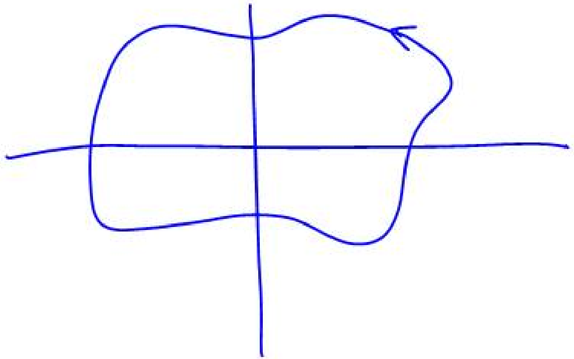
$$\frac{df}{dz} = \operatorname*{limit}_{\Delta z \to 0} \left\{ \frac{f(z+\Delta z) - f(z)}{\Delta z} \right\}$$

$$= \frac{\partial f}{\partial x} \quad \text{but also} \quad \frac{1}{i}\frac{\partial f}{\partial y}$$

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \& \quad -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

For sense $\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ "Cauchy Riemann eq'ns"

Contour Integration

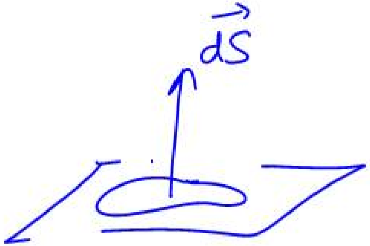


Assume f analytic in the contour

$$\oint_C f(z)\,dz = 0$$ Cauchy's Theorem

$$\oint g(z)\,dz = \oint (u+iv)(dx+idy)$$

$$= \oint [u+iv]\,dx - [-iu+v]\,dy$$



$$\oint \vec{A}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{A})\cdot d\vec{S}$$

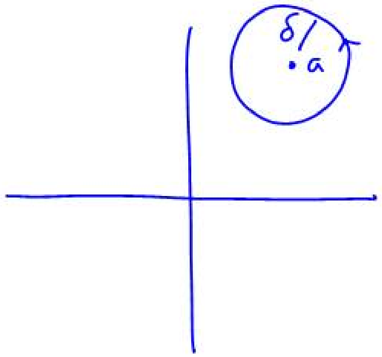
$$= \int \left(\frac{\partial}{\partial x}[v-iu] + \frac{\partial}{\partial y}[u+iv]\right) dS$$

$= 0$ by Cauchy Rieman eq'ns

More general meromorphic fns can be written near $z=a$

$$f(z) = \sum_{n\geq 0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

("poles"



$$z = a + \delta e^{i\theta}$$

$$dz = i\delta e^{i\theta} \delta\theta$$

$$\oint f(z)\,dz = i\delta \int_0^{2\pi} d\theta \Big[\sum_n a_n \delta^n e^{i(n+1)\theta} + \sum_n b_n \delta^{-n} e^{i(1-n)\theta}$$

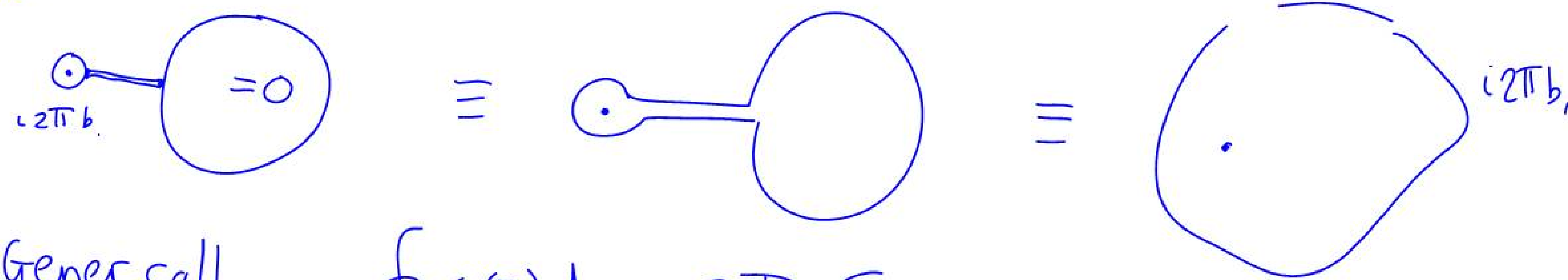
$$\int_0^{2\pi} e^{im\theta}\,d\theta = 0 \quad m \neq 0$$

$$= 2\pi \quad m = 0$$

$\Rightarrow \oint f(z)\,dz = 2\pi i \; b_1$

↑ this is pole's residue $\dfrac{b_1}{z-a}$

Merge contours



Generally: $\oint f(z)\,dz = 2\pi i \sum \text{residues}$

$+i\varepsilon$ PRESCRIPTION

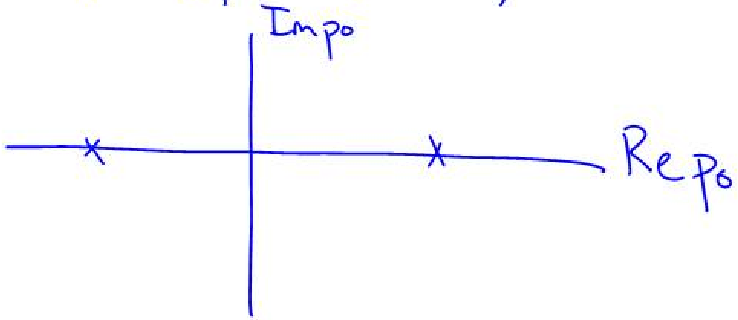
Scalar (free) Green's fn

$$G_F(x-x') = \frac{1}{(2\pi)^4}\int d^4p \, \frac{1}{p^2-m^2} \, e^{-ip\cdot(x-x')}$$

$$\frac{1}{\left(p_0+\sqrt{|\mathbf{p}|^2+m^2}\right)} \quad \frac{1}{\left(p_0-\sqrt{|\mathbf{p}|^2+m^2}\right)}$$

p_0 integral sees poles!

We move to the complex plane

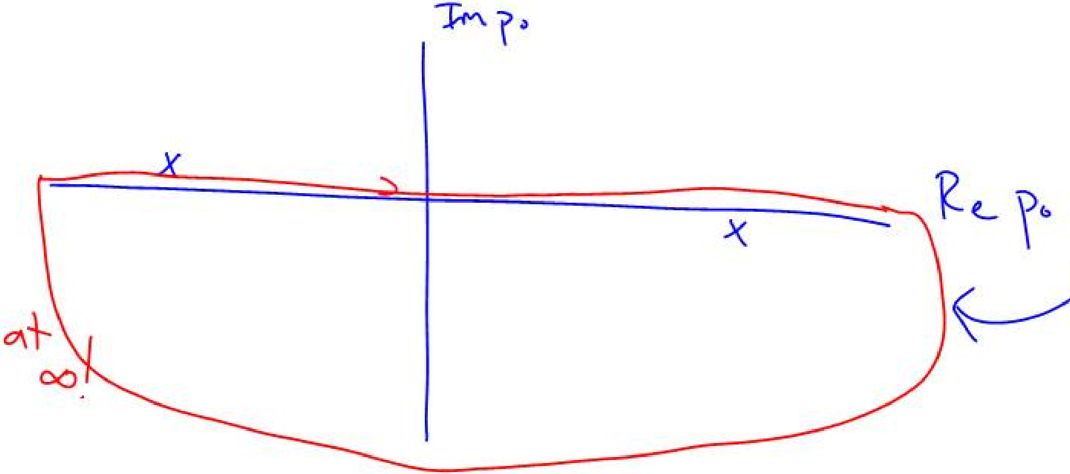


Use Feynman – Stuckelberg interpretation:

+E states moving forwards in t are pts

−E states " backward in t " anti-pts

→ when $t > t'$ so looking forward propagation we will complete the contour below the axis & include pt



$$e^{-ip_0(t-t')}$$

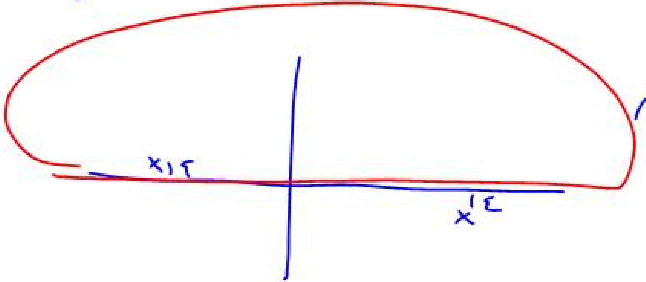
↑ −ve imag

$$\sim e^{-E(t-t')}$$

↑ +ve

vanishes at $E \to \infty$

$t<t'$: backwards prop. – complete above axis & include anti-pt



$$e^{-ip_0(t-t')} \sim e^{E(t-t')}$$

$\uparrow$ −ve

vanishes as $E \to \infty$

Thus in G_F

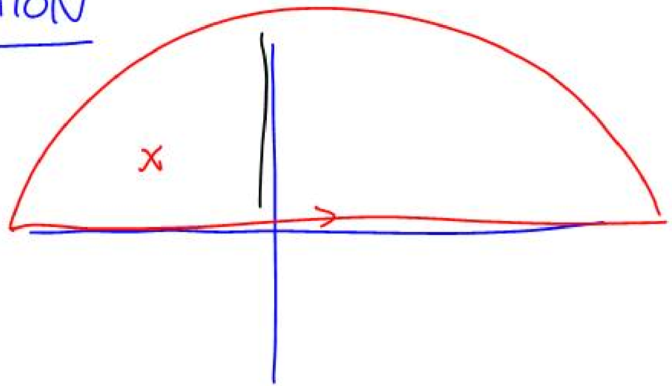
$$\frac{1}{(p_0+\sqrt{|p|^2+m^2}-i\epsilon)} \qquad \frac{1}{(p_0-\sqrt{p^2+m^2}+i\epsilon)}$$

$$\to \quad G(x-x') = \frac{1}{(2\pi)^4}\int d^4p \, \frac{1}{p^2-m^2+i\epsilon'} \, e^{-ip\cdot(x-x')}$$

$$\epsilon' = 2\epsilon\sqrt{|p|^2+m^2}$$

WICK ROTATION

eg



① GAUGE THEORIES

QED

MINIMAL SUBSTITUTION

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Ampère's lecture 1820

$\downarrow$

1892 Lorenz to derive

$$S = \int L \, dt \qquad L = \tfrac{1}{2} m |\dot{x}|^2 + q \dot{\vec{x}} \cdot \vec{A} - q\phi$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \;, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{P}_{gen} = \frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{x} + q\vec{A}$$

"minimal sub"

4-vector form: $A^\mu = (\phi, \vec{A})$

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$\sim \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B & B \\ -E_y & B & 0 & B \\ -E_z & -B & B & 0 \end{pmatrix}$$

$$f^\mu = q\, u_\nu\, F^{\mu\nu}$$

$$\frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \gamma(c, \vec{v})$$

$$\rightarrow \quad p^\mu = m u^\mu + q A^\mu$$

$$\xrightarrow{QM} i(\partial^\mu - i q A^\mu)$$ "adhoc rule"

Here gauge invariance is an oddity. $A^\mu \to A^\mu + \frac{1}{q}\partial^\mu \omega(x^\mu)$

$\vec{E}, \vec{B}$ & $F^{\mu\nu}$ are unchanged

GLOBAL TRANSFORMATIONS OF DIRAC EQ'N

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

$$\psi \to e^{i\omega}\psi$$

$$\bar{\psi} \to e^{-i\omega}\bar{\psi}$$

$\leftarrow \psi^\dagger \gamma^0$

If ω is independent of x^μ this is an invariance.

Note: this is an abelian transform

$$e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)} = e^{i\beta} e^{i\alpha}$$

GAUGE TRANSFORMATIONS

$$\psi \to e^{i\omega(x^\mu)}\psi \qquad \text{(local or gauge transform)}$$

Get extra term: $\delta\mathcal{L} = -\bar{\psi}\gamma^\mu(\partial_\mu \omega)\psi$

In full QED it does work

$$\mathcal{L} = \bar{\Psi}\left[i\gamma^\mu \underbrace{(\partial_\mu - iqA_\mu)}_{D^\mu}\right]\Psi \quad *$$

$$\Psi \to e^{i\omega(x^\mu)}\Psi \quad \& \quad A^\mu \to A^\mu + \frac{1}{q}\partial^\mu\omega$$

$$\delta L = -\bar{\Psi}\gamma^\mu(\partial_\mu\omega)\Psi + \bar{\Psi}\gamma^\mu(\partial_\mu\omega)\Psi = 0 \quad \checkmark$$

Now reverse logic: require the symmetry → forcing us to introduce A^μ & it's ~~transform~~ property → QED by magic!

We also add all gauge invariant & Lorentz invariant terms

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad * \quad \& \quad m\bar{\Psi}\Psi$$ are all dim 4 - terms possible

NB/ We predicted $m_L = 0$ since $A^\mu A_\mu$ is not gauge invariant

<u>Trick.</u> $F^{op}_{\mu\nu} 1 = \frac{i}{q} [D_\mu , D_\nu]$

$$= \frac{i}{q} \cancel{[\partial_\mu , \partial_\nu]} + [\partial_\mu , \bar{A}_\nu] + [A_\mu , \partial_\nu] + ie \cancel{[A_\mu , \bar{A}_\nu]}$$

$$= \partial_\mu A_\nu - \cancel{A_\nu \partial_\mu} + \cancel{A_\mu \partial_\nu} - \partial_\nu A_\mu$$

when acts on 1 gives 0.

NON ABELIAN GAUGE INVARIANCE

Is only there are 2+ identical pts then we can change basis

$$u, d \longrightarrow \psi = \frac{1}{\sqrt{2}}(\bar{u} \pm \bar{d})$$

Generically

$$\underset{\binom{u}{d}}{\psi_{\underset{1 \text{ or } 2}{i}}} \longrightarrow \underbrace{\left(e^{i\omega^a T^a}\right)_i^{\;j}}_{U^j{}_i} \psi_j$$

real parameters, generators

eg $SU(2)$ $\quad T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$T^3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow T^i = T^{i\dagger}$$

$$[T^a, T^b] = i f^{abc} T^c$$

$\Rightarrow$ non-abelian nature $\quad U_1 U_2 = e^{i\omega T^1} e^{i\omega^2 T^2} \neq U_2 U_1$

$$\sim (1 + i\omega^1 T^1)(1 + i\omega^2 T^2 + \dots)$$

Global versions of these transformations are an invariance of

$$\mathcal{L} = \bar{\Psi}^i (i\gamma^\mu \nabla_\mu - m) \Psi_i$$

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$U^\dagger$ $\qquad U$

$$\begin{pmatrix} \gamma^\mu \partial_\mu & \\ & \ddots \end{pmatrix}$$

Invariance since $U^\dagger U = 1$

$$\sim e^{i\omega T^a}$$

Now promote to local symmetry. We get a non-invariant term

$$\bar{\Psi} U^{-1} i \gamma^\mu (\partial_\mu U) \Psi$$

note here for SU(2) there are 3 $\omega^a(x^\mu)$ – we will need 3 A^μ to cancel derivative terms

Introduce $A^\mu = A^{a\mu} T^a$ ← summing over a!

We will add terms like

$$-\bar{\Psi} \underset{U^{-1}}{\gamma^\mu}\ g \underset{U}{T^a} \underset{U^{-1}}{A^a_\mu} \underset{U}{\Psi}$$

need to make invariant

So $\quad A_\mu \to U A_\mu U^{-1} + \frac{i}{g} (\partial_\mu U) U^{-1}$

{ this term will cancel the term from the free Dirac eq'n.

So our gauge invariant theory is

$$\mathcal{L} = \bar{\Psi}^i \left(i \not{D} - m \mathbb{1} \right)^j{}_i \Psi_j$$

$$D_\mu = \partial_\mu \mathbb{1} + i g T^a A^a{}_\mu$$

sign is just convention on charge

Note under GT $\quad D_\mu \to U D_\mu U^{-1}$

We construct

$$F^{\mu\nu} = \frac{-i}{g}\left[D_\mu, D_\nu\right]$$

$$= -\frac{i}{g}\cancel{\left[\partial_\mu, \partial_\nu\right]} + \underbrace{\left[\partial_\mu, A_\nu\right]}_{\partial_\mu A_\nu} + \underbrace{\left[A_\mu, \partial_\nu\right]}_{-\partial_\nu A_\mu} + ig\left[A_\mu, A_\nu\right]$$

$$F^{a\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu$$

Note $F^{\mu\nu} \xrightarrow{GT} U F^{\mu\nu} U^{-1}$

$$\mathrm{Tr}\, U^{-1} B U$$

$$\mathrm{Tr}\, B$$

An invariant

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\, F^{\mu\nu} F_{\mu\nu}$$

$$= -\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu}$$

$$\mathrm{Tr}\, T^a T^b = \frac{1}{2}\delta$$

$$\mathcal{L}_{NAC} = -\frac{1}{2}\,\mathrm{Tr}\,F^{\mu\nu}F_{\mu\nu} + \bar{\Psi}\,(i\not{D} - m)\,\Psi$$