The Chiral Anomaly

Some global symmetries in QFT are "anomalous" in the quantum theory. eg Chiral symmetry in QCD

$$\mathcal{L} = i\bar{\Psi}_L \not{D} \Psi_L + i\bar{\Psi}_R \not{D} \Psi_R + m(\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L)$$

When $m=0$: $SU(N_f)_L \otimes SU(N_f)_R$
N_f flavours

$$\left.\begin{array}{l} \Psi_L \to U \Psi_L \\ \Psi_R \to \hat{U} \Psi_R \end{array}\right\} \text{these are fine ...}$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix} \quad U(1)_L \otimes U(1)_R$$

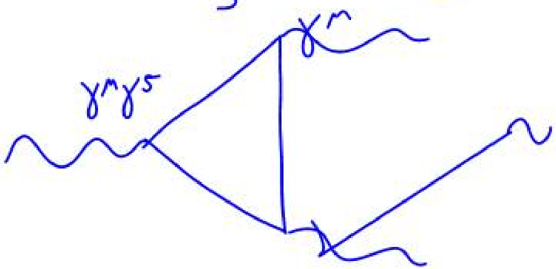
$$\Psi_L \to e^{i\alpha} \Psi_L$$

$$\Psi_R \to e^{i\beta} \Psi_R$$

$\alpha = \beta$: $U(1)_B$ ✓ $\qquad \alpha = -\beta$ $U(1)_A$ is anomalous.

$$U(1)_A: \quad \psi \to e^{i\alpha \underbrace{\gamma_5}_{\begin{pmatrix}1 & \\ & -1\end{pmatrix}}} \psi \sim \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$$

The breaking was first seen in one loop diagrams



$$\varepsilon_{\sigma\nu\tau\rho} F^{\sigma\nu} F^{\tau\rho} \neq 0$$

If we tried to gauge $U(1)_A$ we would get a $A^\mu \gamma\gamma$ 3-pt vertex that is forbidden by gauge invariance.

Fujikawa's Non-Perturbative Proof

First I'm going to Fourier expand fields in path integral

$$\Psi(x) = \sum_n a_n \phi_n(x) \qquad \text{with} \qquad \phi_n(x) = u_r e^{-ikx}$$

ϕ_n: counts r & k $\qquad u_r$: $\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \cdots \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$$a_n = \int d^4x \, \bar{\phi}_n \Psi$$

note orthogonality $\int \bar{\phi}_i \phi_j d^4x = \delta_{ij}$

$$\bar{\Psi}(x) = \sum_n \varphi_n^\dagger(y) \bar{b}_n$$

The key point is that $D[\Psi] D[\bar{\Psi}] \longrightarrow \prod_n da_n \prod_m d\bar{b}_m$

Under $U(1)_A$ $\quad \Psi \to \Psi + i\omega\gamma_5\Psi + \ldots$

$$a_n \to a_n - i\omega \int d^4x \, \bar{\phi}_n \gamma_5 \Psi \equiv a_n'$$

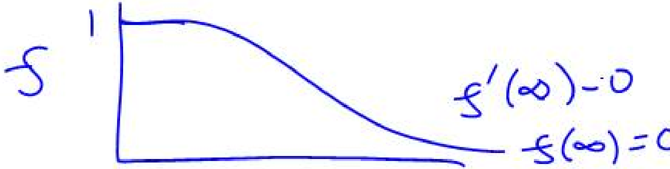
$\prod da_n \to \prod da_r'$ we get a det $\left|\frac{\partial a_i}{\partial a_i'}\right|$

$$\underset{\text{small } \omega}{\det} = \det \begin{bmatrix} 1+iK_{11} & 0 & & & \\ 0 & 1+iK_{22} & & & \\ & & 1\ldots & & \\ & & & 1. & \\ & & & & \ddots \end{bmatrix}$$

$K_{11} = \omega \int \bar{\phi}_1 \gamma_5 \phi_1 \, d^4x$

drop K_{ij} terms since in det they square to small!

$$= 1 + i \sum_n K_n$$



$$\det = e^{+i\sum_n \omega \int \bar{\phi}_n \gamma_5 \phi_n \, d^4x}$$

Regulator: multiply by a smooth, Lorentz & gauge invariant fn: $f\left[\frac{(i\not{D})^2}{g}\right.$

Now $\cancel{D}^2 = \gamma^\mu \gamma^\nu D_\mu D_\nu$

$$= \frac{1}{2}[\gamma^\mu, \gamma^\nu] D_\mu D_\nu + \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} D_\mu D_\nu$$

$$= D^2 + \underbrace{\tfrac{1}{2}\gamma^\mu \gamma^\nu [D_\mu, D_\nu]}_{\frac{1}{2}\frac{i}{g} F_{\mu\nu}\gamma^\mu\gamma^\nu}$$

Taylor expand $f\left(\frac{(i\cancel{D})^2}{g}\right)$

$$f(i\;\;) = f\left(-\frac{D^2}{g}\right) - f'\left(\frac{-D^2}{g}\right)\frac{i}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu$$
$$+ f''\left(-\frac{D^2}{g}\right)\left(-\frac{1}{8}\right)F_{\mu\nu}\gamma^\mu\gamma^\nu F_{\lambda\sigma}\gamma^\lambda\gamma^\sigma + \ldots\ldots$$

$$\det \simeq e^{i \sum_n w \int \bar{\phi}_n \gamma_5 f(-\not{D}^2/g)\, \phi_n \, d^4x}$$

$$\sum_n \to \sum_{r=1}^{4} \int \frac{d^4k}{(2\pi)^4}, \qquad \bar{\phi}_n \to \bar{u}_r e^{-ikx}, \qquad \phi_n \to u_r e^{ikx}$$

$$(1\,0\,0\,0)\begin{pmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \\ \cdots & & \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = a_1$$

The sum over u_r gives $\mathrm{Tr}\, M$

$$\simeq e^{iw \iint \frac{d^4k}{(2\pi)^4} d^4x \, \mathrm{Tr}\, \gamma_5 f}$$

Now $\mathrm{Tr}\,\gamma_5 = 0$, $\mathrm{Tr}\,\gamma_5 \gamma^\mu \gamma^\nu = 0$, $\mathrm{Tr}\,\gamma_5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma = -4i\varepsilon^{\mu\nu\lambda\sigma}$

$\mathrm{Tr}\,\gamma_5 \ldots = 0$ all higher

$$\det = e^{i\omega \int \frac{d^4k}{(2\pi)^4} d^4x \left(-\frac{1}{8}\right) F_{\mu\nu} F_{\lambda\sigma} f''(-4 i \epsilon^{\mu\nu\lambda\sigma})}$$

Now $\int \frac{d^4k}{(2\pi)^4} f''\left(\frac{-D^2}{g}\right) \xrightarrow[k]{\text{shift}} \int \frac{d^4k'}{(2\pi)^4} f''(-k^2)$

Wick rotate

$$= \frac{-i}{16\pi^2} \int k^2 \, dk^2 f''(k^2)$$

$$= \frac{-i}{16\pi^2} \left\{ \left[k^2 f'\right]_0^\infty - \int_0^\infty f'(k^2)\, dk^2 \right\}$$

$$= -\frac{i}{16\pi^2} \left[f(0) - f(\infty) \right]$$

$$= -\frac{i}{16\pi^2}$$

Thus under $U(1)_A$

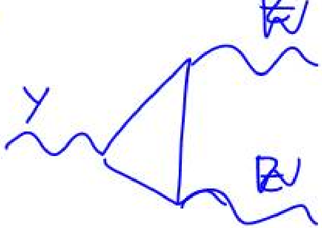
$$\Delta \mathcal{L} = +\frac{\omega}{32\pi^2} F \hat{F}$$

Also change $D[\bar{\psi}]$

$$\rightarrow \Delta \mathcal{L} = +\frac{\omega}{16\pi^2} F \hat{F}$$

$\rightarrow$ this is not a symmetry!

$U(1)_a \rightsquigarrow U(1)_Y$ whic is chiral.

You have to check 

vanish.

They do because $\sum_f Y_f = 0$

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in Peskin Schroeder.