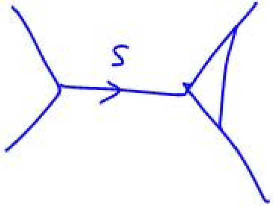
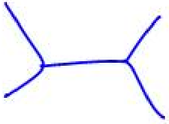
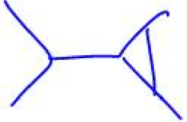
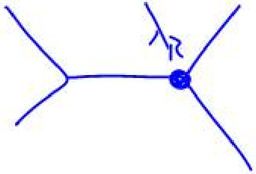
RENORMALIZING $\frac{\lambda}{3!}\phi^3$



$$i\mathcal{M} = \frac{-i\lambda}{(s-m^2)} \underbrace{\left(-i\lambda - i\lambda^3 \Delta F(s) + \dots\right)}_{-i\lambda_R}$$

Standard book keeping is to move the correction into an "effective" tree level vertex

 $+$  $\equiv$ 

$$\lambda_R = 1 + \lambda^2 \Delta F(s) + \dots$$

$$\equiv \frac{\lambda}{Z_1} \qquad Z_1 = 1 - \lambda^2 \Delta F$$

We' also can move Z into the mass $\Sigma(p^2)$ instead we move $Z^{1/2}$ to each

Generically $\quad \Sigma(p^2) \sim A p^2 + B \qquad$ A & B can be divergent & B can depend on p^2

$$A(p^2 - m^2) + A m^2$$

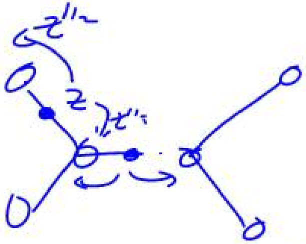
For the 1st term :

$$= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\left(-iA\cancel{(p^2-m^2)}\right)\frac{i}{\cancel{p^2-m^2}}$$

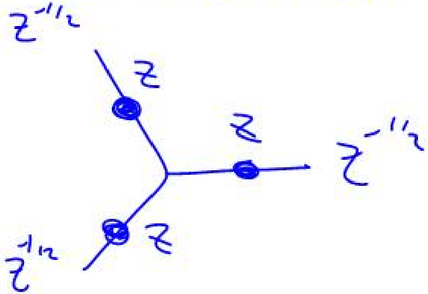
$$= \frac{i}{p^2-m^2}(1+A) \equiv \frac{iZ}{p^2-m^2}$$

We can't absorb Z into M .... so instead we move $Z^{1/2}$ to each end of the propagator. For external states we renormalize the normalization of the wave function – this is the $Z^{-1/2}$ Feynman rule

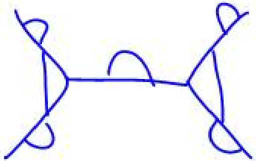
$$Z^{-1/2} \quad \frac{iZ}{p^2-m^2} \quad Z^{-1/2}$$



This still leaves $Z^{1/2}$ on the internal ends of propagators which we suck into the coupling λ_R



$$\Rightarrow \quad \lambda_R = \frac{Z^{3/2}\lambda_0}{Z_1(s)}$$



WITHOUT s DEPENDENCE WE'D BE DONE... BUT NOW HAVE TO FIX AT SOME s. ~ "SCHEMES"

eg "on mass shell scheme"

$$\Sigma(p^2) \underset{\substack{\text{Taylor}\\ \text{expand}}}{=} \underbrace{\Sigma(m_R^2)}_{\substack{= m^2 \text{ at}\\ \text{this order in}\\ \text{expansion.}}} + \underbrace{\left.\frac{\partial \Sigma}{\partial p^2}\right|_{p^2 = m_R^2}}_{A \equiv Z-1} (p^2 - m_R^2) + \underbrace{\Sigma_R(p^2)}_{\substack{\text{vanishes as}\\ p^2 \to m_R^2}}$$

$$— + \text{—✳—} + \text{—✳✳—}$$

$$\frac{1}{p^2 - m^2 - \Sigma^*}$$

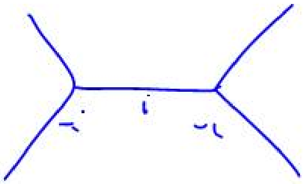
$\Sigma(m_R^2)$ & $\Sigma_R(p^2)$ are resummed into the propagator as we did in \Sigma

$$\frac{i}{p^2 \underbrace{- m^2 - \Sigma(m_R^2)}_{m_R^2} - \underbrace{\Sigma_R(p^2)}_{\text{vanish s } p^2 = m_R^2}}$$

A or Z upstairs has been sucked to the ends.

Now measure λ_R at $s=m_R^2$ too.

$$\lambda_R(m_R) = \frac{\lambda_0 Z^{3/2}}{Z_1(m_R^2)} \qquad \lambda_R(s) = \frac{\lambda_0 Z^{3/2}}{Z_1(s)}$$



$$i\mathcal{M} = \frac{-i\lambda_0^2 Z^3}{(s-m_R^2-\Sigma_R(s))}\frac{1}{Z_1^2}$$

$$= \frac{-i\lambda_R^2}{(s-m_R^2-\Sigma_R(s))}\frac{Z_1^2(m_R)}{Z_1^2(s)}$$

$$\simeq 1 \cancel{\mp} 2\lambda_0^2\Delta F(s) + 2\lambda_0^2\Delta F(m_R^2)$$

Nothing here now diverges!

An alternative way to work is to use <u>counter terms</u> - divergent Lagrangian terms whose tree level diagrams cancel loops..

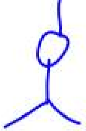
$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{CT}$$

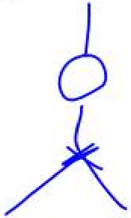
$$\mathcal{L}_R = \tfrac{1}{2}\left(\partial_\mu \phi_R \partial^\mu \phi_R - m_R^2 \phi_R^2\right) - \frac{\lambda_R}{3!}\phi_R^3$$

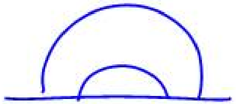
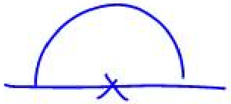
$$\mathcal{L}_{CT} = \frac{1}{2}(Z-1)\,\partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2}(Z_m - 1)\, m_R^2 \phi_R^2 - (Z_1 - 1)\frac{\lambda_R}{3!}\phi_R^3$$

So eg the vertex

$$(1 - Z_1)\lambda_R \sim \frac{\lambda_R}{Z_1}$$

The wave function renormalization  still generate $Z^{1/2}$ factors to be cancelled by external Feynman rules...

There's also mixed diagrams  in there!

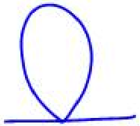
& At higher loops  is cancelled vs 

One ought to prove that just m_R, λ_R, ϕ_R can be used to eliminate all divergences at all loop level.... Bogoliobov, Parasiuk, Hepp & Zimmerman proved it!

WHAT THEORIES ARE RENORMALIZABLE

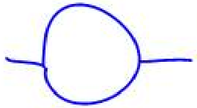
The superficial degree of divergence in a diagram is

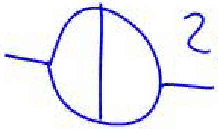
$$\omega = \underbrace{4L}_{d^4k \text{ per loop}} - \underbrace{I_F}_{\substack{\text{internal fermion} \\ \not{p}/p^2}} - 2\underbrace{I_B}_{\substack{\text{internal boson} \\ 1/p^2}} + \sum_{\text{vertices}} \underbrace{\delta_v}_{\substack{\text{number of} \\ \text{derivatives in} \\ \text{the term}}}$$

eg  $\sim \underbrace{4}_{\text{loop}} \underbrace{-2}_{\substack{\text{internal} \\ \text{boson}}} \sim 2 \qquad \Lambda^2$

An expression for the number of loops is

$$L = I_B + I_F + 1 - V$$

eg  $1 = 2 + 1 - 2$

 $2 = 5 + 1 - 4$

Now $\omega = 3I_f + 2I_B + 4 + \sum_{\text{vert}} (\delta_v - 4)$

Next consider a vertex

$$\eta_v = \underset{\text{\# derivatives}}{\delta_v} + \tfrac{3}{2}\underset{\text{\# fermions}}{f_v} + \underset{\text{\# bosons}}{b_v}$$

Energy dimensions

$$S = \int \underset{\substack{\text{length}^{+4} \\ \text{energy}^{-4}}}{d^4x}\ \underset{\dim 4}{\mathcal{L}}$$

$\mathcal{L} = \bar{\psi} \not\partial \psi$

$\rightarrow \psi \dim 3/2$

$(\partial_\mu \phi)^2$

$\phi \rightarrow \dim 1$

Sum over the vertices

$$\sum_v \eta_v = \sum_v \delta_v + \underbrace{3I_f + 2I_B}_{\substack{\text{internals couple} \\ \text{to 2 vertices}}} + \underbrace{\tfrac{3}{2}E_f + E_B}_{\substack{\text{external to just} \\ \text{one}}}$$

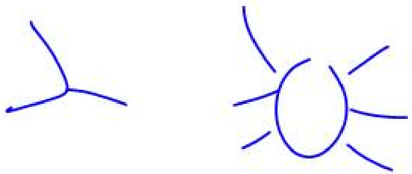
Finally $\omega = 4 - \frac{3}{2}E_F - E_B + \sum_{\text{Vert}} (\eta_v - 4)$

<u>Non-renormalizable $\mathcal{L}$ terms</u> : $\eta_v > 4$

eg $\lambda \phi^6$
$G \bar{\psi}\psi\bar{\psi}\psi$

For fixed E_F, E_B the degree of divergence grows with loop level

Worse at higher loop level divergences appear for graphs with more external legs



eventually you need ∞ number of measurements to fix theory!

Renormalizable $\mathcal{L}$ terms: $\eta_v = 4$

Divergences in same processes at same level of divergence no matter how many loops $\rightarrow$ finite number of renormalizable couplings ✓

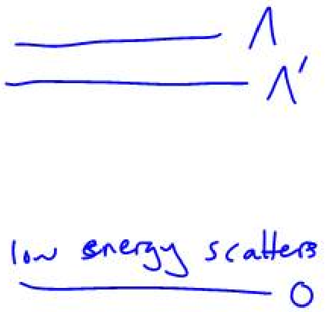
eg Any term in the Standard Model $\mathcal{L}$ $\quad \bar{\Psi} \cancel{A} \psi$

<u>Super-renormalizable</u> $\quad \eta_v < 4 \quad$ Fewer divergences at higher loop level

$m^2 \phi^2$

Wilsonian Renormalization

Accept a $\mathcal{L}$ only describes physics below some large scale Λ

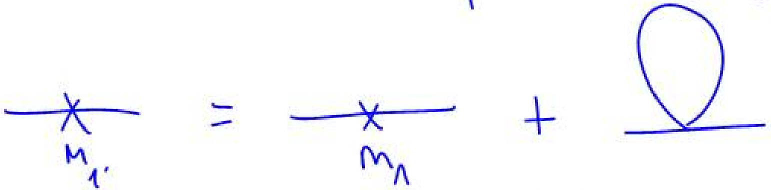


Now consider working with a lower Λ'

What action gives the same low energy scattering results?

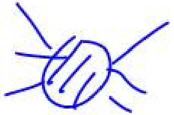
You have to do loops for energies between Λ & Λ'

eg



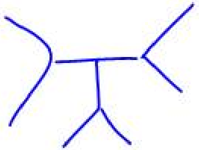
$$\lambda \int_{\Lambda'}^{\Lambda} \frac{d^4k}{k^2}$$

This is a running parameter ... but no ∞ in calculation.

Note you get contributions to multi-leg scatterings 

but these give q^2/Λ^2 which people drop since small.

So Wilsonian approach has a direction down in energy.



If we write all couplings as dimensionless.

eg $m^2\phi^2$ $\quad m^2 = \tilde{m}^2\Lambda^2 \quad \xrightarrow{\Lambda\to\Lambda'} \quad \underbrace{\tilde{m}^2\frac{\Lambda^2}{\Lambda'^2}}_{\tilde{m}'^2}\Lambda'^2$ dimensionless coupling grows

$g\bar{\psi}A\psi \quad g = g \quad \longrightarrow \quad g$ In this case quantum effects can be important

$G\bar{\psi}\psi\bar{\psi}\psi \quad G = \hat{G}\Lambda^{-2} \quad \longrightarrow \quad G\frac{\Lambda'^2}{\Lambda^2}(\Lambda')^{-2}$ This term becomes weak in IR.

Regularization

How should we book-keep for divergences?

Cut-off

$$\int_0^\Lambda \frac{d^4k}{k^2} \sim \Lambda^2 \qquad \Lambda \to \infty$$

Pauli Villars

$$\frac{1}{k^2-m^2} \longrightarrow \sum_{i=0}^{\infty} a_i \frac{1}{k^2-m_i^2} \qquad a_0=1 \quad m_0=m$$

Expand in m^2/k^2:

$$\sum_i a_i \frac{1}{k^2} + \sum_i a_i \frac{m_i^2}{k^4} + \dots$$

somewhere in expansion divergences are killed

So now by hand cook so $\sum_i a_i = 0$, $\sum_i a_i m_i^2 = 0$ ...

But neither work for Gauge Theories

- $\Lambda \sim p^\mu \sim \partial^\mu \longrightarrow D^\mu$ which changes when $A^\mu \to A^\mu + \frac{1}{g}\partial^\mu \omega$
- Gauge bosons are massless ... so can't in GI way add heavy copies!

DIMENSIONAL REGULARIZATION

Note: $\int_0^\Lambda \frac{d^4k}{k^4} \sim \ln \Lambda^2$ but $\int_0^\Lambda \frac{d^3k}{k^4} \sim$ finite

So work in $d = 4 - 2\varepsilon$ dimensions & take $\varepsilon \to 0$ at the end ..... $\frac{1}{\varepsilon}$