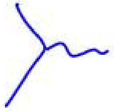
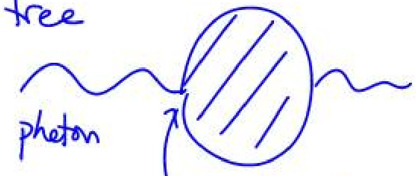
QED

 $\dfrac{i}{\not{p}-m}$  $\dfrac{-i g_{\mu\nu}}{q^2}$  $-ie\gamma_\mu$? $\mu^\epsilon e$

& $\partial_\mu J^\mu = 0$

WARD IDENTITIES – results that follow from $\partial^\mu J_\mu = 0$ and apply at all orders in perturbation theory.

eg full photon propagator



$$G^{\mu\nu}(q^2) = G_{0\rho}^{\mu}(q^2)\left[g^{\rho\nu} - i\int d^4x\, e^{-iq\cdot x} \langle 0|T\, J^\rho(x)\, J^\nu(0)|0\rangle\right]$$

Now h.t with q_μ ... $\quad q_\mu G_0^{\mu\rho} = q_\mu \dfrac{-ig^{\mu\rho}}{q^2} \sim q^\rho$

$$-iq^\rho \int d^4x\, e^{-iqx} \langle 0| T J_\rho(x) A_\nu(0)|0\rangle$$

$$= \int d^4x\, \partial^\rho [e^{-iq\cdot x}] \langle 0| T J_\rho(x) A_\nu(0)|0\rangle$$

Integrate by parts

$$= -\int d^4x\, e^{-iq\cdot x}\, \partial^\rho \langle 0| T J_\rho(x) A_\nu(0)|0\rangle$$

step fns

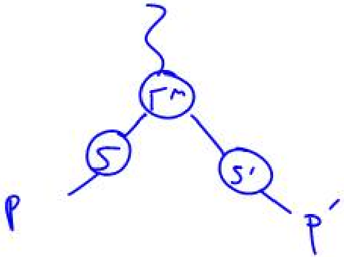
Now $\quad \partial_\rho \langle 0|T J_\rho(x) A_\nu(0)|0\rangle = \partial_\rho \Big\{ \langle 0| J_\rho(x) A_\nu(0)|0\rangle\, u(x_0)$
$\qquad + \langle 0| A_\nu(0) J_\rho(x)|0\rangle\, u(-x_0) \Big\}$

$= \langle 0 | [J_\rho(x), A_\nu(0)] | 0 \rangle \, \delta(x_0)$

$= 0$ since commute

$$q_\nu G^{\mu\nu}(q^2) = q_\mu G_0^{\mu\nu}(q^2)$$

eg Another follows from



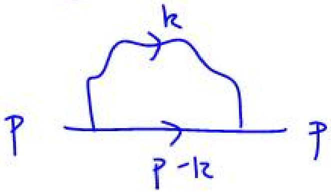
$$= S(p')\,\Gamma^\mu(p,p')\,S(p)\,(2\pi)^4\,\delta^4(p+q-p')$$

$$\equiv \int d^4x\, d^4y\, e^{i(p'\cdot z - p\cdot y - q\cdot x)} \langle 0 | T J_\mu(x)\psi(y)\bar\psi(z) | 0 \rangle$$

Hit with q^μ $\Rightarrow$ $\boxed{q_\mu \Gamma^\mu = i\left(S^{-1}(p') - S'(p)\right)}$

Henceforth $m_e = 0$ & only collect $1/\varepsilon$ divergences

Electron Self Energy



$= -i\Sigma$

$$\Sigma = i\int \boxed{\frac{d^dk}{(2\pi)^d}} \left(-ie\mu^{\varepsilon}\gamma^\mu\right)\cdot\frac{i(\not p-\not k)}{(p-k)^2}\cdot\left(-ie\mu^{\varepsilon}\gamma^\nu\right)\cdot\frac{-ig_{\mu\nu}}{k^2}$$

Feynman Parameterize:

$$= -i\mu^{2\varepsilon}e^2\int \int_0^1 d\alpha\, d\beta\, \delta(\alpha+\beta-1)\,\frac{\gamma^\mu(\not p-\not k)\gamma_\mu}{\left[\alpha(p^2+k^2-2p\cdot k)+\beta k^2\right]^2}$$

Do β integral $\alpha+\beta=1$

$$D = k^2 - 2\alpha\, p\cdot k + \alpha p^2$$

Shift $k^\mu = k'^\mu + \alpha p^\mu$

$$D = k'^2 + \cancel{\alpha^2 p^2} - \cancel{2}\alpha^2 p^2 + \alpha p^2$$

$$= k'^2 + \alpha(1-\alpha) p^2$$

$$= k'^2 - A^2$$

$$\Sigma = -i\mu^{2\varepsilon} e^2 \int \frac{d^d k'}{(2\pi)^d} \int_0^1 d\alpha \; \frac{\gamma^\mu \left(\overset{\text{odd integral } =0}{-\not{k}'} + \not{p}(1-\alpha) \right) \gamma_\mu}{(k'^2 - A^2)^2}$$

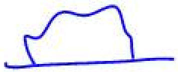
Wick Rotate: gain i, $k'^2 - A^2 \to -k'^2 - A^2$

$$(1-\alpha)\gamma^{\mu}\not{p}\gamma_{\mu} = -2(1-\varepsilon)\not{p}(1-\alpha)$$

$$\Sigma_1 = \mu^{2\varepsilon} e^2 \int_0^1 d\alpha \; 2(\alpha-1)(1-\varepsilon)\not{p} \underbrace{\int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2+A^2]^2}}_{\frac{1}{16\pi^2}\left[\frac{1}{\varepsilon} + \text{finite} + O(\varepsilon)\right]}$$

$$\Sigma_{d,1} = \frac{e^2}{16\pi^2} \underbrace{\int_0^1 d\alpha \; 2(\alpha-1)}_{2\left[\frac{\alpha^2}{2}-\alpha\right]_0^1 = -1} \not{p}\,\frac{1}{\varepsilon}$$

$$= \frac{-e^2}{16\pi^2}\frac{1}{\varepsilon}\not{p} + \text{finite}$$

 $= \delta m + (Z_2 - 1)(\not p - m) + \Sigma_R(p^2 - m^2)$

$\delta m \sim \Sigma(p^2 = m^2)$

↳ vanishes at $p^2 = m^2$

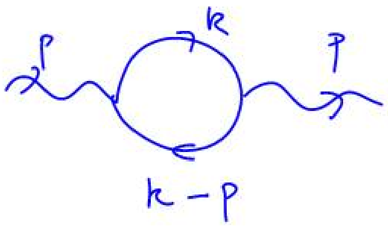
$\delta m + \Sigma_R(p^2, m^2)$:  +  +  +

$$\to \frac{i}{\not p - m} \to \frac{i}{\not p \underbrace{- m - \delta m}_{-m_{phys}} - \Sigma_R}$$

$Z_2 - 1$.
$$\cancel{\frac{i}{\not p - m}} + \frac{i}{\not p - m}\left(+i\,(Z_2 - 1)\right)\cancel{(\not p - m)}\,\cancel{\frac{i}{\not p - m}}$$

we move $Z_2^{1/2}$ to each end of prop & kill one with external wave fn renorm & the other goes into vertex renormalization.

Photon Self Energy



We need to track back to path integral

$$e^{iS} = e^{iS_0}\left(1 + \dots + \int d^4x\, d^4y\; \underbrace{\bar{\Psi}\gamma^\mu\Psi(x)}_{f.r.}\;\; \underbrace{\bar{\Psi}\gamma^\nu\Psi(y)}_{A,} + \dots\right)$$

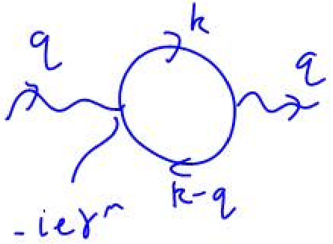
Here we Wick contract to make 2 fermion propagators

$\langle \Psi(x)\bar{\Psi}(y)\rangle$ – 3 anti-commutations $\longrightarrow$ – sign!

EXTRA FEYNMAN RULE: FERMION LOOPS GET (–) sign

(GHOSTS)

- Also think about Dirac algebra in loop



$$\Gamma_{ab} S_{bc} \Gamma_{cd} S_{da} = M_{aa} = \mathrm{Tr}\, M$$

So we trace over loop γ-matrix structure

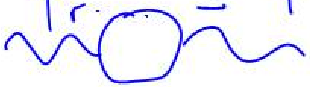
- Lorentz symmetry: restricts the answer to the form

$$i\Pi^{\mu\nu}(q^2) = iA(q^2)\, g^{\mu\nu} + iB(q^2) q^\mu q^\nu$$

& we can use Ward Identity $q_\mu \Pi^{\mu\nu} = 0$

$$\Rightarrow \Pi^{\mu\nu}(q^2) = \left(g^{\mu\nu} q^2 - q^\mu q^\nu\right) \Pi(q^2)$$

$\mathrm{Tr} = 4\left[-(k-q)\cdot k\, g^{\mu\nu} + (k-q)^\mu k^\nu + (k-q)^\nu k^\mu\right]$



$$\equiv i\Pi^{\mu\nu} = -\mu^{2\varepsilon}\int \frac{d^dk}{(2\pi)^d}\, \mathrm{Tr}\left[(-ie\gamma^\mu)\frac{i(\not{k}-\not{q})}{(k-q)^2}(-ie\gamma^\nu)\frac{i\not{k}}{k^2}\right]$$

fermion loop!

Feynman Parameterize

$$\Pi^{\mu\nu} = i\mu^{2\varepsilon} e^2 \int \frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha\, d\beta\, \delta(\alpha+\beta-1)\, \frac{\mathrm{Tr}\,\not{k}\gamma^\mu(\not{k}-\not{q})\gamma^\nu}{\left[\alpha(k-q)^2 + \beta k^2\right]^2}$$

Do β integral

$$D = k^2 - 2\alpha k\cdot q + \alpha q^2$$

$$\mathrm{Tr}\,\gamma^\mu(\not{k}-\not{q})\gamma^\nu\not{k} = (k-q)^\rho k^\sigma \underbrace{\mathrm{Tr}\,\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma}_{4\left[g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}\right]}$$

$$J\Pi'(q^2) = 4\left[-(k-q)\cdot k\, g^{\mu\nu} + (k-q)^\mu k^\nu + (k-q)^\nu k^\mu\right]$$

Shift: $k^\mu = k'^\mu + \alpha q^\mu$

$$D \longrightarrow k'^2 + \alpha^2 q^2 - 2\alpha^2 q^2 + \alpha q^2$$

$$N \longrightarrow -4\left[g^{\mu\nu}\left(k'^2 + \alpha^2 q^2 - \alpha q^2\right) - 2k'^\mu k'^\nu - 2\alpha^2 q^\mu q^\nu + 2\alpha q^\mu q^\nu \right.$$

$$\left. + \text{ terms linear in } k' \text{ that integrate to zero!}\right]$$

We get to $\Pi(q^2)$ ~~from~~ just the $q^\mu q^\nu$ terms

$$-q^\mu q^\nu \Pi(q^2) = -4i\, q^\mu q^\nu e^2 \mu^{2\epsilon} \int \frac{d^4k'}{(2\pi)^4} \int_0^1 d\alpha \, \frac{2\alpha(1-\alpha)}{\underbrace{(k'^2 - \alpha^2 q^2 + \alpha q^2)^2}_{(k^2+A^2)^2}}$$

Wick Rotate: gain $i\, dk_0$

$$\Pi(q^2) = -4\, e^2 \mu^{2\varepsilon} \underbrace{\int_0^1 d\alpha \; 2\alpha(1-\alpha)}_{2\left[\frac{\alpha^2}{2} - \frac{\alpha^3}{3}\right]_0^1 = 1/3} \left[\frac{1}{16\pi^2} \frac{1}{\varepsilon} + \cdots \right]$$

$$= -\frac{1}{12\pi^2} e^2 \frac{1}{\varepsilon}$$

- Renormalization

$$\frac{-i\, g^{\mu\nu}}{q^2} = \frac{-i}{q^2} \left[\underbrace{\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right)}_{\substack{\text{transverse} \\ \eta^\mu}} + \underbrace{\frac{q^\mu q^\nu}{q^2}}_{\text{longitudinal}} \right]$$

 $\sim \Pi(q^2)\left(-g^{\mu\nu}q^2 + q^\mu q^\nu\right)$

$$\Pi(q^2) \simeq \Pi(0) + q^2\,\Pi_R(q^2)$$

$$\Pi(0) \simeq 1 - Z_3$$

$\Pi(0)$: 

transverse

$$\frac{Z_3}{q^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)$$

move $Z_3^{1/2}$ to each end of propagator → wave fn

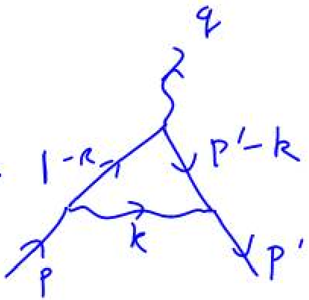
→ coupling renormalization

$q^2\Pi_R(q^2)$: 

$$-\frac{i}{q^2(1-\Pi_R(q^2))}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)$$

The longitudinal piece does not change.

VERTEX FN

$-ie\,\Gamma^\mu(p, p') =$ 

Feyn Para. α, β, γ

Assume external on-shell

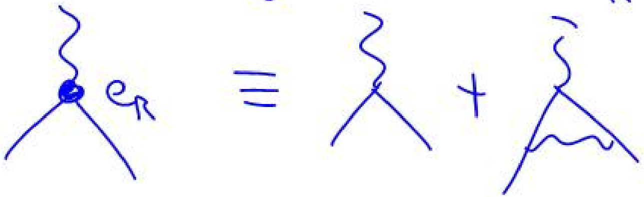
Shift momenta

Lot's terms $\to$ just pick divergent

Wick rotate i $(-1)^3$

$$\Gamma^\mu = \frac{1}{16\pi^2} e^2 \gamma^\mu \frac{1}{\epsilon} + \dots$$

We absorb divergence into e_R

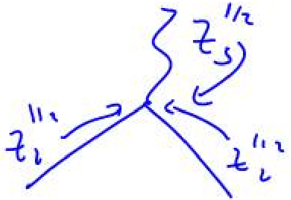


$$= -\frac{ie}{Z_1}\gamma^\mu$$

$$Z_1 = 1 - \frac{e^2}{16\pi^2}\frac{1}{\epsilon} + \dots$$

(Note $Z_1 = Z_2$ which follows from Ward Identity)

In addition we suck Z_2 & Z_3 factors from legs



$$-i e_R \gamma_\mu = -ie\gamma_\mu \frac{Z_2 Z_3^{1/2}}{Z_1}$$

Now at some eg $q^2=0$ replace e_R with measured value.

One can be lazier

eg $$Z_3 = 1 - \frac{\alpha}{3\pi}\left[\frac{1}{\epsilon} + \ln 4\pi - \gamma_E - \ln p_e^2/\mu^2\right]$$

You might want to evaluate at $p_e^2 = m_e^2$ but people do the unphysical $p_e^2 = \mu^2$ & absorb $\frac{1}{\epsilon} + \ln 4\pi - \gamma_E$: $\overline{MS}$ .... in MS only absorb $\frac{1}{\epsilon}$ .... e_R is not directly observable