RUNNING COUPLINGS

QED β FUNCTION

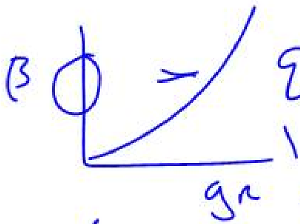
We have seen in $d = 4 - 2\varepsilon$

$$g_B = \mu^{\varepsilon} \frac{\cancel{Z_1}}{\cancel{Z_2} Z_3^{1/2}} g_R \qquad Z_3^{1/2} = \left(1 - \frac{4}{3}\frac{g_R^2}{16\pi^2}\frac{1}{\varepsilon}\right)^{1/2}$$

In MS-scheme

$$g_B \to \cancel{g_B} = \mu^{\varepsilon} g_R \left(1 + \frac{a}{2\varepsilon} g_R^2\right) \qquad a = \frac{4}{3}\frac{1}{16\pi^2}$$

We will act with $\mu \frac{d}{d\mu}$



$$\rightarrow \epsilon g_R \cancel{\mu^{\epsilon}} \rightsquigarrow \underbrace{g_R \cancel{\mu^{\epsilon}}\, \mu \frac{dg_R}{d\mu}}_{\mu} + \frac{a}{2\epsilon}\left(\epsilon \cancel{\mu^{\epsilon}} g_R^3 + 3 \cancel{\mu^{\epsilon}} \mu g_R^2 \frac{dg_R}{d\mu}\right)$$

$$\mu \frac{dg_R}{d\mu} = -\epsilon g_R - \frac{a}{2\epsilon}\left[\epsilon g_R^3 + 3 g_R^2 \left(-\epsilon g_R\right)\right]$$

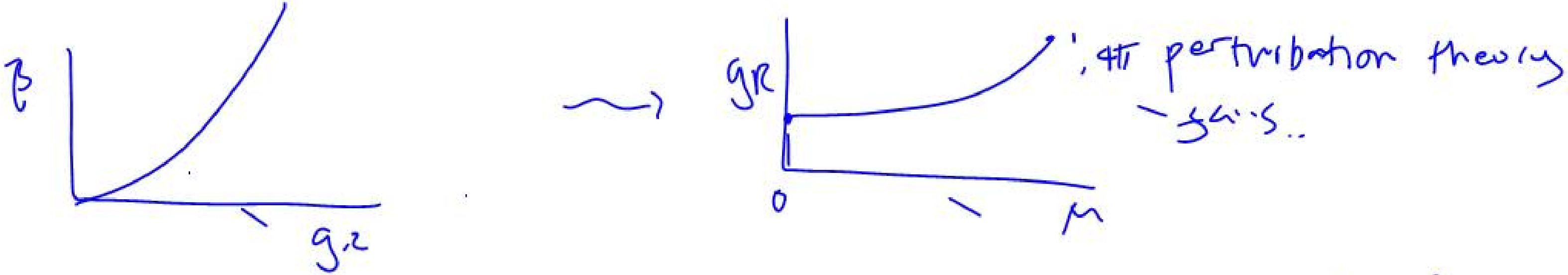
↑ here I've subbed to leading order in g_R from the same eq'n

$$= -\epsilon g_R + a g_R^3 + \dots$$

Now $\epsilon \to 0$

$$\underbrace{\mu \frac{dg_R}{d\mu}}_{\equiv\ \beta} = a g_R^3 = \frac{4}{3} \frac{g_R^3}{16\pi^2} = b \frac{g_R^3}{16\pi^2}$$

Generically $b = \frac{4}{3} \sum_i Q_i^2$



Probably QED is sick & must be replaced at high energy.

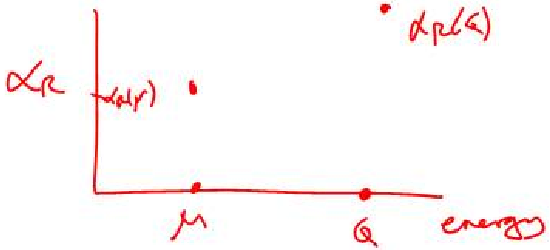
Let's solve $\beta = \frac{\partial g_R}{\partial \ln \mu} = \frac{b\, g_R^3}{16\pi^2}$

$$\frac{d \ln \mu}{d\mu} = \frac{1}{\mu}$$

$$\mu \frac{\partial}{\partial \mu} = \frac{\partial}{\partial \ln \mu}$$

or $\alpha_R = \frac{g_R^2}{4\pi}$, $\frac{\partial \ln \mu^2}{\partial \ln \mu} = 2$

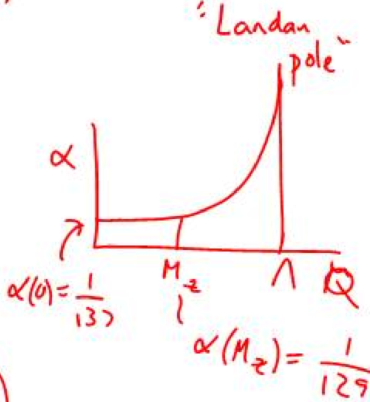
$$\frac{\partial \alpha_R}{\partial \ln \mu^2} = b\, \frac{\alpha_R^2}{4\pi} \Rightarrow \alpha_\mu(\mu^2) = \frac{\alpha_R(Q^2)}{1 + \frac{\alpha_R(Q^2)}{4\pi}\, b \ln \mu^2/Q^2}$$



Now $\displaystyle \frac{1}{\alpha_R(\mu)} = \frac{1}{\alpha_R(Q^2)} \cdot \frac{1}{4\pi} b \ln Q^2$

People usually pick a reference $\mu = \Lambda$ where $\alpha_R(\Lambda) = \infty$

$$\Rightarrow \quad \alpha_R(Q^2) = \frac{4\pi}{b \ln Q^2/\Lambda^2}$$



RG Scheme Dependence

Generically

$$\beta = \overset{g_R}{\overset{\downarrow}{g_R}} \left[b_0 \frac{g_R^2}{16\pi^2} + b_1 \left(\frac{g_R^2}{16\pi^2} \right)^2 + \dots \right]$$

Now if shift scheme

$$g_R' = g_R + \frac{A g_R^3}{16\pi^2}$$

& $\beta' = g_R' \left(b_0' \dfrac{g_R'^2}{16\pi^2} + b_1' \left(\dfrac{g_R'^2}{16\pi^2}\right)^2 + \cdots \right]$

We have $\beta' = \dfrac{d g_R'}{d\ln\mu} = \dfrac{d}{d\ln\mu}\left(g_R + \dfrac{A}{16\pi^2} g_R^3 + \cdots \right)$

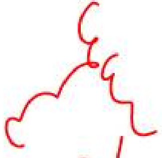
$$= \beta\left(1 + \frac{3A}{16\pi^2} g_R^2 + \cdots\right)$$

$$= \left(1 + \frac{3A}{16\pi^2} g_R^2 + \cdots\right)\left(g_R^3 \frac{b_0}{16\pi^2} + g_R^5 \frac{b_1}{(16\pi^2)^2} + \cdots \right)$$

$$= g_R^3 \frac{b_0}{16\pi^2} + g_R^5 \left(\frac{3A}{16\pi^2}\frac{b_0}{16\pi^2} + \frac{b_1}{(16\pi^2)^2} + \cdots \right)$$

Now $g_R^3 = g_R'^3\left(1 - \dfrac{A g_R'^2}{16\pi^2}\right)^3$ here "$g_R = g_R'$" to the order we want

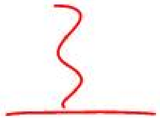
$$= g_R'^3 - 3A g_R'^5/(16\pi^2) + \cdots$$



$$\beta' = g_R^3 \frac{b_0}{16\pi^2} + \frac{g_R^5}{(16\pi^2)^2} b_1 + \dots$$

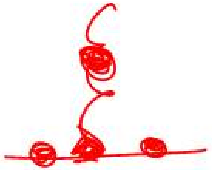
The first two terms are scheme independent...

NAGT Renormalization

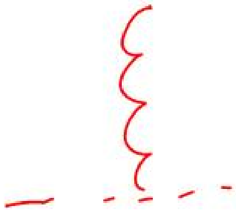


The non-abelian symmetry ensures g renormalizes the same however we compute it...

eg



$$g_R = \frac{Z_3^{1/2} Z_2}{Z_1} g_B$$



$$g_R = \frac{Z_3^{1/2} \tilde{Z}_3}{\tilde{Z}_1} g_0$$

$$g_R = \frac{Z_3^{3/2}}{Z_1^{YM}} g_B$$

$$g_R = \frac{Z_3^{2}}{Z_{4S}} g_B$$