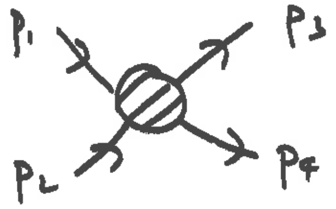
4 ϕ^3 THEORY

This theory is unbounded but good pertb. toy.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - m^2 \phi^2 - \frac{\lambda}{3!}\phi^3$$

real scalar

Let's look at

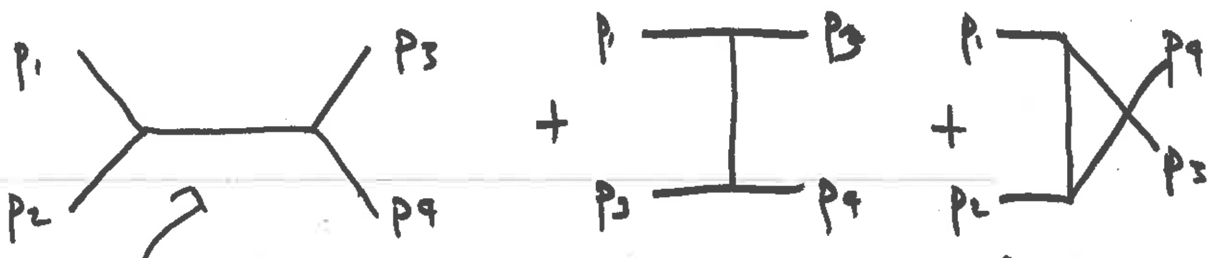


scattering amplitude

Feynman rules to n^{th} order:

1. Draw all graphs with n vertices
(2. $1/\sqrt{Z}$ for each external line)
3. $\dfrac{i}{k^2 - m^2 + i\epsilon}$ for internal propagators
4. $i\lambda$ at vertex
5. $(2\pi)^4 \delta(k_1 + k_2 + k_3)$ per vertex
6. $\int \dfrac{dk_i^4}{(2\pi)^4}$ on any internal momentum k_i

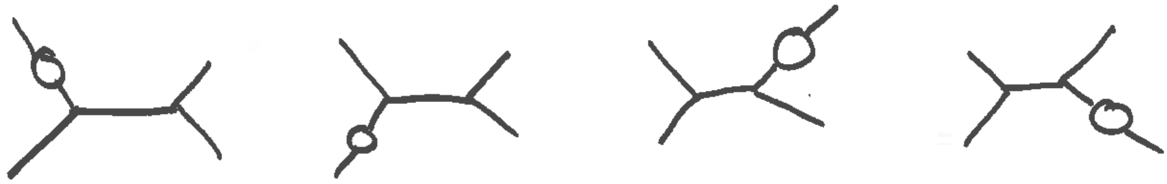
$O(\lambda^2)$:



$$\int \frac{d^4k}{(2\pi)^4} (2\pi)^4 \delta^4(p^2 + p^1 - k)(2\pi)^4 \delta^4(p^3 + p^4 - k) \to (2\pi)^4 \delta^4(p^1 + p^2 - p^3 - p^4)$$

$$\& \quad (+i\lambda)^2 \frac{i}{(s - m^2)} \to \frac{-i\lambda^2}{s - m^2}$$

$O(\lambda^4)$:



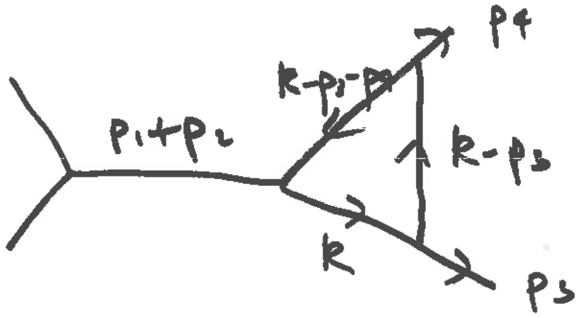
"self energy"

"vertex" "box"

4 integrals, 4 vertex δ^4 – 1 δ fn remains to conserve p^μ

– so one $\int$ remains to be done

VERTEX CORRECTION



$$i\mathcal{M} = \frac{\lambda^4}{(p_1+p_2)^2-m^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2)}\frac{1}{(k-p_3)^2-m^2}\frac{1}{(k-p_3-p_4)^2-m^2}$$

$$p_i^2 = m^2 \qquad (p_1+p_2)^2 = (p_3+p_4)^2 = s$$

$$= \frac{\lambda^4}{s-m^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}\frac{1}{k^2-2k\cdot p_3}\frac{1}{k^2+s-2k\cdot(p_3+p_4)-m^2}$$

FEYNMAN PARAMETERIZATION

We need the integral in the form $\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-A^2)^n}$

so we use some tricks...

$$\frac{1}{a_1} = \int_0^1 d\alpha_1 \frac{\delta(1-\alpha_1)}{a_1 \alpha_1}$$

$$\frac{1}{a_1 a_2} = \int_0^1 \int_0^1 d\alpha_1 \, d\alpha_2 \frac{\delta(1-\alpha_1-\alpha_2)}{(a_1\alpha_1 + a_2\alpha_2)^2}$$

$$= \int_0^1 d\alpha_1 \frac{1}{(a_1\alpha_1 + a_2(1-\alpha_1))^2}$$

$$= \left[\frac{-1}{a_1-a_2} \frac{1}{\alpha_1 a_1 + a_2(1-\alpha_1)}\right]_0^1$$

$$= \frac{1}{a_1-a_2}\left(-\frac{1}{a_1} + \frac{1}{a_2}\right) = \frac{1}{\cancel{a_1-a_2}} \frac{\cancel{a_1-a_2}}{a_1 a_2} \checkmark$$

In general

$$\frac{1}{a_1 a_2 \ldots a_n} = (n-1)! \int_0^1 d\alpha_1 d\alpha_2 \ldots d\alpha_n \frac{\delta(1-\sum_i \alpha_i)}{(\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_n a_n)^n}$$

Back to the vertex correction

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2} \frac{1}{k^2 - 2k.p_3} \frac{1}{k^2 + s - 2k.(p_3+p_4) - m^2}$$

$$= 2\int \frac{d^4k}{(2\pi)^4} d\alpha\, d\beta\, d\alpha \frac{\delta(1-\alpha-\beta-\gamma)}{\left[\underset{+m^2-m^2}{\alpha(k^2-2k.p_3)} + \beta(k^2-2k.(p_3+p_4)+s-m^2) + \gamma(k^2-m^2)\right]^3}$$

Do γ integral to set $\alpha+\beta+\gamma = 1$

$$= 2\int \frac{d^4k}{(2\pi)^4} d\alpha\, d\beta \frac{1}{[k^2 - m^2 - 2k.(p_3(\alpha+\beta) + p_4\beta) + s\beta + \alpha m^2]^3}$$

Now shift momentum $(d^4k \to d^4k')$

$$k^\mu \to k'^\mu = k^\mu - p_3^\mu(\alpha+\beta) - \beta p_4^\mu$$

The cross term in k^2 cancel the odd term in k

$$D = k'^2 - m^2 - (p_3(\alpha+\beta) + \beta p_4)^2 + s\beta + m^2\alpha$$

$\uparrow$ two terms in k^2 & $k.p$ contribute $+1$ & -2!

$$= k'^2 - m^2 - (\alpha+\beta)^2 m^2 - \beta^2 m^2 - 2\beta(\alpha+\beta)\underset{\frac{1}{2}(s-2m^2)}{p_3.p_4} + s\beta + m^2\alpha$$

$$s = (p_3+p_4)^2 = 2m^2 + 2p_3.p_4$$

$$= \underbrace{k'^2 + s\beta(1-\alpha-\beta) - m^2(1-\alpha(1-\alpha))}_{-A^2}$$

So $$iM = \frac{\lambda^4}{s-m^2} \int \frac{d^4k'}{(2\pi)^4} d\alpha\, d\beta \frac{1}{(k'^2-A^2)^3}$$

Now Wick rotate : gain $+i$ & $(k'^2 - A^2)^3 \to -(k_E'^2 + A^2)^3$

$$iM = \frac{-i\lambda^4}{(s-m^2)} \int \frac{d^4k'_E}{(2\pi)^4} \frac{1}{(k_E'^2 + A^2)^3}$$

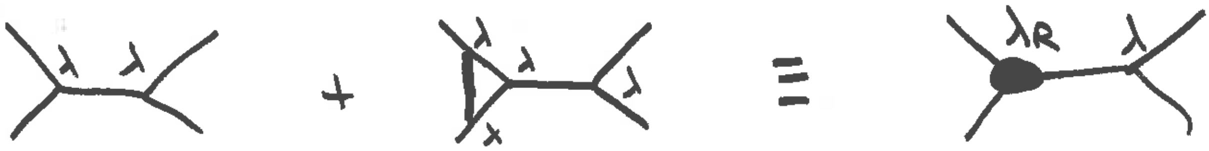
$$= \frac{-i\lambda^4}{s-m^2} \int \frac{k_E'^2 \, dk_E'^2}{16\pi^2} \frac{1}{(k_E'^2 + A^2)^3}$$

$$\sim \int \frac{dk_E'^2}{k_E'^4} \sim \frac{1}{k_E'^2}\Bigg|^{\Lambda}$$

the integration is not divergent here but it does depend on s through A!

$$= \frac{-i\lambda^4}{(s-m^2)} \Delta F(s)$$

The standard trick is to move this s-dependence into an <u>effective</u> tree level coupling



$$\lambda_R(s) = \lambda(1 + \lambda^2 \Delta F(s) + \dots)$$

$$\equiv \frac{\lambda}{Z_1(s)} \qquad Z_1 = 1 - \lambda^2 \Delta F(s)$$

⑤ ϕ^3 SELF ENERGY

We absorbed the vertex correction into a λ_R tree Feynman rule

We now need to worry about self energy corrections

↑ external propagators removed $\longrightarrow$ $-i\,\Sigma(p^2)$

Generically $\quad \Sigma(p^2) \sim A\,p^2 + B$

$\qquad\qquad A(p^2-m^2) + Am^2$

A & B can be divergent & B can depend on p^2

For first term

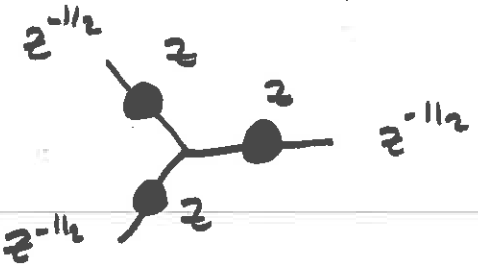
$$= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\,\left(-iA(p^2-m^2)\right)\frac{i}{p^2-m^2} + \dots$$

$$= \frac{i}{p^2-m^2}\,(1+A) \equiv \frac{iZ}{p^2-m^2}$$

This can not be absorbed into the mass. Instead we use the wave function normalization of external state

External wave fns have $Z^{-1/2}$ to cancel. This was the second Feynman rule!

We still have some factors of Z inside diagrams that we suck into λ_R



$$\Rightarrow \quad \lambda_R = \frac{Z^{3/2}\lambda_0}{Z_1(s)}$$

More particularly we need a <u>scheme</u> where we relate λ_R to a physical observable. EG "on mass shell scheme"

$$\Sigma(p^2) = \Sigma(M_R^2) + \frac{\partial \Sigma}{\partial p^2}\bigg|_{p^2=M_R^2} (p^2 - M_R^2) + \Sigma_R(p^2)$$

Taylor Expand

$A = Z - 1$

$M_R^2 = M^2$ to this order in expansion

vanishes ~~quadratically~~ as $(p \to M_R^2)$

$\Sigma(M_R^2)$ & $\Sigma_R(p^2)$ get resummed into the mass as we saw in $\lambda\phi^4$ ... the resummed propagator is

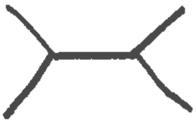
$$\frac{i\,Z}{p^2 \underbrace{- M^2 - \Sigma(M_R^2)}_{-M_R^2} - \Sigma_R(p^2)}$$

vanishes at $p^2 = M_R^2$

this is the physically measured mass

Now we measure the coupling at $s = m_R^2$

$$\underset{\text{measured value.}}{\lambda_R(m_R)} = \frac{\lambda_0 Z^{3/2}}{Z_1(m_R)}$$

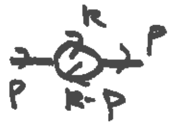


$$-i\mathcal{M} = \frac{-i\lambda_0^2 Z^3}{(s - m_R^2 - \Sigma_R(s))} \frac{1}{Z_1^2}$$

$$= \frac{-i\lambda_R^2}{s - m_R^2 - \Sigma_R(s)} \underbrace{\frac{Z_1^2(m_R)}{Z_1^2(s)}}_{(1 + 2\underbrace{\lambda_0^2}_{=\lambda_R^2 \text{ at this order}} \Delta F_{\bullet}(s) - 2\lambda_0^2 \Delta F_{\bullet}(m_R^2))}$$

The divergent pieces in $\Delta F_{\bullet}$ & the divergent Z are now not in the answer.

SELF ENERGY COMPUTATION



$$e^{i\int d^4x\, \mathcal{L}} \rightarrow 1 + \dots + \frac{1}{2!}\left(\frac{\lambda}{3!}\right)^2 \int d^4x\, \phi^3 \int d^4y\, \phi^3$$

combinatorics : $\dfrac{6\times 2\times 3}{(2\times 1).(3\times 2\times 1)^2} = \dfrac{1}{2}$ (Go back & check it was 1 for vertex!)

So

$$-i\Sigma(p^2) = \frac{1}{2}\lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2)(k^2-2k.p+p^2-m^2)}$$

we won't set $p^2=m^2$ since the diagram might be an internal leg of a diagram... Feynman parameterize:

$$\Sigma(p^2) = \frac{i}{2}\lambda^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha\, d\beta \frac{\delta(1-\alpha-\beta)}{[\beta(k^2-m^2)+\alpha(k^2-2k.p+p^2-m^2)]^2}$$

$$= \frac{i}{2}\lambda^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha \frac{1}{(k^2-m^2-2\alpha k.p+\alpha p^2)^2}$$

Shift : $k^\mu = k'^\mu + \alpha p^\mu$

$$= \frac{i}{2}\lambda^2 \int \frac{d^4k'}{(2\pi)^4} \int_0^1 d\alpha \frac{1}{(\underbrace{k'^2-m^2-\alpha^2p^2+\alpha p^2}_{-A^2 = -(m^2-p^2\alpha(1-\alpha))})^2}$$

Wick Rotate :

$$= -\frac{\lambda^2}{32\pi^2} \int_0^1 d\alpha \int_0^{\Lambda^2} dk_E'^2\, k_E'^2 \Big/ (k_E'^2+A^2)^2$$

$$\Sigma(p^2) = \frac{-\lambda^2}{32\pi^2}\int_0^1 d\alpha \int_0^{\Lambda^2} dk_E^2 \left[\frac{1}{k_E^2+A^2} - \frac{A}{(k_E^2+A^2)^2}\right]$$

$$= \frac{\lambda^2}{32\pi^2}\int_0^1 d\alpha \ \ln\left[\frac{\Lambda^2}{m^2 - \alpha(1-\alpha)p^2}\right] + \text{finite} \ldots$$

constant piece at $p^2 = m_R^2$

$$m_R^2 = m_0^2 + \Sigma(m_R^2)$$

$$= m_0^2 + \frac{\lambda}{32\pi^2}\int_0^1 d\alpha \ \ln\left[\frac{\Lambda^2}{m^2(1-\alpha(1-\alpha))}\right] + \text{finite}$$

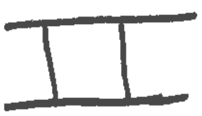
$m = m_R$ at this order in expansion.

$$Z - 1 = \left.\frac{\partial\Sigma}{\partial p^2}\right|_{p^2=m_R^2}$$

$$= \frac{+\lambda^2}{32\pi^2}\int_0^1 d\alpha \ \frac{\alpha(1-\alpha)}{m^2(1-\alpha(1-\alpha))}$$

At this order it's finite - it isn't at higher loops.

Box Diagrams



$$\int k^2 dk^2 \left(\frac{1}{k^2}\right)^4 \sim \frac{1}{\Lambda^4}$$

So they're not divergent... they are messy... let's excuse ouselves!

⑥ RENORMALIZATION

In summary: if we compute an n-point Green's function

$$(2\pi)^4\delta^4(p_1+p_2+\dots p_n)\, G^{(n)}(p_1\dots p_{n-1},\lambda_0,m_0,\Lambda)$$
$$= \int d^4x_1\dots d^4x_n\, e^{i(p_1\cdot x_1\dots p_n x_n)}\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle$$

The result depends on Λ. We shuffle within the expression to get $\lambda_R(\lambda_0,\Lambda)$ etc so

$$G^n_R(p_1\dots p_{n-1},\lambda_R,m_R) = Z^{-n/2}\, G^n(p_1\dots p_n,\lambda_0,m_0,\Lambda)$$

↑ we fix these by a measurement at some scale

Folks often talk about "truncated" or "one-particle irreducible" Green functions - graphs that don't fall in two by one cut

$$\Gamma^n_R(p_1\dots p_{n-1},\lambda_R,m_R) = Z^{n/2}\,\Gamma^n(p_1\dots p_{n-1},\lambda_0,m_0,\Lambda)$$

A lot of people think about renormalization using <u>counter terms</u> though - these are divergent Λ dependent Lagrangian terms whose tree level values cancel one loop divergences.

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{CT}$$

$$\mathcal{L}_R = \tfrac{1}{2}(\partial_\mu \phi_R \partial^\mu \phi_R - m_R^2 \phi_R^2) - \frac{\lambda_R}{3!}\phi_R^3$$

$$\mathcal{L}_{CT} = \tfrac{1}{2}(Z-1)\,\partial_\mu \phi_R \partial^\mu \phi_R - \tfrac{1}{2}(Z_m-1)\,m_R^2 \phi_R^2 - (Z_1-1)\frac{\lambda_R}{3!}\phi_R^3$$

So eg the vertex

 +  $= \lambda_R$

λ_R $\quad Z_1$ $\qquad\qquad (1-Z_1)\lambda_R$

The wave function renormalisation from  still generate the $Z^{N/2}$ factors. Note there's also  in there!

At higher loops  is cancelled by  & so on.... one needs to prove that just m_R, λ_R & ϕ_R can get rid of all divergences at all orders.... Bogoliubov, Parasiuk, Hepp & Zimmerman did....

Most theories are not renormalizable ... generically the superficial degree of divergence in a diagram is

$$\omega = 4L - I_F - 2I_B + \sum_{\text{vertices}} \delta_v$$

$4L$: dk per loop

I_F: internal fermions $\not p/p^2$

$2I_B$: internal bosons $1/p^2$

δ_v: number of derivatives in the F.R.

The number of loops is given by

$$L = I_B + I_F + 1 - V$$

2-2+1

So

$$\omega = 3I_F + 2I_B + 4 + \sum_{\text{vert}} (\delta_v - 4)$$

5+1-4=2

Now consider your vertex and define

$$\eta_v = \delta_v + \tfrac{3}{2} f_v + b_v$$

δ_v: # derivatives

f_v: # fermions

b_v: # bosons

Sum this over all vertices in a diagram

$$\eta_v = \sum_v \delta_v + 3I_f + 2I_B + \tfrac{3}{2}E_f + E_B$$

$3I_f + 2I_B$: internals couple to 2 vertices

$\tfrac{3}{2}E_f + E_B$: externals link to one vertex

So

$$\omega = 4 - \tfrac{3}{2}E_f - E_B + \sum_{\text{vert}} (\eta_v - 4)$$

<u>Non-renormalisable</u> $\eta_v > 4$ For fixed E_F & E_B degree of divergence grows with loop level.

Worse at higher loop divergences appear ~~for~~ graphs with higher $\frac{3}{2}E_F + E_B$ .... eventually need ∞ number of counter terms / couplings to absorb ∞s.

eg $\lambda\phi^5$
$g\bar{\psi}\gamma^\mu\psi\partial_\mu\phi$

<u>Renormalisable</u> $\eta_v = 4$ Divergences in same diagrams & at same level of divergence at different orders

All SM interactions

$\rightarrow$ finite number of renormalised objects

<u>Super-renormalisable</u> $\eta_v < 4$ Fewer diagrams divergent at higher orders.

<u>Link to dimension</u>

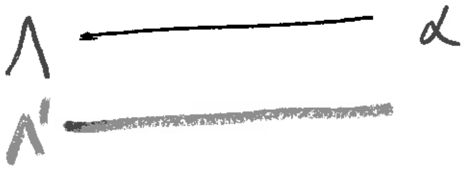
super-renormalisable terms have +ve dimension couplings
eg $m^2\phi^2$

renormalisable terms are dimensionless eg $\bar{\psi} g A \psi$

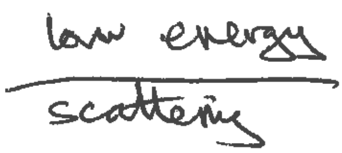
non-renormalisable terms have -ve dimension $G\bar{\psi}\psi\bar{\psi}\psi$
$\uparrow$
dim -2.

Wilsonian Renormalization

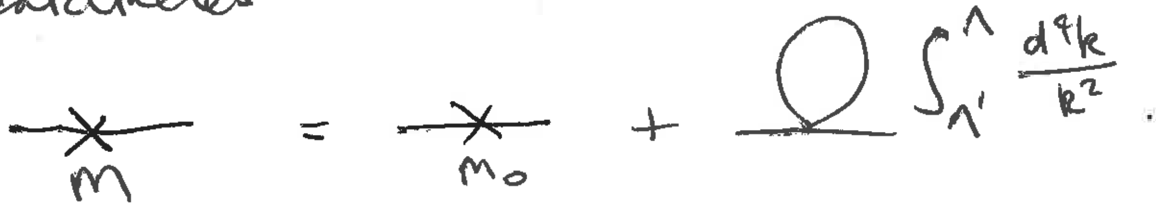
Renormalization felt like a "cheat" for a long while hiding ∞s but Wilson's approach removes that. Accept a $\mathcal{L}$ only describes physics below some scale Λ



Now consider working with a lower Λ' - what action at the lower Λ' gives the same answer for the low energy scattering?

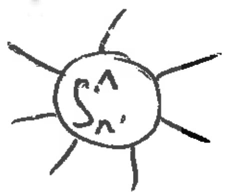


[redacted] loops with energies between Λ & Λ' must be evaluated and treated as shifts of the parameters



This shift is finite in size so there are no ∞s.

You get non-renormalizable terms eg



but these have $\frac{q^2_{low}}{\Lambda^2}$ effects & we drop them (in this framework you can therefore only go down in energy not back up).

If we write all couplings as dimensionless

eg $m^2 = \tilde{m}^2 \Lambda^2 \xrightarrow{\Lambda \to \Lambda'} \tilde{m}^2 \frac{\Lambda^2}{\Lambda'^2} \Lambda'^2$

$g = g \longrightarrow g$

$G = G \Lambda^{-2} \longrightarrow G \frac{\Lambda'^2}{\Lambda^2} \frac{1}{\Lambda'^2}$

$\tilde{m}$ grows as $\Lambda' \to 0$

g stays the same

G falls to zero.

this is the case where quantum loop running is important deciding whether the coupling survives to $\Lambda' \to 0$ or not.

We now see why the standard model only has renormalizable & super-renormalizable operators even if its UV has everything present.