# 10 RUNNING COUPLINGS

## QED $\beta$ FUNCTION

We have ~~seen~~ that in $d = 4 - 2\epsilon$

$$g_B = \mu^\epsilon \frac{\cancel{Z_1}}{\cancel{Z_2} Z_3^{1/2}} g_R \qquad Z_3 = \left(1 - \frac{4}{3}\frac{g_R^2}{16\pi^2}\frac{1}{\epsilon}\right)^{1/2}$$

In MS-scheme: $\quad g_B = \mu^\epsilon g_R \left(1 + \frac{a}{2\epsilon} g_R^2\right) \qquad a = \frac{4}{3}\frac{1}{16\pi^2}$

This contains the famous running coupling result. To see this we will $\mu \frac{d}{d\mu}$ – think of $\mu$ as the scale of the interaction although $\mu$ is a fictious scale & we will eventually get to a $\mu$ independent result.

$$0 = \epsilon g_R \mu^\epsilon + \mu^\epsilon \mu \frac{dg_R}{d\mu} + \frac{a}{2\epsilon}\left[\epsilon \mu^\epsilon g_R^3 + 3\mu^\epsilon \mu g_R^2 \frac{dg_R}{d\mu}\right]$$

move this term to left $\quad \div \mu^\epsilon$

$$\mu \frac{dg_R}{d\mu} = -\epsilon g_R - \frac{a}{2\epsilon}\left[\epsilon g_R^3 + 3 g_R^2 (-\epsilon g_R + \dots)\right]$$

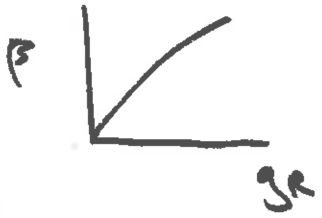
here I've subbed the eq'n into itself to leading order in $g_R$...

$$= -\epsilon g_R + a g_R^3$$

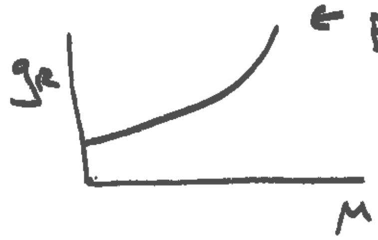
Now when $\epsilon \to 0 \qquad \mu \frac{dg_R}{d\mu} = a g_R^3$

$$\mu \frac{dg_R}{d\mu} = \frac{4}{3}\frac{g_R^3}{16\pi^2} = b\,\frac{g_R^3}{16\pi^2}$$

Generically with many charged fermions $b = \sum_i \frac{4}{3} Q_i^2$




$\Rightarrow$




← perturbation theory fails

We believe such theories are sick & need a UV completion before strong coupling sets in (or could strong coupling physics solve the problem?)

Let's solve
$$\beta = \frac{\partial g_R}{\partial \ln\mu} = b\,\frac{g_R^3}{16\pi^2}$$

or
$$\frac{\partial \alpha_R}{\partial \ln\mu^2} = b\,\frac{\alpha_R^2}{4\pi}$$

$\alpha_R = \frac{g_R^2}{4\pi}$

$\frac{\partial \ln\mu^2}{\partial \ln\mu} = 2$

$$\Rightarrow \quad \alpha_R(\mu) = \frac{\alpha_R(Q^2)}{1 + \frac{\alpha_R(Q^2)}{4\pi}\, b \ln \mu^2/Q^2}$$

$\alpha_R(Q^2)$ & $Q^2$ are integration constants

Now
$$\frac{1}{\alpha_R(\mu)} = \frac{1}{\alpha_R(Q)} + \frac{1}{4\pi}\, b \ln \mu^2/Q^2$$

$$\frac{1}{\alpha_R(\mu)} - \frac{1}{4\pi}\, b\ln\mu^2 = \underbrace{\frac{1}{\alpha_R(Q^2)} - \frac{1}{4\pi}\, b\ln Q^2}_{\text{this just a const}} \equiv -\frac{b}{4\pi}\ln\Lambda^2$$

$$\Rightarrow \quad \alpha_R(Q^2) = \frac{4\pi}{b \ln Q^2/\Lambda^2}$$

This answer, as promised, is independent of $\mu$. $\Lambda$ is the scale where $\alpha_R \to \infty$. "Landau pole"

## RG Scheme Dependence

Generically in some RG-scheme

$$\beta = g_R\left[\beta_0 \frac{g_R^2}{16\pi^2} + \beta_1\left(\frac{g_R^2}{16\pi^2}\right)^2 + \cdots\right]$$

Now if one shifts scheme $\quad g_R' = g_R + A\, g_R^3/16\pi^2$

$$\& \quad \beta' = g_R'\left(\beta_0' \frac{g_R'^2}{16\pi^2} + \beta_1'\left(\frac{g_R'}{16\pi^2}\right)^2 + \cdots\right)$$

We have

$$\beta' = \frac{dg_R'}{d\ln\mu} = \frac{d}{d\ln\mu}\left(g_R + \frac{A}{16\pi^2} g_R^3 + \cdots\right)$$

$$= \beta\left(1 + \frac{3A}{16\pi^2} g_R^2\right)$$

$$= \left(1 + \frac{3A}{16\pi^2} g_R^2\right)\left(g_R^3 \frac{\beta_0}{16\pi^2} + g_R^5 \frac{\beta_1}{(16\pi^2)^2}\right)$$

$$= g_R^3 \frac{\beta_0}{16\pi^2} + g_R^5\left(\frac{3A}{16\pi^2}\frac{\beta_0}{16\pi^2} + \frac{\beta_1}{(16\pi^2)^2}\right)$$

Now $\quad g_R^3 = \left(g_R' - \frac{A g_R^3}{16\pi^2}\right)^3 \quad g_R^3 = g_R'^3$ at this order

$$= g_R'^3 - \frac{3A}{16\pi^2} g_R'^5 + \cdots$$

So $\quad \beta' = g_R'^3 \frac{\beta_0}{16\pi^2} + \frac{g_R'^5}{(16\pi^2)^2}\beta_1 + \cdots \quad$ The first two terms are scheme independent.

## NAGT Renormalization

Review Feynman Rules.

The non-abelian symmetry ensures $g$ renormalizes the same however we compute it

eg  $g_R = \dfrac{Z_3^{1/2} Z_2}{Z_1} g_0$

 $g_R = \dfrac{Z_3^{1/2} \tilde{Z}_5}{\tilde{Z}_1} g_0$

 $g_R = \dfrac{Z_3^{3/2}}{Z_1^{4M}} g_0$

 $g_R = \dfrac{Z_3^{2}}{Z_5} g_0$

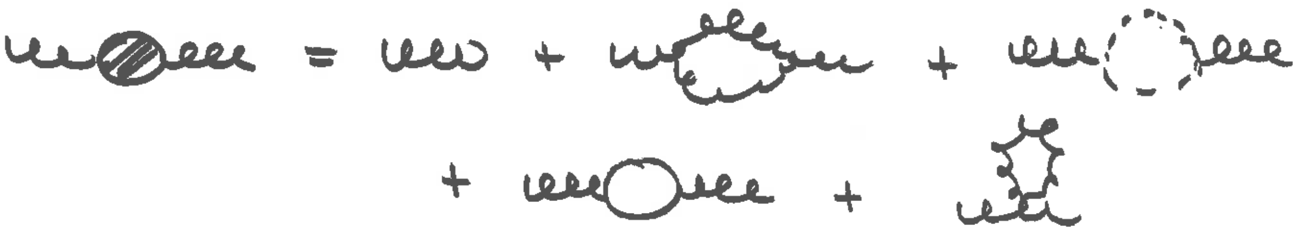
Tadpoles

$\sim \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu}{k^2}$

$\sim 0.$

Let's briefly discuss the components for the first diagram

Gluon Propagator:



Keeping just the poles

$$\Pi_{\mu\nu}^{ab} = \delta^{ab} \frac{g^2}{16\pi^2 \epsilon} (g_{\mu\nu} p^2 - p_\mu p_\nu)\left(5 - \overset{?}{\frac{2N_f}{3}}\right)$$

Group theory: 


$\mathrm{Tr}\, T^a T^b = \frac{1}{2} \delta^{ab}$

 &  $\sim -f^{acd} f^{dcb}$

$= C_A$ quadratic casimir in Adjoint

$= N$ for $SU(N)$

this loop integral is zero in dim. reg.

$$\Rightarrow \quad Z_3 = 1 + \frac{g^2}{16\pi^2 \epsilon}\left(5 - \frac{2N_f}{3}\right) \qquad \text{For } SU(3)$$

## FERMION SELF ENERGY

$\sim (T^c)_{ad} (T^c)_{db}$

$= C_2(F)$ Quadratic casimir in fundamental rep

$= \dfrac{N^2-1}{2N}$ $SU(N)$

Otherwise as in QED: $\quad Z_2 = 1 - \dfrac{g^2}{12\pi^2 \epsilon}$

## VERTEX CORRECTIONS

$\sim \dfrac{C_A}{2}$ $\qquad \sim \left(C_F - \dfrac{C_A}{2}\right)$

$$\Lambda^a_\mu = \frac{g^2}{16\pi^2\epsilon}\,\frac{13}{3}\,\gamma_\mu T^a \qquad \text{For } SU(3)$$

$$Z_1 = 1 - \frac{g^2}{16\pi^2\epsilon}\,\frac{13}{3}$$

Now

$$g_R = \frac{Z_3^{1/2} Z_2}{Z_1}\,\mu^{-\epsilon} g_0$$

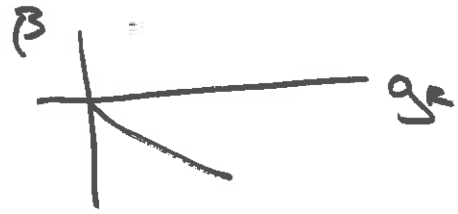
$$= \mu^{-\epsilon} g_0 \left(1 + \frac{g_0^2}{16\pi^2\epsilon}\left[\frac{5}{2} - \frac{N_f}{3} - \frac{4}{3} + \frac{13}{3}\right] + \dots\right)$$

$$= g_0 \mu^{-\epsilon}\left(1 - \frac{g_0^2}{32\pi^2\epsilon}\left(-11 + \frac{2N_f}{3}\right)\right)$$

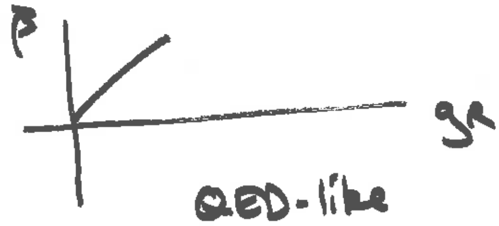
Computing $\beta$ as in QED gives

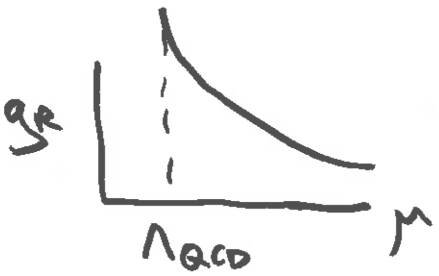
$$\beta = \mu \frac{\partial g_R}{\partial \mu} = \frac{g_R^3}{16\pi^2}\left(-11 + 2N_f/3\right)$$

$N_f < 33/2$




$N_f > 33/2$



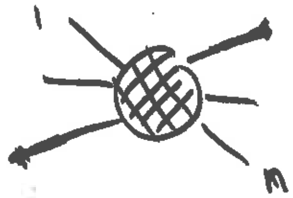




"Asymptotic freedom" $\Rightarrow$ quark confine
$\Rightarrow$ jet production

$$\alpha_R(\mu^2) = \frac{\alpha_0}{1 + \frac{\alpha_0}{4\pi}\beta_0 \ln \mu^2/\mu_0^2}$$

Thresholds: quarks basically contribute when $q^2 > 4m^2$ where they can be pair produced.

# CALLAN - SYMANZIK EQ'N




$$\Gamma_0 \text{ (bare)} = \prod_{i=1}^{n} Z_i^{1/2} \, \Gamma_R \text{ (renormalized quantities)}$$

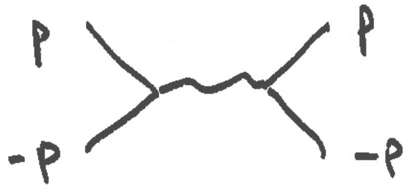
We introduced the arbitrary scale $\mu$ .... so ....

$$\Rightarrow 0 = \frac{\partial}{\partial \ln\mu} \Gamma_R + \beta \frac{\partial \Gamma_R}{\partial g_R} + M_{R_j} \gamma_{m_j} \frac{\partial \Gamma_{R_j}}{\partial M_{R_j}} + \sum_i \gamma_i \Gamma_R$$

$$\gamma_{m_j} = \frac{1}{M_{R_j}} \frac{\partial M_{R_j}}{\partial \ln\mu}$$

$$\gamma_i = \frac{1}{2 Z_i^{1/2}} \frac{\partial Z_i}{\partial \ln\mu}$$

In the very high E limit we can drop all masses...
further in processes like




Then dimensionally $\quad \Gamma_R = p^d \, \tilde{\Gamma}_R$

$\uparrow$ "naive engineering dim of Green's fn"

$\Gamma_R$ also depends on $\mu^2/p^2$ so

$$\frac{\partial}{\partial \ln p} \Gamma_R = \left( d - \frac{\partial}{\partial \ln\mu} \right) \Gamma_R$$

"anomalous dimension of $\Gamma_R$"

Now have

$$\left( -\frac{d}{d\ln p} + \beta \frac{d}{dg_R} + \sum_i \gamma_i + d \right) \Gamma_R = 0$$

This tells you how $\Gamma_R$ changes from one $p$ to another.