Prof Nick Evans

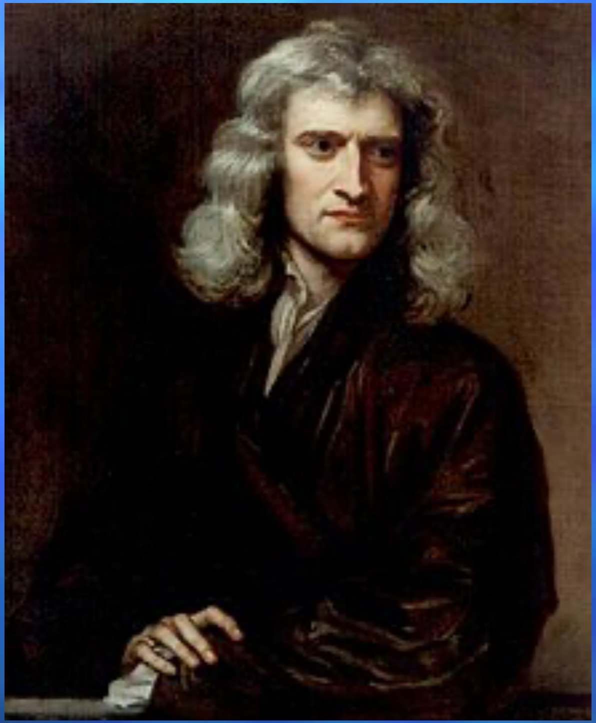
I'm interested in:

The Strong Nuclear Force
Composite models of the Higgs boson
String Theory

We're going to try to teach in segments of 20 minutes of lecture + 20 minutes of problem solving…

www.southampton.ac.uk/~evans/QuantumSchool/

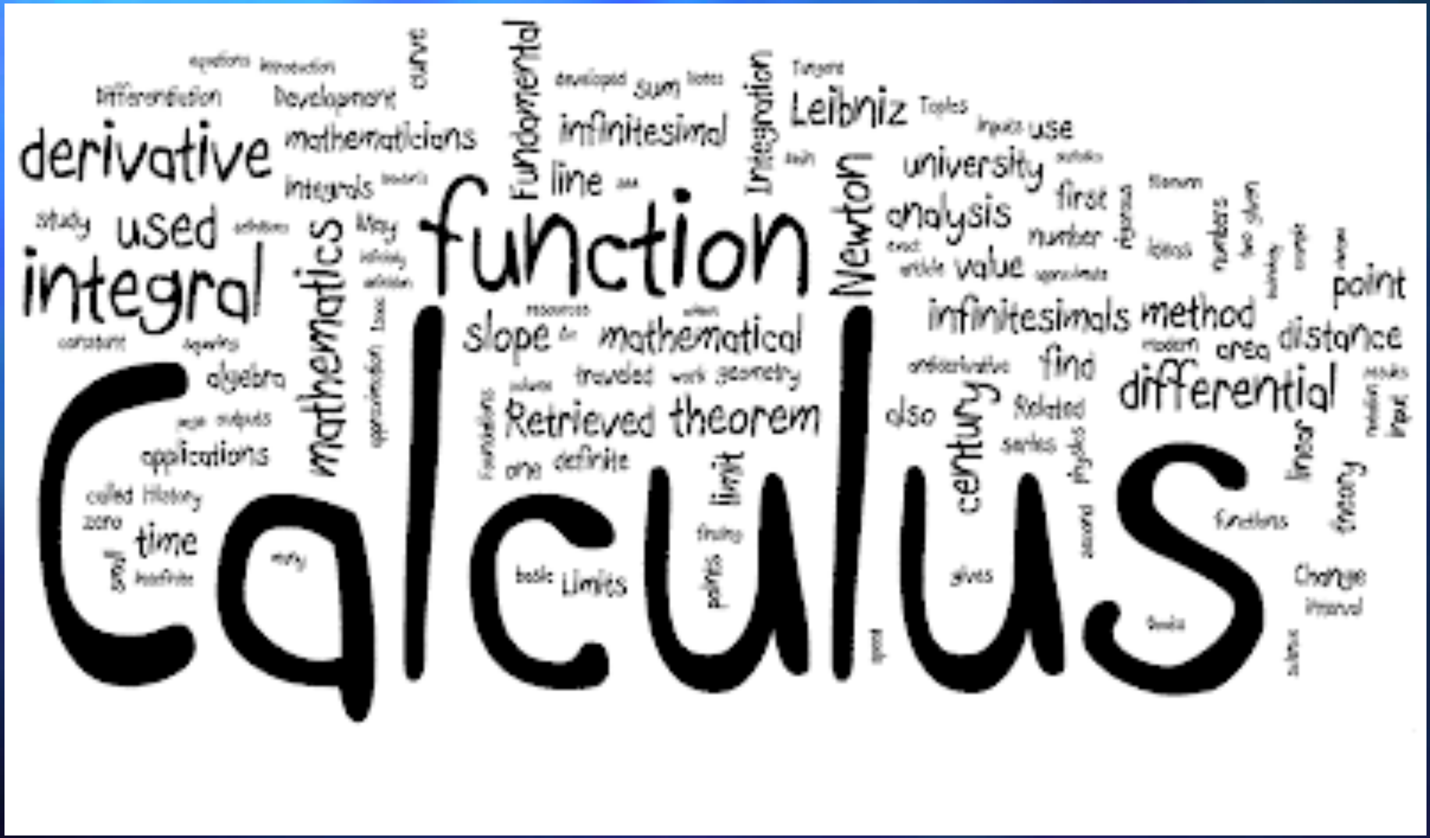
An Introduction to Calculus



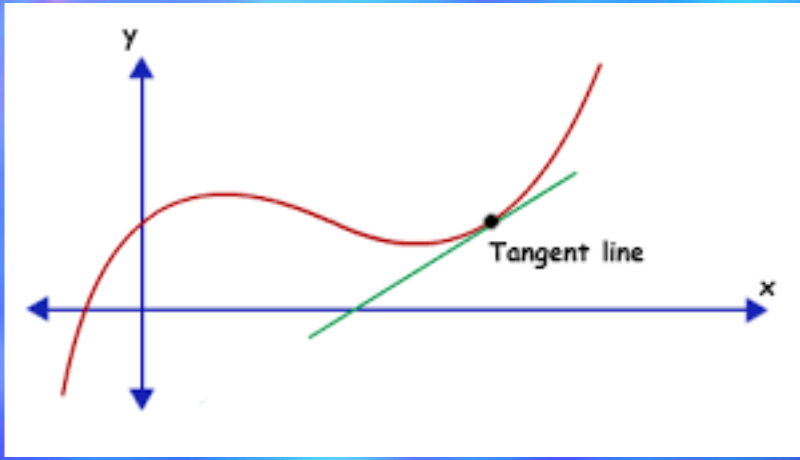
Prof Nick Evans
University of Southampton

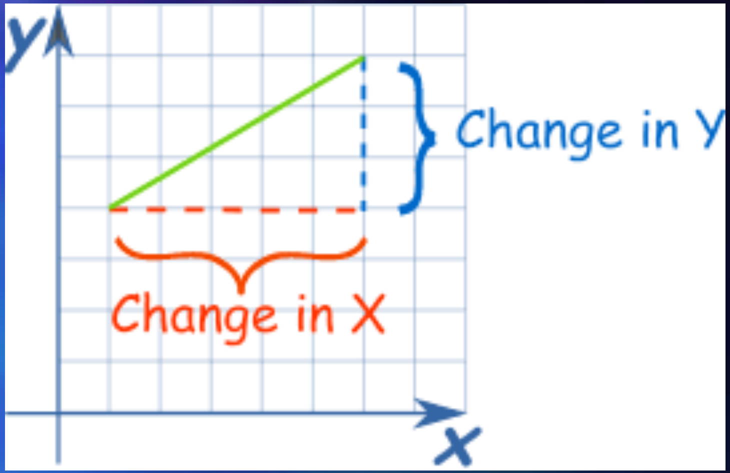
Calculus is a key component of physics – it lets us calculate how things *change* in time or space…

So worth getting the basics under our belt



Derivative = Gradient





Locally any curve is a straight line and we can compute it's gradient as

$$\frac{df(x)}{dx} = \frac{\lim}{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Derivative of x^n

$$\frac{df(x)}{dx} = \frac{\lim}{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

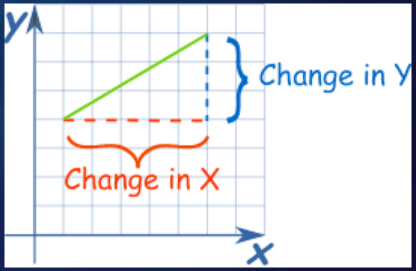
Expand

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

$$(x+\delta x)^n = x^n + nx^{n-1}\delta x + \ldots$$

$$\frac{df(x)}{dx} = \frac{\lim}{\delta x \to 0} \frac{x^n + nx^{n-1}\delta x - x^n}{\delta x}$$

$$\frac{dx^n}{dx} = nx^{n-1}$$



Derivative of cos x

$$\frac{df(x)}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

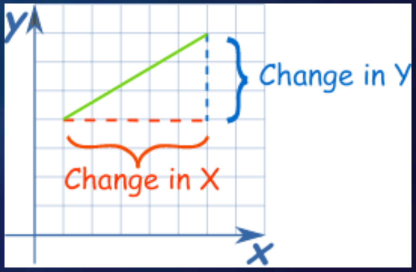
$$\frac{d\cos x}{dx} = \lim_{\delta x \to 0} \frac{\cos(x+\delta x) - \cos x}{\delta x}$$

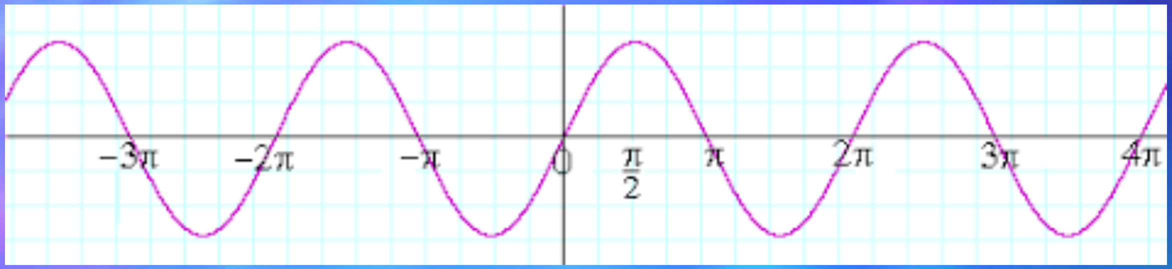
$$= \frac{\cos x \cos \delta x - \sin x \sin \delta x - \cos x.}{\delta x}$$

$$= \frac{\cos x - \sin x \ \delta x - \cos x.}{\delta x}$$

$$= -\sin x$$

$$\frac{d\sin x}{dx} = \cos x$$

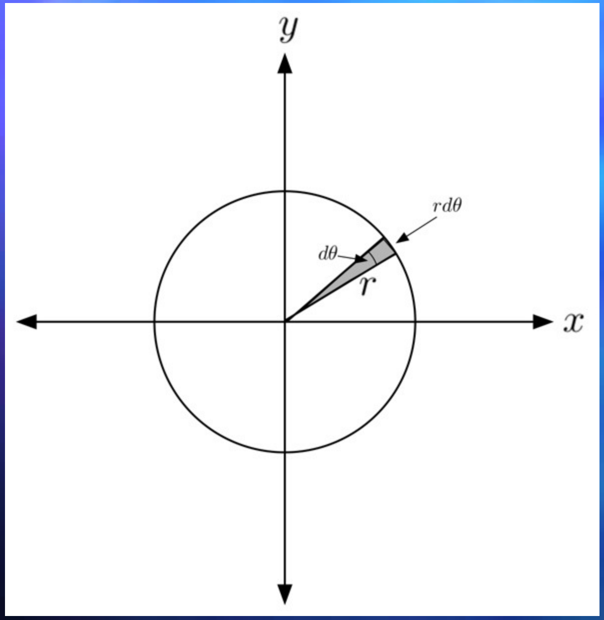




When we said

$\sin dx = dx$

we assumed radians…



In degrees the gradient between 1 and 90 degrees is roughly 1/90… not 1… Here we see

$$\sin \delta\theta = O/H = 2\pi r \quad \delta\theta / \theta_{max} \quad / r$$

So we need $\theta_{max} = 2\pi$

Euler's Number e = 2.718

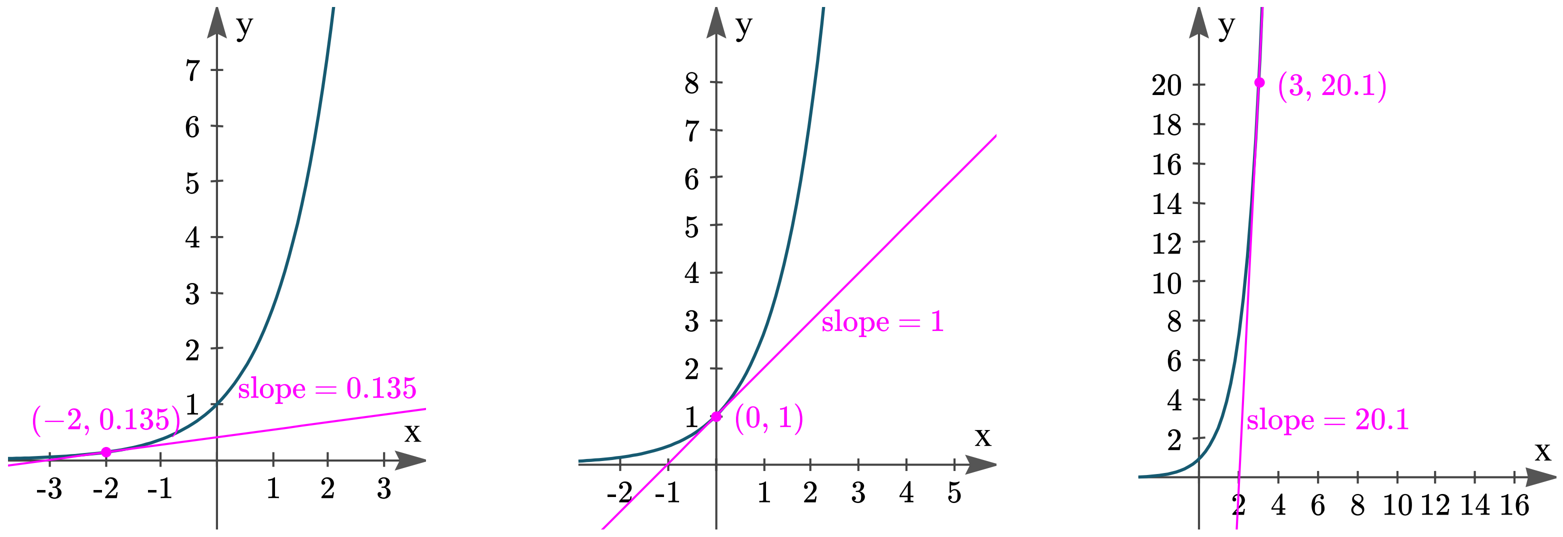
The number e, also known as **Euler's number**, is a mathematical constant approximately equal to 2.71828, and can be characterized in many ways. It is the base of the natural logarithm.[1][2][3] It is the limit of $(1 + 1/n)^n$ as n approaches infinity, an expression that arises in the study of compound interest. It can also be calculated as the sum of the infinite series[4][5]

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots$$

The derivative of e^x is quite remarkable. The expression for the derivative is the same as the expression that we started with; that is, e^x!

$$\frac{d(e^x)}{dx} = e^x$$

What does this mean? It means the slope is the same as the function value (the y-value) for all points on the graph.



When $x = -2$, $y = \text{slope} \approx 0.135$. When $x = 0$, $y = \text{slope} = 1$. When $x = 3$, $y = \text{slope} \approx 20.1$.

$$\frac{d(e^x)}{dx} = e^x$$

Derivatives in Physics

Conservative forces can be written in terms of a potential

$$F = -\frac{dV(x)}{dx}$$

Eg gravitational potential near the earth's surface

$$V = mgh \qquad F = -mg$$

Eg An extended spring

$$V = \frac{1}{2}ke^2 \qquad F = -ke$$

Derivatives in Physics

Conservation of energy

$$E = K + V$$

$$E = \frac{1}{2}mv^2 + V(x)$$

$$\frac{dE}{dt} = \frac{1}{2}m\frac{dv^2}{dv}\frac{dv}{dt} + \frac{dV(x)}{dx}\frac{dx}{dt}$$

Here I’ve used the chain rule…

$$\frac{dE}{dt} = mva + \frac{dV(x)}{dx}v$$

Definitions of v and a

$$\frac{dE}{dt} = v\left(ma + \frac{dV(x)}{dx}\right)$$

Factorised

$$\frac{dE}{dt} = v\left(ma - F\right) = 0$$

Newton’s 2nd law

… and breath…

Time to try Q1, Q2 and Q3 on the problem sheet on the web site below..

Do use the chat to ask questions!

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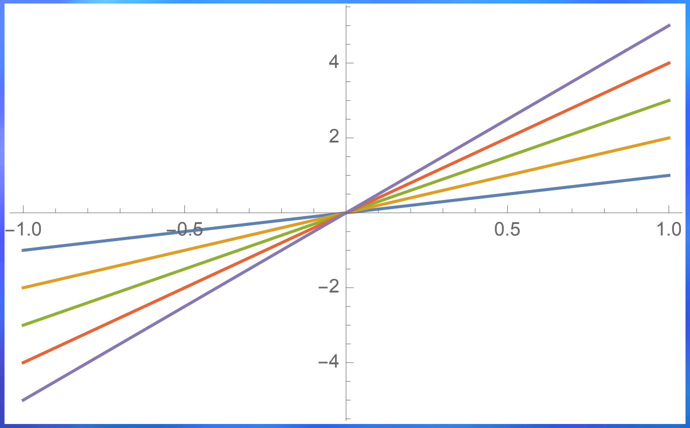
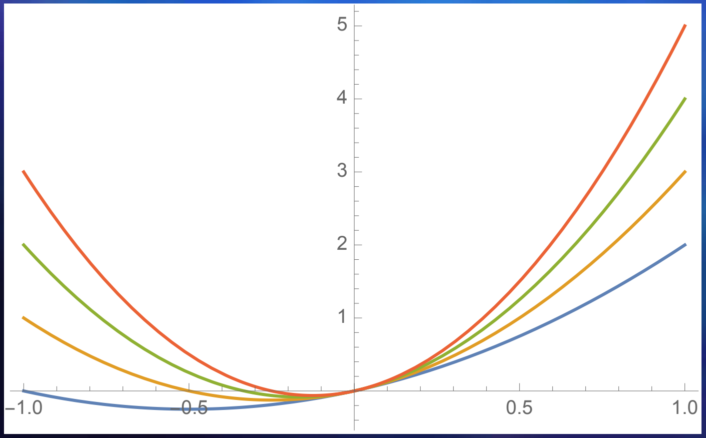
Taylor Expansion

Consider the lines going through the origin… there are an infinite number.. eg



Straight lines: $y=mx$

They all have the same value at x=0… but they have different values of the gradient..



Quadratics: $y=mx + c\,x^2$

They all have the same value and gradient at x=0… but they have different values of the second derivative..

Taylor Expansion

Hopefully you are starting to see that the full set of curves that go through the origin are defined by the set of derivatives

f'(0)... f''(0)... f'''(0).... Etc

Each choice of those infinite number of parameters gives a possible curve... The Taylor expansion directly states this

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + ... + \frac{1}{n!}f^n(0)x^n + ...$$

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + ... + \frac{1}{n!}f^n(0)x^n + ...$$

Evaluate at x= 0 $\quad f(0) = f(0)$

Evaluate $f'(0)$ $\quad f'(0) = f'(0)$

Evaluate $f^n(0)$ $\quad f^n(0) = f^n(0)$

The expansion is normally used though as an expansion… if you only want to know f(x) very close to x=0 then the higher terms in the series are likely small.. You can just keep say three terms and get a good approximation to the function

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + ... + \frac{1}{n!}f^n(0)x^n + ...$$

The most general form is for $f(x_0)$ where x_0 is at some general x value not just x=0.. We just shift by x_0

$$f(x_0) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + ... + \frac{1}{n!}f^n(x_0)(x - x_0)^n + ...$$

Now it’s a good expansion if x lies close to x_0

Example - Mechanics

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \ldots + \frac{1}{n!}f^n(0)x^n + \ldots$$

Let's use it for a particles position as a function of time

$$x \to t \qquad f(x) \to s(t)$$

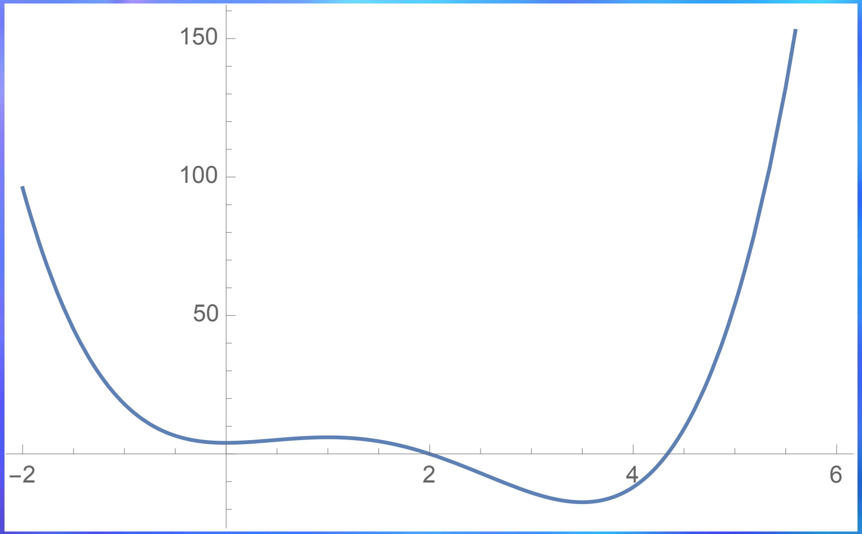
$$s(t) = s(0) + \dot{s}(0)t + \frac{1}{2}\ddot{s}(0)t^2 + \ldots$$

We normally say s(0)=0… the first and second time derivatives at t=0 are u and a…

$$s = ut + \frac{1}{2}at^2$$

Provided all the higher derivatives are strictly zero (constant acceleration) then this is exact for all t…

Example – Why are sine waves so common in nature?



Imagine this is some generic potential V(x).

At equilibrium the particle will sit in the bottom of a well…

When we excite it a little it will oscillate about the minimum…

We can Taylor expand the potential about the minimum

$$V(x) = V(x_0) + \left.\frac{dV}{dx}\right|_{x_0} \delta x + \frac{1}{2} \left.\frac{d^2V}{dx^2}\right|_{x_0} \delta x^2 + \ldots$$

$$V(x) = V(x_0) + \left.\frac{dV}{dx}\right|_{x_0} \delta x + \frac{1}{2} \left.\frac{d^2V}{dx^2}\right|_{x_0} \delta x^2 + \ldots$$

Since x_0 is a minimum the first derivative is zero…

$$V(x) = V(x_0) + \frac{1}{2} \left.\frac{d^2V}{dx^2}\right|_{x_0} \delta x^2 + \ldots$$

Now remember

$$F = -\frac{dV(x)}{dx}$$

So

$$F = -\left.\frac{d^2V}{dx^2}\right|_{x_0} \delta x + \ldots$$

There is a restoring force proportional to the displacement -> SHM

$$F = - \left. \frac{d^2V}{dx^2} \right|_{x_0} \delta x + \ldots$$

$$m\ddot{\delta x} = -V^{''}\delta x$$

$$\ddot{\delta x} = -\omega^2 \delta x$$

$$\omega^2 = \frac{V^{''}}{m}$$

$$\delta x = A \sin(\omega t + \phi)$$

… and breath…

Time to try Q4 and Q5 on the problem sheet on the web site below..

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Integration

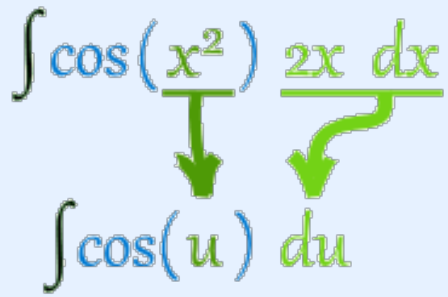
Integration is the inverse of differentiation… so for example

$$\int \cos x \; dx = \sin x + \text{constant}$$

Alas there is no way to compute these from first principles… traditionally mathematicians would differentiate things and then make books of the results that could be used to reverse the calculation… now we use computer programs to remember for us…

Sometimes you can guess a change of variables that takes you an answer you already know…

Example: $\int \cos(x^2)\ 2x\ dx$



Now integrate:

$$\int \cos(u)\ du = \sin(u) + C$$

And finally put $\mathbf{u=x^2}$ back again:

$$\sin(x^2) + C$$

Integration by Parts

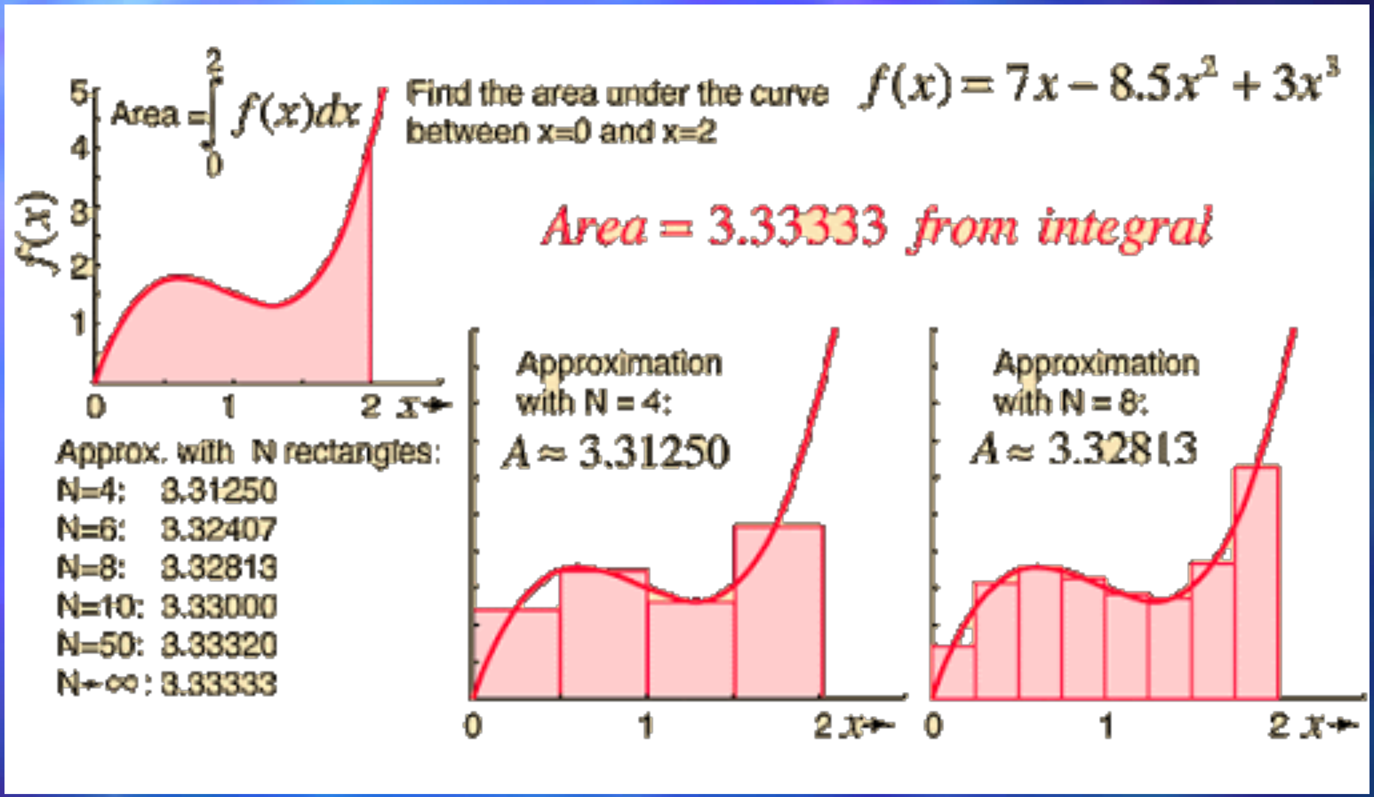
This trick lets you massage some products to a form you know…

$$\frac{d(u\ v)}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$

$$u.v = \int \frac{du}{dx}v\ dx\ + \int u\frac{dv}{dx}\ dx$$

$$\int \frac{du}{dx}v\ dx = u\ v - \int u\frac{dv}{dx}\ dx + \text{constant}$$

Integration is Area Under a Curve



$$\int_0^2 (7x - 8.5x^2 + 3x^3)\ dx = \left[\frac{7}{2}x^2 - \frac{8.5}{3}x^3 + \frac{3}{4}x^4\right]_0^2$$

$$= \left(\frac{7\times 2^2}{2} - \frac{8.5\times 2^3}{3} + \frac{3\times 2^4}{4}\right) - 0 = 3.33333$$

Of course there's a lot more to say but this gets us started....

Have a go at Q6 – we'll provide answer to all the questions before we meet next time…

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