BUSSTEPP 2002

Lattice Methods in Field Theory: Problem Sheet 1 (Tutors)

Question 1. Lattice derivatives

Use the definitions of the forward and backward lattice derivatives to verify the following relations:

$$[\nabla_\mu, \nabla_\nu] = [\nabla^*_\mu, \nabla^*_\nu] = [\nabla_\mu, \nabla^*_\nu] = 0$$

and

$$\nabla_\mu\{f(x)g(x)\} = \{\nabla_\mu f(x)\}g(x) + f(x)\{\nabla_\mu g(x)\} + a\{\nabla_\mu f(x)\}\{\nabla_\mu g(x)\}$$

(*no* summation over μ in the second result)

Assuming translational invariance in all spacetime dimensions, show that:

$$\begin{aligned}\sum_{x\in\Lambda_E} g(x)\Delta f(x) &= -\sum_{x\in\Lambda_E}\sum_{\mu=0}^{3}\{\nabla_\mu g(x)\}\{\nabla_\mu f(x)\} \\ &= -\sum_{x\in\Lambda_E}\sum_{\mu=0}^{3}\{\nabla^*_\mu g(x)\}\{\nabla^*_\mu f(x)\}\end{aligned}$$

where $\Delta = \sum_{\mu=0}^{3}\nabla_\mu\nabla^*_\mu = \sum_{\mu=0}^{3}\nabla^*_\mu\nabla_\mu$ is the discretised Laplacian

Answer 1. Use the definitions

$$\nabla_\mu f(x) = \frac{1}{a}[f(x+a\hat{\mu}) - f(x)], \qquad \nabla^*_\mu f(x) = \frac{1}{a}[f(x) - f(x-a\hat{\mu})]$$

and check explicitly

$$\begin{aligned}[\nabla_\mu, \nabla^*_\nu]f(x) &= \nabla_\mu\nabla^*_\nu f(x) - \nabla^*_\nu\nabla_\mu f(x) \\ &= \frac{1}{a^2}\Big\{f(x+a\hat{\mu}) - f(x) - f(x-a\hat{\nu}+a\hat{\mu}) + f(x-a\hat{\nu}) \\ &\quad - f(x+a\hat{\mu}) + f(x+a\hat{\mu}-a\hat{\nu}) + f(x) - f(x-a\hat{\nu})\Big\} \\ &= 0\end{aligned}$$

Check similarly for $[\nabla_\mu, \nabla_\nu]$, $[\nabla^*_\mu, \nabla^*_\nu]$.

For second result:

$$\begin{aligned}\nabla_\mu\{f(x)g(x)\} &= \frac{1}{a}\{f(x+a\hat{\mu})g(x+a\hat{\mu}) - f(x)g(x)\} \\ &= \frac{1}{a}\Big\{[f(x)+a\nabla_\mu f(x)][g(x)+a\nabla_\mu g(x)] - f(x)g(x)\Big\} \\ &= \{\nabla_\mu f(x)\}g(x) + f(x)\{\nabla_\mu g(x)\} + a\{\nabla_\mu f(x)\}\{\nabla_\mu g(x)\}\end{aligned}$$

For the last part, first observe that

$$\begin{aligned}\sum_{x\in\Lambda_E} g(x)\{\nabla_\mu f(x)\} &= \frac{1}{a}\sum_{x\in\Lambda_E}\{g(x)f(x+a\hat{\mu}) - g(x)f(x)\} \\ &= \frac{1}{a}\sum_{x\in\Lambda_E}\{g(x-a\hat{\mu})f(x) - g(x)f(x)\} \\ &= -\sum_{x\in\Lambda_E}\{\nabla^*_\mu g(x)\}f(x)\end{aligned}$$

where we used translation invariance to change $g(x)f(x+a\hat{\mu})$ into $g(x-a\hat{\mu})f(x)$ inside the summation (this works with both periodic or antiperiodic boundary conditions). Now check:

$$\sum_{x\in\Lambda_E} g(x)\Delta f(x) = \sum_{x\in\Lambda_E}\sum_{\mu=0}^{3} g(x)\nabla_\mu\nabla_\mu^* f(x) = -\sum_{x\in\Lambda_E}\sum_{\mu=0}^{3}\{\nabla_\mu^* g(x)\}\{\nabla_\mu^* f(x)\}$$

and similarly for $\Delta = \nabla_\mu^*\nabla_\mu$.

Question 2. Generating functional and Feynman propagator

Verify the relation

$$\exp(W[J]) = \exp\left\{\frac{1}{2}(J, K^{-1}J)\right\}$$

where $(\cdot,\cdot)$ denotes the scalar product of lattice scalar fields

Use the following steps to derive an expression for $\exp\{\frac{1}{2}(J,K^{-1}J)\}$ in terms of the scalar field propagator on the lattice.

(a) Recall that $K = -\nabla_\mu^*\nabla_\mu + m^2$ and compute $KJ(x)$ using the Fourier transform for $J(x)$. Deduce that K acts by multiplication with $\hat{p}^2 + m^2$, where

$$\hat{p}_\mu = \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right)$$

(b) Use the result in (a) and the Fourier transform for $J(x)$ to show that

$$(K^{-1}J)(x) = a^4\sum_{y\in\Lambda_E} G(x-y)J(y)$$

where

$$G(x-y) = \frac{1}{a^4L^3T}\sum_{p\in\Lambda_E^*}\frac{\mathrm{e}^{ip\cdot(x-y)}}{\hat{p}^2+m^2}$$

Write down the expression for $\mathrm{e}^{W[J]}$ in terms of $G(x-y)$.

Answer 2. The generating functional is

$$\begin{aligned}\mathrm{e}^{W[J]} &= \langle \mathrm{e}^{(J,\phi)}\rangle = \frac{1}{Z_E}\int\prod_{x\in\Lambda_E} d\phi(x)\mathrm{e}^{-S_E[\phi]+(J,\phi)} \\ &= \frac{1}{Z_E}\int\prod_{x\in\Lambda_E} d\phi(x)\exp\left\{-\frac{1}{2}(\phi,K\phi)+(J,\phi)\right\}\end{aligned}$$

In the exponent [using $(\theta, A\phi) = (A^{\mathrm{T}}\theta,\phi)$]:

$$\begin{aligned}-\frac{1}{2}(\phi,K\phi)+(J,\phi) &= -\frac{1}{2}\left(\phi-(K^{-1})^{\mathrm{T}}J, K(\phi-K^{-1}J)\right)+\frac{1}{2}(J,K^{-1}J) \\ &= -\frac{1}{2}(\phi',K\phi')+\frac{1}{2}(J,K^{-1}J)\end{aligned}$$

using $\phi' = \phi - K^{-1}J$. Now shift integration variables (Jacobian is 1):

$$\mathrm{e}^{W[J]} = \frac{1}{Z_E}\int\prod_{x\in\Lambda_E} d\phi'(x)\mathrm{e}^{-S_E[\phi']}\mathrm{e}^{\frac{1}{2}(J,K^{-1}J)} = \mathrm{e}^{\frac{1}{2}(J,K^{-1}J)} \tag{1}$$

since the ϕ' integration produces $Z_E = (\det K)^{-1/2}$.

(a) Action of K on $J(x)$:

$$\begin{aligned} KJ(x) &= \frac{1}{a^4L^3T}\sum_{p\in\Lambda_E^*}(-\nabla_\mu^*\nabla_\mu+m^2)\mathrm{e}^{ip\cdot x}\tilde{J}(p) \\ &= \frac{1}{a^4L^3T}\sum_{p\in\Lambda_E^*}\left[\sum_{\mu=0}^{3}\frac{1}{a^2}\big(-\mathrm{e}^{ip\cdot(x+a\hat{\mu})}-\mathrm{e}^{ip\cdot(x-a\hat{\mu})}+2\mathrm{e}^{ip\cdot x}\big)+m^2\mathrm{e}^{ip\cdot x}\right]\tilde{J}(p) \\ &= \frac{1}{a^4L^3T}\sum_{p\in\Lambda_E^*}\left[\sum_{\mu=0}^{3}\Big(\frac{-\mathrm{e}^{ip\cdot a\hat{\mu}}-\mathrm{e}^{-ip\cdot a\hat{\mu}}+2}{a^2}\Big)+m^2\right]\mathrm{e}^{ip\cdot x}\tilde{J}(p) \end{aligned}$$

This simplifies using $p\cdot a\hat{\mu}=ap_\mu$ and

$$\frac{-\mathrm{e}^{ip\cdot a\hat{\mu}}-\mathrm{e}^{-ip\cdot a\hat{\mu}}+2}{a^2}=\frac{2}{a^2}\big(1-\cos(ap_\mu)\big)=\frac{4}{a^2}\sin^2(ap_\mu/2)=\hat{p}_\mu^2$$

so that

$$KJ(x)=\frac{1}{a^4L^3T}\sum_{p\in\Lambda_E^*}(\hat{p}^2+m^2)\mathrm{e}^{ip\cdot x}\tilde{J}(p)$$

(b) Now easy to invert:

$$K^{-1}J(x)=\frac{1}{a^4L^3T}\sum_{p\in\Lambda_E^*}\frac{\mathrm{e}^{ip\cdot x}}{\hat{p}^2+m^2}\tilde{J}(p)=a^4\sum_{y\in\Lambda_E}\left(\frac{1}{a^4L^3T}\sum_{p\in\Lambda_E^*}\frac{\mathrm{e}^{ip\cdot(x-y)}}{\hat{p}^2+m^2}\right)J(y)$$

so that

$$G(x-y)=\frac{1}{a^4L^3T}\sum_{p\in\Lambda_E^*}\frac{\mathrm{e}^{ip\cdot(x-y)}}{\hat{p}^2+m^2} \tag{2}$$

Finally subsitute for K^{-1} in equation 1 using the result in 2:

$$e^{W[J]}=\exp\left\{\frac{a^8}{2}\sum_{x,y\in\Lambda_E}J(x)G(x-y)J(y)\right\}$$

Question 3. Zero 3-momentum propagator of a free scalar field in configuration space

Let

$$C(x_0) = a^3 \sum_{\mathbf{x}} \langle \phi(x_0, \mathbf{x}) \phi(0) \rangle = a^3 \sum_{\mathbf{x}} G(x_0, \mathbf{x})$$

This is a 2-point correlator where a particle is created at the origin and propagates to any point on the timeslice labelled by t where $x_0 = ta$. The sum on $\mathbf{x}$ projects onto zero 3-momentum. Show that

$$C(x_0) = \frac{1}{aT} \sum_{p_0} \frac{e^{ip_0 x_0}}{\hat{p}_0^2 + m^2}$$

Now for simplicity let $T \to \infty$ so that

$$\frac{1}{aT} \sum_{p_0} \to \int_{-\pi/a}^{\pi/a} \frac{dp_0}{2\pi}$$

leaving us to evaluate:

$$C(x_0) = \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} \frac{dp_0 e^{ip_0 x_0}}{\hat{p}_0^2 + m^2}$$

We can do this by contour integration.

(a) Show that the integrand has poles at $p_0 = \pm i\overline{m} + 2n\pi/a$, for $n \in \mathbb{Z}$, where

$$\sinh\left(\frac{a\overline{m}}{2}\right) = \frac{am}{2}$$

(b) For $x_0 > 0$ close the contour in the upper half plane with lines C_1 from $\pi/a + i0$ to $\pi/a + i\infty$ and C_2 from $-\pi/a + i\infty$ to $-\pi/a + i0$. The contributions from C_1 and C_2 cancel because of the periodicity of the integrand. For $x_0 < 0$ close the contour in the lower half plane in a similar way. Hence show that

$$C(x_0) = \frac{e^{-\overline{m}|x_0|}}{2m(1 + m^2a^2/4)^{1/2}}$$

This result shows that (at least on a Euclidean lattice with infinite time extent) the free scalar two-point function decays exponentially with time. Viewed as a function of the timeslice label t, the exponent fixes $\overline{m}a$ (lattice calculations produce dimensionless numbers). Compare this to the general discussion of two-point functions later in the lectures.

Answer 3. Using $\sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} = L^3 \delta_{\mathbf{p},\mathbf{0}}$,

$$C(x_0) = a^3 \sum_{\mathbf{x}} \frac{1}{a^4 L^3 T} \sum_{p \in \Lambda_E^*} \frac{e^{ip_0 x_0 + i\mathbf{p}\cdot\mathbf{x}}}{\hat{p}_0^2 + \hat{\mathbf{p}}^2 + m^2} = \frac{1}{aT} \sum_{p_0} \frac{e^{ip_0 x_0}}{\hat{p}_0^2 + m^2}$$

(a) Look for zeros of $\hat{p}_0^2 + m^2 = 4\sin^2(ap_0/2)/a^2 + m^2$. If $p_0 = \pm i\overline{m}$, then we want

$$-4\sinh^2(a\overline{m}/2)/a^2 + m^2 = 0 \qquad \text{or} \qquad \sinh\left(\frac{a\overline{m}}{2}\right) = \frac{am}{2}$$

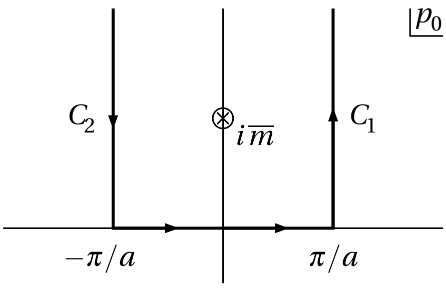
Note: $\overline{m} = E(\mathbf{p} = \mathbf{0})$ is called the *physical mass*, where the dispersion relation is

$$\sinh^2\left(\frac{aE(\mathbf{p})}{2}\right) = \frac{a^2m^2}{4} + \sum_{i=1}^{3}\sin^2\left(\frac{ap_i}{2}\right)$$

Accounting for periodicity we find poles for

$$p_0 = \pm i\overline{m} + \frac{2n\pi}{a}, \qquad n \in \mathbb{Z}$$

(b) For $x_0 > 0$ close the contour as shown and use $\int_{C_1} = -\int_{C_2}$. The residue at $i\overline{m}$ is found by,



$$\begin{aligned}
f(p_0) &= \frac{4}{a^2}\sin^2\left(\frac{ap_0}{2}\right) + m^2 \\
\frac{\partial f}{\partial p_0} &= \frac{4}{a^2}2\sin\left(\frac{ap_0}{2}\right)\cos\left(\frac{ap_0}{2}\right)\frac{a}{2} \\
\left.\frac{\partial f}{\partial p_0}\right|_{p_0 = i\overline{m}} &= \frac{4i}{a}\sinh\left(\frac{a\overline{m}}{2}\right)\sqrt{1+\sinh^2(a\overline{m}/2)} \\
&= 2im\sqrt{1+m^2a^2/4}
\end{aligned}$$

From the residue theorem

$$C(x_0) = \frac{1}{2\pi}2\pi i \frac{\mathrm{e}^{-\overline{m}x_0}}{2im\sqrt{1+m^2a^2/4}}$$

Including the case $x_0 < 0$ gives

$$C(x_0) = \frac{\mathrm{e}^{-\overline{m}|x_0|}}{2m\sqrt{1+m^2a^2/4}}$$

The exponential falloff is governed by the physical mass. The same is true more generally by the spectral decomposition of two-point functions: two-point functions are used to extract particle masses.

For finite T you would get the modification $\mathrm{e}^{-\overline{m}x_0} \to \mathrm{e}^{-\overline{m}x_0} + \mathrm{e}^{-\overline{m}(Ta-x_0)}$.

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Lattice Methods in Field Theory: Problem Sheet 2 (Tutors)

Question 1. Naive discretisation of the Yang-Mills action

Consider the transformation law of the nonabelian gauge potential in the continuum

$$A_\mu(x) \to g(x)A_\mu(x)g^{-1}(x) + g(x)\partial_\mu g^{-1}(x), \qquad g(x) \in SU(N), \qquad x \in \mathbb{R}^4$$

and its naive transcription to the lattice:

$$A_\mu(x) \to g(x)A_\mu(x)g^{-1}(x) + g(x)\nabla_\mu g^{-1}(x), \qquad g(x) \in SU(N), \qquad x \in \Lambda_E$$

Show that the continuum transformation law is *not* reproduced using the naive transcription. Hint: apply a gauge transformation $g(x) = g_1(x) \cdot g_2(x)$ in the continuum and on the lattice and compare the results.

Answer 1. Continuum: let $A_\mu^g = gA_\mu g^{-1} + g\partial_\mu g^{-1}$ and let $g = g_1 \cdot g_2$ (where g_2 acts first) and compare

$$\begin{aligned} A_\mu^{g_1 g_2} &= g_1 g_2 A_\mu g_2^{-1} g_1^{-1} + g_1 g_2 (\partial_\mu g_2^{-1} \cdot g_1^{-1} + g_2^{-1} \partial_\mu g_1^{-1}) \\ (A_\mu^{g_2})^{g_1} &= g_1 (g_2 A_\mu g_2^{-1} + g_2 \partial_\mu g_2^{-1}) g_1^{-1} + g_1 \partial_\mu g_1^{-1} = A_\mu^{g_1 g_2} \end{aligned}$$

$\longrightarrow$ compatible in the continuum.

Lattice: let $A_\mu^g = gA_\mu g^{-1} + g\nabla_\mu g^{-1}$ and use a result from Problem Sheet 1, Q1

$$\begin{aligned} A_\mu^{g_1 g_2} &= g_1 g_2 A_\mu g_2^{-1} g_1^{-1} + g_1 g_2 \nabla_\mu (g_2^{-1} g_1^{-1}) \\ &= g_1 g_2 A_\mu g_2^{-1} g_1^{-1} + g_1 g_2 (\nabla_\mu g_2^{-1} \cdot g_1^{-1} + g_2^{-1} \cdot \nabla_\mu g_1^{-1}) + a g_1 g_2 \cdot \nabla_\mu g_2^{-1} \cdot \nabla_\mu g_1^{-1} \\ (A_\mu^{g_2})^{g_1} &= g_1 (g_2 A_\mu g_2^{-1} + g_2 \nabla_\mu g_2^{-1}) g_1^{-1} + g_1 \nabla_\mu g_1^{-1} = A_\mu^{g_1 g_2} + O(a) \end{aligned}$$

$\longrightarrow$ not compatible: gauge invariance is broken by lattice artifacts.

Question 2. Wilson plaquette action

For gauge group $SU(N)$, let $A_\mu(x)$ be a given gauge potential in the continuum which defines a link variable $U_\mu(x)$ through

$$U_\mu(x) = \mathrm{e}^{aA_\mu(x)}, \qquad A_\mu^\dagger(x) = -A_\mu(x)$$

Using this definition, derive the following result for the trace of the plaquette:

$$\begin{aligned} \mathrm{Tr}\, P_{\mu\nu}(x) &= \mathrm{Tr}\left\{ U_\mu(x) U_\nu(x+a\hat{\mu}) U_\mu^\dagger(x+a\hat{\nu}) U_\nu^\dagger(x) \right\} \\ &\overset{a\to 0}{=} N + \frac{a^4}{2} \mathrm{Tr}\,(F_{\mu\nu}(x) F_{\mu\nu}(x)) + O(a^5) \end{aligned}$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)]$$

is the field tensor in the continuum.

Hint: write $\mathrm{Tr}\, P_{\mu\nu}$ as $\mathrm{Tr}(S{\cdot}T)$ where

$$S = U_\nu^\dagger(x) U_\mu(x), \qquad T = U_\nu(x{+}a\hat{\mu}) U_\mu^\dagger(x{+}a\hat{\nu})$$

and apply the Baker-Campbell-Hausdorff formula

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\frac{1}{12}[B,[B,A]]+\cdots}$$

separately to S and T. Use relations of the kind

$$A_\nu(x + a\hat{\mu}) = A_\nu(x) + a\partial_\mu A_\nu(x) + O(a^2)$$

and expand the product $S{\cdot}T$ in powers of a before taking the trace.

Answer 2. Write $\mathrm{Tr}\, P_{\mu\nu}(x) = \mathrm{Tr}(S \cdot T)$ as in the question and use:

$$\begin{aligned}
S &= U_\nu^\dagger(x) U_\mu(x) = \mathrm{e}^{-aA_\nu(x)} \mathrm{e}^{aA_\mu(x)} \\
&\overset{\mathrm{BCH}}{=} \exp\left\{ a(A_\mu - A_\nu) + \frac{1}{2} a^2 [A_\mu, A_\nu] + O(a^3) \right\} \\
T &= U_\nu(x + a\hat{\mu}) U_\mu^\dagger(x + a\hat{\nu}) = \mathrm{e}^{aA_\nu(x+a\hat{\mu})} \mathrm{e}^{-aA_\mu(x+a\hat{\nu})} \\
&\overset{\mathrm{BCH}}{=} \exp\left\{ -a(A_\mu - A_\nu) + a^2(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2} a^2 [A_\mu, A_\nu] + O(a^3) \right\} \\
S \cdot T &\overset{\mathrm{BCH}}{=} \exp\left\{ a^2 F_{\mu\nu} + O(a^3) \right\}
\end{aligned}$$

Now expand the exponential and take the trace:

- use $\mathrm{Tr}\, F_{\mu\nu} = 0$
- traces over 'nested' commutators vanish: Lie algebra reduces these to (vanishing) traces over generators
- $O(a^3)$ term comprises such nested commutators

Finally:

$$\mathrm{Tr}(S \cdot T) = \mathrm{Tr}\left(1 + a^2 F_{\mu\nu} + O(a^3) + \frac{a^4}{2} F_{\mu\nu} F_{\mu\nu} + \cdots \right) = N + \frac{a^4}{2} \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu}) + O(a^5)$$

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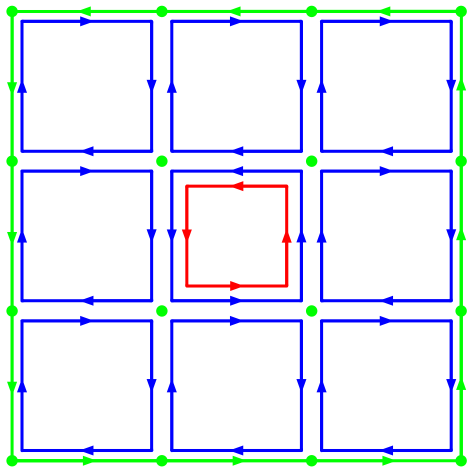
Lattice Methods in Field Theory: Problem Sheet 3 (Tutors)

Question 1. **Strong coupling expansion at order β^2**

Consider the strong coupling expansion of a Wilson loop $W(R,T)$ of size $R \times T$ for gauge group $SU(3)$. Compute the contribution at order β^2 by evaluating a graph like:



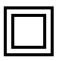
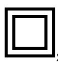
Combine the result with that obtained at order β and derive the expression for the string tension σ by observing that

$$\langle W(R,T)\rangle = \mathrm{e}^{-V(R)T}, \qquad V(R) \approx \sigma R$$

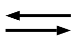
Hint: use the expressions for group integrals in $SU(3)$, in particular:

$$\int dU\, U_{ij} U_{kl} U_{mn} = \frac{1}{3!}\varepsilon_{ikm}\varepsilon_{jln}$$

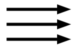
Answer 1. Consider a tiling like the one shown in the question.

- Each plaquette contributes $\beta/2N = \beta/6$ (set $N = 3$ from now on)
- RT possibilities to place  on surface
- Can swap 'inner' and 'outer' plaquettes in , giving symmetry factor of $1/2$ (quadratic term in exponential series gives $1/2!$)

$$\longrightarrow \quad \frac{1}{2} RT \left(\frac{\beta}{6}\right)^{RT+1}$$

- Integration over each pair of links  gives $1/3$

$$\longrightarrow \quad \left(\frac{1}{3}\right)^{(R+1)T+R(T+1)-4}$$

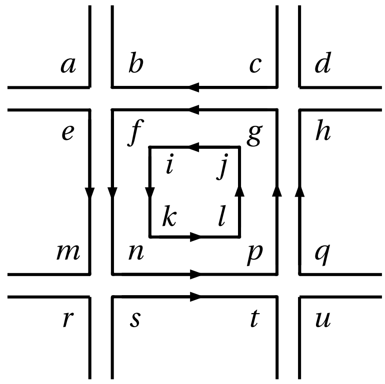
- Integration over 3 links  gives 1/3!

$$\longrightarrow \quad \left(\frac{1}{3!}\right)^4$$

- Every site where 8 links meet contributes a factor 3 (just as in lowest order contribution; also works at edges, corners)

$$\longrightarrow \quad 3^{(R+1)(T+1)-4}$$

- Every site where 10 links meet contributes a factor of 3!. Group integrals give



$$\begin{aligned} & \delta_{ea}\delta_{ab}\ \delta_{cd}\delta_{dh}\ \delta_{qu}\delta_{ut}\ \delta_{sr}\delta_{rm} \\ & \times \varepsilon_{ifb}\varepsilon_{jgc}\ \varepsilon_{jgh}\varepsilon_{lpq}\ \varepsilon_{lpt}\varepsilon_{kns}\ \varepsilon_{knm}\varepsilon_{ife} \\ = \ & \varepsilon_{ifb}\varepsilon_{ifb}\ \varepsilon_{jgc}\varepsilon_{jgc}\ \varepsilon_{lpq}\varepsilon_{lpq}\ \varepsilon_{kns}\varepsilon_{kns} \\ = \ & 3!\ 3!\ 3!\ 3! \end{aligned}$$

$$\longrightarrow \quad (3!)^4$$

- Recall $1/N = 1/3$ in definition of Wilson loop.

Putting everything together:

$$\frac{1}{2}RT\left(\frac{\beta}{6}\right)^{RT+1}\left(\frac{1}{3}\right)^{(R+1)T+R(T+1)-4}\left(\frac{1}{3!}\right)^4 3^{(R+1)(T+1)-4}\ (3!)^4\,\frac{1}{3} = RT\left(\frac{\beta}{18}\right)^{RT}\left(\frac{\beta}{12}\right)$$

and combining with the lowest order result:

$$\begin{aligned} \langle W(R,T)\rangle &= \left(\frac{\beta}{18}\right)^{RT}\left(1 + RT\frac{\beta}{12} + O(\beta^2)\right) \\ &= \mathrm{e}^{RT\ln(\beta/18)}\left(\mathrm{e}^{RT\beta/12} + O(\beta^2)\right) \\ &= \exp\left\{RT\left[\ln(\beta/18) + \beta/12\right] + O(\beta^2)\right\} \approx \mathrm{e}^{-\sigma RT} \end{aligned}$$

so that

$$\sigma = -\ln\left(\frac{\beta}{18}\right) - \frac{\beta}{12} + O(\beta^2)$$

Question 2. Ginsparg-Wilson relation and exact chiral symmetry

Let Q be an arbitrary lattice transcription of the free Dirac operator in Euclidean spacetime. Assume that Q satisfies the Ginsparg-Wilson relation,

$$\{Q, \gamma_5\} = aQ\gamma_5 Q$$

Show that the fermion lattice action

$$S[\overline{\psi}, \psi] = a^4 \sum_{x\in\Lambda_E} \overline{\psi}(x)(Q\psi)(x)$$

is invariant under the global infinitesimal transformation

$$\psi \to \psi + \varepsilon\gamma_5(1 - \frac{1}{2}aQ)\psi, \qquad \overline{\psi} \to \overline{\psi} + \varepsilon\overline{\psi}(1 - \frac{1}{2}aQ)\gamma_5$$

Compare to the usual chiral transformation in the continuum.

Answer 2. Using

$$\delta\psi = \varepsilon\gamma_5(1-\frac{1}{2}aQ)\psi \qquad \text{and} \qquad \delta\overline{\psi} = \varepsilon\overline{\psi}(1-\frac{1}{2}aQ)\gamma_5$$

you find

$$\begin{aligned}
\delta S &= a^4 \sum_{x\in\Lambda_E} \{\delta\overline{\psi}Q\psi + \overline{\psi}Q\delta\psi\} \\
&= \varepsilon a^4 \sum_{x\in\Lambda_E} \overline{\psi}\Big\{(1-\frac{1}{2}aQ)\gamma_5 Q + Q\gamma_5(1-\frac{1}{2}aQ)\Big\}\psi \\
&= \varepsilon a^4 \sum_{x\in\Lambda_E} \overline{\psi}(\gamma_5 Q + Q\gamma_5 - aQ\gamma_5 Q)\psi \\
&= 0 \quad \text{(by the G-W relation } \gamma_5 Q + Q\gamma_5 = aQ\gamma_5 Q\text{)}
\end{aligned}$$

Question 3. Neuberger's operator

Follow the steps outlined below to show that Neuberger's operator satisfies the Ginsparg-Wilson relation.

Recall that Neuberger's operator is defined through

$$Q_{\mathrm{N}} = \frac{1}{a}(1 - A(A^\dagger A)^{-1/2}), \qquad A = 1 - aQ_{\mathrm{W}} \tag{1}$$

where

$$Q_{\mathrm{W}} = \frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - \frac{a}{2}\nabla^*_\mu\nabla_\mu$$

is the free massless Wilson-Dirac operator. The combination $U = A(A^\dagger A)^{-1/2}$ satisfies

$$U^\dagger U = 1, \qquad \gamma_5 U\gamma_5 = U^\dagger \tag{2}$$

(a) Show that any operator which satisfies the properties in equation 2 also satisfies

$$\gamma_5(1-U)^{-1}\gamma_5 = 1 - (1-U)^{-1} \tag{3}$$

Hint: recall that $\gamma_5^{-1} = \gamma_5$.

(b) Use the result of equation 3 and the definition of Q_{N} in equation 1 to show that

$$Q_{\mathrm{N}}\gamma_5 + \gamma_5 Q_{\mathrm{N}} = aQ_{\mathrm{N}}\gamma_5 Q_{\mathrm{N}}$$

which is the Ginsparg-Wilson relation.

Answer 3.

(a) Rewrite in several steps ...

$$\begin{aligned}
\gamma_5(1-U)^{-1}\gamma_5 &= \gamma_5^{-1}(1-U)^{-1}\gamma_5^{-1} = (\gamma_5(1-U)\gamma_5)^{-1} \\
&= (1-\gamma_5 U\gamma_5)^{-1} = (1-U^\dagger)^{-1} \\
&= UU^{-1}(1-U^\dagger)^{-1} = U((1-U^\dagger)U)^{-1} \\
&= U(U-1)^{-1} = -U(1-U)^{-1} = (1-U-1)(1-U)^{-1}
\end{aligned}$$

so that $\gamma_5(1-U)^{-1}\gamma_5 = 1-(1-U)^{-1}$.

(b) Use the definition $Q_{\mathrm{N}} = \frac{1}{a}(1 - A(A^\dagger A)^{-1/2}) = \frac{1}{a}(1 - U)$ in the result from (a):

$$\gamma_5 a^{-1} Q_{\mathrm{N}}^{-1} \gamma_5 = 1 - a^{-1} Q_{\mathrm{N}}^{-1}$$

Multiply by $a\gamma_5$ on right:

$$\gamma_5 Q_{\mathrm{N}}^{-1} = a\gamma_5 - Q_{\mathrm{N}}^{-1} \gamma_5$$

then multiply by Q_{N} on both sides

$$Q_{\mathrm{N}} \gamma_5 + \gamma_5 Q_{\mathrm{N}} = a Q_{\mathrm{N}} \gamma_5 Q_{\mathrm{N}}$$

which is the G-W relation.

Question 1. Quark matrix

Consider the fermionic part of the QCD lattice action with Wilson fermions,

$$S_F[U,\overline{\psi},\psi] = \sum_{x\in\Lambda_E} \overline{\psi}(x)(Q\psi)(x) = \sum_{x,y\in\Lambda_E} \overline{\psi}(x)Q_{xy}\psi(y)$$

where the quark matrix is given by:

$$Q_{xy} = \delta_{xy} - \kappa \sum_{\mu=0}^{3} \left\{ \delta_{y,x+\hat{\mu}}(1-\gamma_\mu)U_\mu(x) + \delta_{y,x-\hat{\mu}}(1+\gamma_\mu)U_\mu^\dagger(y) \right\}$$

(a) Show that $\gamma_5 Q^\dagger \gamma_5 = Q$ (recall that $\gamma_\mu = \gamma_\mu^\dagger$ and $\{\gamma_\mu,\gamma_5\} = 0$)

(b) Use the result in (a) to show that $(\det Q)$ is real

Answer 1. Note that $Q^\dagger$ here means transpose colour, spin *and* site indices and take complex conjugate.

For the first part:

$$\begin{aligned}
(\gamma_5 Q^\dagger \gamma_5)_{xy} = \gamma_5 (Q^\dagger)_{xy}\gamma_5 &= \gamma_5 \delta_{yx}\gamma_5 - \kappa \sum_{\mu=0}^{3} \left[\delta_{x,y+\hat{\mu}}\gamma_5(1-\gamma_\mu^\dagger)\gamma_5 U_\mu^\dagger(y) + \delta_{x,y-\hat{\mu}}\gamma_5(1+\gamma_\mu^\dagger)\gamma_5 U_\mu(x) \right] \\
&= \delta_{xy} - \kappa \sum_{\mu=0}^{3} \left[\delta_{y,x+\hat{\mu}}(1-\gamma_\mu)U_\mu(x) + \delta_{y,x-\hat{\mu}}(1+\gamma_\mu)U_\mu^\dagger(y) \right] \\
&= Q_{xy}
\end{aligned}$$

We interchanged the two terms in $[\cdots]$ in going from the first to the second line.

For the second part:

$$\det Q = \det(\gamma_5 Q^\dagger \gamma_5) = (\det\gamma_5)^2 \det(Q^\dagger) = \det(\gamma_5^2)(\det Q)^* = (\det Q)^*$$

Question 2. Fixed point of Neuberger's construction

Starting from a Dirac operator Q (satisfying $Q^\dagger = \gamma_5 Q \gamma_5$), Neuberger's operator, Q_{N}, is constructed according to:

$$A = 1 - aQ, \qquad U = \frac{A}{\sqrt{A^\dagger A}}, \qquad Q_{\mathrm{N}} = \frac{1}{a}(1-U).$$

Show that if Q already satisfies the Ginsparg-Wilson relation, then it is reproduced by the Neuberger construction, $Q_{\mathrm{N}} = Q$.

Answer 2. Note that

$$A^\dagger A = (1 - aQ^\dagger)(1 - aQ) = (1 - a\gamma_5 Q\gamma_5)(1 - aQ) = 1 - a\gamma_5 Q\gamma_5 - aQ + a^2\gamma_5 Q\gamma_5 Q = 1$$

since Q already satisfies GW. So, $U = A$ and $Q_\mathrm{N} = Q$.

Question 3. Ginsparg-Wilson eigenvalues

If a Dirac operator Q, satisfying $Q^\dagger = \gamma_5 Q\gamma_5$, also satisfies the Ginsparg-Wilson relation, $\{Q, \gamma_5\} = aQ\gamma_5 Q$, show that Q's eigenvalues lie on the circle $(1 + \mathrm{e}^{i\theta})/a$.

Answer 3. Consider an eigenvector e with eigenvalue λ, $Qe = \lambda e$. The Dirac condition on Q gives,

$$e^\dagger\gamma_5 Q = e^\dagger Q^\dagger\gamma_5 = \lambda^* e^\dagger\gamma_5.$$

The GW relation gives,

$$Q\gamma_5 e = (aQ\gamma_5 Q - \gamma_5 Q)e = \lambda(aQ\gamma_5 - \gamma_5)e,$$

or

$$Q\gamma_5 e = \frac{-\lambda}{1 - a\lambda}\gamma_5 e.$$

Combining these two results,

$$e^\dagger\gamma_5 Q\gamma_5 e = \lambda^* e^\dagger e = \frac{-\lambda}{1 - a\lambda} e^\dagger e.$$

Since $e^\dagger e > 0$, we find,

$$\lambda^* = \frac{-\lambda}{1 - a\lambda},$$

which is solved (for example by letting $a\lambda = 1 + \xi$ and finding ξ) to get

$$\lambda = \frac{1}{a}(1 + \mathrm{e}^{i\theta}).$$

Question 4. Step-scaling

In quenched $SU(3)$ Yang-Mills, at large β, so that one-loop perturbation theory holds, by how much should you increase β to halve the lattice spacing a?

Answer 4. For large β (small g_0), use the 1-loop result,

$$-a\frac{\partial g_0}{\partial a} = -b_0 g_0^3.$$

Integrate to get,

$$\ln(a_2/a_1) = \frac{1}{2b_0}\left(\frac{1}{g_1^2} - \frac{1}{g_2^2}\right).$$

In SU(3), $b_0 = 11/16\pi^2$, and $\beta = 6/g_0^2$, so

$$\ln(a_2/a_1) = \frac{1}{2}\frac{16\pi^2}{11}\frac{1}{6}(\beta_1 - \beta_2).$$

For $a_2 = a_1/2$, this gives $\Delta\beta = \beta_2 - \beta_1 = 33\ln 2/4\pi^2 = 0.579$.