Lattice Phenomenology

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Abstract

I discuss the application of lattice simulations to particle physics phenomenology. After a general introduction, I discuss the application of lattice computations to B-physics, I also present a brief review of recent attempts to compute the matrix elements relevant for two-hadron final states, such as those in $K \to \pi\pi$ decays or the decays of B-mesons into two light mesons.

1 Introduction

It is an embarrassment that for many physical processes, the leading error in extracting fundamental information from experimental data is a theoretical one, namely our inability to compute the long-distance QCD effects. During the last 15 years or so, lattice simulations have been used to quantify these effects. In these lectures I will present an introduction to lattice simulations and illustrate their value by discussing applications to a wide variety of quantities in B-physics (and in the final section to $K \to \pi\pi$ decays). For students who wish to study lattice field theory in detail I am happy to recommend the books by Creutz [1] (for a detailed introduction to the formulation of lattice field theory) and by Montvay and Münster [2]. I also draw to your attention the notes from the previous time I lectured at the Spanish Summer School in 1995 in Almuñecar [3].

Lattice phenomenology starts with the evaluation of correlation functions of the form

$$\langle 0 \,|\, O(x_1, x_2, \cdots, x_n) \,|\, 0 \rangle = \frac{1}{Z} \int [dA_\mu]\,[d\psi]\,[d\bar{\psi}]\; e^{iS}\, O(x_1, x_2, \cdots, x_n)\;, \tag{1}$$

where $O(x_1, x_2, \cdots, x_n)$ is a multilocal operator composed of quark and gluon fields and Z is the partition function:

$$Z = \int [dA_\mu]\,[d\psi]\,[d\bar{\psi}]\; e^{iS}\;. \tag{2}$$

These formulae are written in Minkowski space whereas lattice calculations are performed in Euclidean space, and so one makes the replacement $\exp(iS) \to \exp(-S)$. The functional integral is evaluated by discretising space-time and using Monte-Carlo integration.

The physics which can be studied depends on the choice of the multi-local operator O, and we will see many examples of such operators in these lectures.

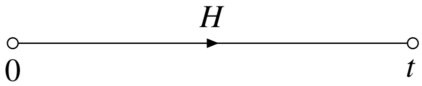

Figure 1: Pictorial representation of the two-point correlation function in eq.(6).

1.1 Two-Point Correlation Functions

We start with the simple case in which the operator O in eq.(1) is a bilocal one, and consider two-point correlation functions of the form:

$$C_2(t) = \int d^3x\, e^{i\vec{p}\cdot\vec{x}} \langle 0 \,|\, J(\vec{x},t)\, J^\dagger(\vec{0},0)\,|\, 0\rangle \, , \tag{3}$$

where J and $J^\dagger$ are any interpolating operators for the hadron H which we wish to study and the time t is taken to be positive. For simplicity we assume the H is the lightest hadron which can be created by $J^\dagger$.

Although t is taken to be positive, lattice simulations are frequently performed on periodic lattices, in which case both time-orderings contribute.

Inserting a complete set of states $\{n\}$ between the operators in eq.(3):

$$\begin{aligned} C_2(t) &= \sum_n \int d^3x\, e^{i\vec{p}\cdot\vec{x}} \langle 0 \,|\, J(\vec{x},t) \,|\, n\rangle \; \langle n \,|J^\dagger(\vec{0},0)\,|\, 0\rangle \, , & (4) \\ &= \int d^3x\, e^{i\vec{p}\cdot\vec{x}} \langle 0 \,|\, J(\vec{x},t)\,|\, H\rangle \; \langle H \,|J^\dagger(\vec{0},0)\,|\, 0\rangle + \cdots & (5) \end{aligned}$$

where the ellipses represent contributions from heavier states with the same quantum numbers as H. Finally, using translational invariance we have:

$$C_2(t) = \frac{1}{2E}\, e^{-iEt} \; |\, \langle 0 | J(\vec{0},0)\, | H(p) \rangle \,|^2 + \cdots \, , \tag{6}$$

where the energy is given by $E = \sqrt{m_H^2 + \vec{p}^{\,2}}$. The term on the right hand side of eq.(6) is represented pictorially by the diagram in fig.1.

In Euclidean space, of course, we need to make the replacement $\exp(-iEt) \to \exp(-Et)$. By fitting $C_2(t)$ to the form above, both the energy (or, if $\vec{p} = 0$, the mass) and the modulus of the matrix element

$$|\, \langle 0 | J(\vec{0},0)\, | H(p) \rangle \,|$$

can be evaluated.

As an example consider the case in which J is the axial current, $J = \bar{u}\gamma^\mu\gamma^5 d$, in which case the decay constant of the π–meson can be evaluated:

$$|\, \langle 0 \,|\bar{u}\gamma^\mu\gamma^5 d\,|\, \pi^-(p) \rangle \,| = f_\pi\, p^\mu \tag{7}$$

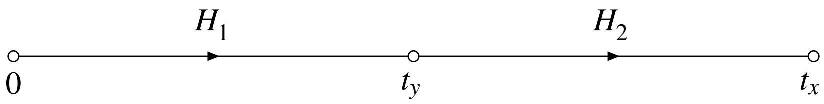

Figure 2: Pictorial representation of the three-point correlation function in eq.(8).

and the physical value of $f_\pi \simeq 132\,\mathrm{MeV}$.

Since the correlation function decreases exponentially with Et, for sufficiently large times the contribution of the lightest state will dominate. Much effort is being devoted to choosing appropriate interpolating operators J, i.e. operators which will enable $C_2(t)$ to be dominated by the lightest state for relatively small times. These are generally non-local or *smeared* operators, whereas physical quantities, such as f_π, are obtained from matrix elements of local operators. It is therefore frequently convenient to compute correlation functions of two different operators.

1.2 Three-Point Correlation Functions

Consider now a three-point correlation function of the form:

$$C_3(t) = \int d^3x\, d^3y\, e^{i\vec{p}\cdot\vec{x}}\, e^{i\vec{q}\cdot\vec{y}}\, \langle 0\,|\, J_2(\vec{x},t_x)\, O(\vec{y},t_y)\, J_1^\dagger(\vec{0},0)\,|\,0\rangle\ , \tag{8}$$

where $J_{1,2}$ are interpolating operators for two (possibly different) particles and we assume that $t_x > t_y > 0$ (see fig.2).

For sufficiently large times t_y and $t_x - t_y$

$$\begin{aligned} C_3(t_x,t_y) \;\; &\simeq \;\; \frac{e^{-E_1 t_y}}{2E_1}\,\frac{e^{-E_2(t_x-t_y)}}{2E_2}\,\langle 0\,|\,J_2(0)\,|\,H_2(\vec{p})\,\rangle\, \times \\ &\qquad \langle H_2(\vec{p})\,|\,O(0)\,|\,H_1(\vec{p}+\vec{q})\,\rangle\,\langle H_1(\vec{p}+\vec{q})\,|\,J_1^\dagger(0)\,|\,0\,\rangle\,, \end{aligned} \tag{9}$$

where $E_1^2 = m_1^2 + (\vec{p}+\vec{q})^2$ and $E_2^2 = m_2^2 + \vec{p}^{\,2}$. From the evaluation of the two-point functions we know the masses of the two hadrons and matrix elements of the form $|\langle 0|\,J\,|H(\vec{p})\rangle|$. Thus from the evaluation of three-point functions we obtain matrix elements of the form $|\langle H_2|O|H_1\rangle|$.

Important examples of physical quantities of phenomenological significance which can be determined using three-point functions are the semileptonic and rare radiative decays of hadrons of the form $B \to \pi, \rho, D, D^* +$ leptons or $B \to K^*\gamma$. The operators in this case are quark bilinears.

I postpone a discussion of nonleptonic decays until the third lecture (section 8).

1.3 Systematic Uncertainties

Lattice computations are a truly *first principles* technique for evaluating non-perturbative strong interaction effects. The precision of the results is limited however, by the available computing resources, and in this section I discuss the principal sources of uncertainty in lattice simulations.

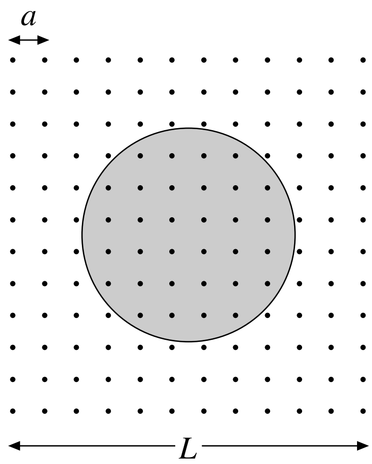

Figure 3: Pictorial representation of a space-time lattice.

In order to study the properties of hadrons in QCD we require

$$L \gg 1\,\mathrm{fm} \qquad \text{and} \qquad a^{-1} \gg \Lambda_{QCD}. \tag{10}$$

Computing resources limit the number of lattice points which can be included and hence the precision of the calculation. Typically, in full QCD, we can currently have up to about 24 points in each spacial direction ($O(50)$ points in quenched simulations) and so compromises have to be made.

The main sources of uncertainty in lattice simulations are:

- **Statistical Errors:** the functional integrals are evaluated by Monte-Carlo sampling. The statistical error is estimated from the fluctuations of computed quantities within different clusters of field configurations.

- **Discretization Errors:** these are uncertainties due to the fact that the lattice spacing a is finite (i.e. non-zero). Current simulations are typically performed with lattice spacings in the range

$$a \sim \left(\frac{1}{20} - \frac{1}{10}\right)\,\mathrm{fm}, \tag{11}$$

  leading to errors of $O(a\Lambda_{QCD})$ of $O(am_Q)$ (with the Wilson formulation of QCD) [1]. Discretization errors are particularly severe in heavy quark physics, where they are of $O(m_Q a)$ of $O(m_Q^2 a^2)$ where m_Q is the mass of the heavy quark. In that case, however, we have some guidance from the HQET. Indeed we can use simulations in the HQET to help reduce the uncertainties.

  A huge effort is being made to reduce discretisation errors. One approach is to perform simulations at several values of a and to extrapolate to $a = 0$. It is also helpful to use *Improved Actions*, i.e. to use a formulation of discretized QCD so that these errors are formally reduced. As an illustration consider two discretizations of the derivative:

$$f'(x) \;=\; \frac{f(x+a) - f(x)}{a} + O(a) \tag{12}$$

$$f'(x) \;=\; \frac{f(x+a) - f(x-a)}{2a} + O(a^2)\,. \tag{13}$$

[1] A useful translation is 0.1 fm$\sim$ 2 GeV.

In eq.(12) the error is the derivative is linear in the lattice spacing whereas in (13) it is of $O(a^2)$. In QCD it has recently become possible to define a lattice formulation such that the errors are quadratic in a. Large scale simulations are currently being performed with such improved actions.

More ambitious still are attempts to derive *perfect actions*. By starting from the continuum action and using the renormalization group, it is, in principle, possible to construct lattice actions with no discretization errors. However the resulting actions are non-local and hence truncations have to be made. Perfect actions have not been implemented in large-scale QCD simulations to date.

- **Finite Volume Effects:** The pion is light (it is the pseudo-Goldstone boson of chiral symmetry breaking) and this implies that it can propagate over large distances. Simulations are performed with heavier pions (typically with unphysically heavy light-quarks, typically with $m_{u,d} \sim m_s$) and the results are subsequently extrapolated to the chiral limit. Typically, in order to suppress finite-volume errors, we impose a requirement such as $m_\pi L > 4$. Up to now we have not been able to perform a simulation with sufficiently light quarks for the $\rho \to \pi\pi$ decay to be possible, and this represents an important milestone for the future.

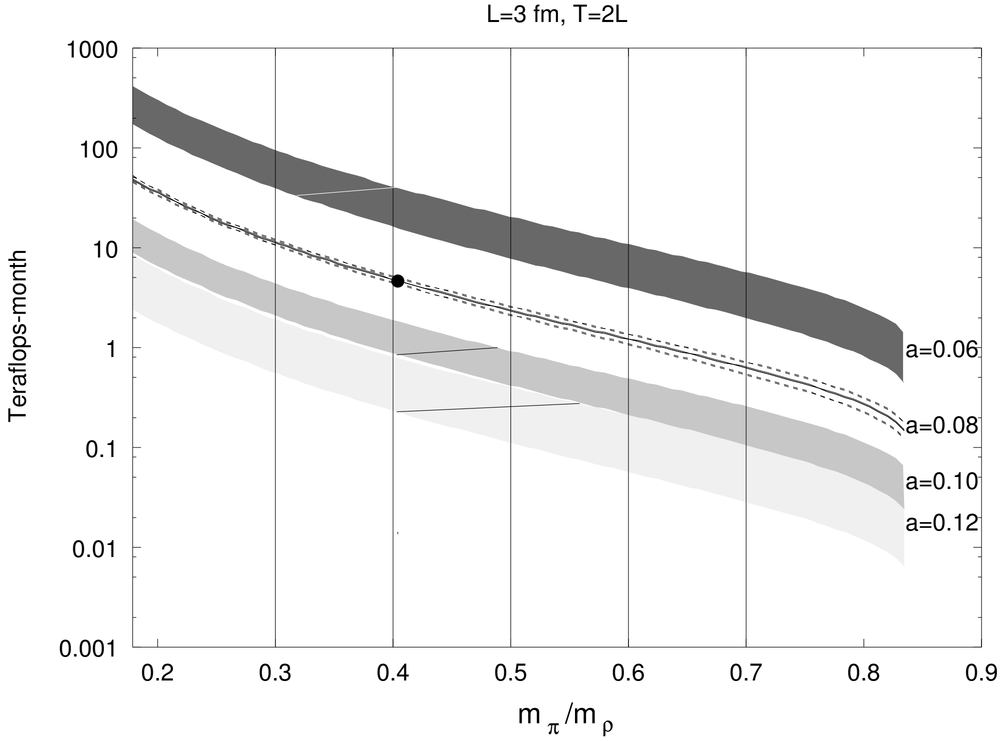

Figure 4: Cost (in teraflop-months) estimate to generate 100 independent configurations vs. m_{ps}/m_v for a lattice with spatial size $L = 3$ fm and temporal size $T = 2L$. The cost is plotted as a function of the quark mass parametrized by m_π/m_ρ.

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- **Quenching:** Most (but increasingly not all) large scale simulations have been performed in the *quenched* approximation, in which the effects of quark loops are neglected. The functional integral over the quark fields can be readily evaluated formally

$$\int [dA_\mu]\,[d\psi]\,[d\bar\psi]\,\exp(-S_{\text{gauge}} - \bar\psi\Delta(A)\,\psi) \sim \int [dA_\mu]\,\det[\Delta(A)]\exp(-S_{\text{gauge}})\;, \quad (14)$$

where S_{gauge} is the gauge-action. The evaluation of the fermion determinant is extremely expensive. For example, a recent detailed investigation of the behaviour of algorithms for Wilson fermions deduced that the number of floating-point operations required to generate an independent field configuration is given by [4, 5]:

$$N^{\text{ops per}}_{\text{indep. conf.}} \simeq 1.7\times 10^7 \cdot (L^3T)^{4.55/4}\left(\frac{1}{a}\right)^{7.25}\left(\frac{1}{m_\pi}\right)^{2.7}\;, \quad (15)$$

where L and T are the spatial and temporal extents of the lattice. This result is illustrated in fig. 4 where the cost (in teraflops-months) is estimated for the generation of 100 independent configurations for a lattice with $L = 3\,\text{fm}$ as a function of the quark mass (parametrized by m_π/m_ρ). The plot illustrates the large cost involved in lattice simulations with realistic parameters. Moreover, it should be noted that the estimate in (15) is indeed just an estimate, based on simulations with $a = 0.08$ fm with a limited range of quark masses and lattice sizes.

- **Renormalization of Lattice Operators:** The lattice formulation of QCD is in terms of a bare theory with ultra-violet cut-off a. Similarly the operators whose matrix elements are computed are bare ones, and one needs to relate these bare operators to renormalized ones, defined in some standard renormalization scheme (such as the $\overline{MS}$ one). The key-point is that, provided that both a^{-1} and the renormalization scale μ are much larger than Λ_{QCD}, then the renormalization constants depend on short distance physics only. They can therefore be computed perturbatively. Recently, however, non-perturbative techniques have been introduced to evaluate these constants. Nevertheless the renormalization procedure does lead to uncertainties.

2 Elements of Lattice QCD

In this section I introduce very briefly some of the standard disretizations of QCD (for a full discussion see refs.[1, 2]). The quark fields $\psi(x)$ are defined on the sites of the lattice whereas the gluons are introduced in terms of link variables $U_\mu(x)$, defined on the link between x and $x+a\mu$:

$$U_\mu(x) = e^{iagA_\mu(x+a\hat\mu/2)}\;. \quad (16)$$

Under a local gauge transformation $g(x)$:

$$\psi(x)\to g(x)\psi(x) \qquad \text{and} \qquad U_\mu(x)\to g(x)U_\mu(x)g^\dagger(x+a\hat\mu)\;. \quad (17)$$

Thus $U_\mu(x)$ can be considered as an approximation to the path-ordered exponential from x to $x+a\hat\mu$. Links from x to $x-a\hat\mu$ are given by

$$U_{-\mu}(x) = U^\dagger_\mu(x-a\hat\mu)\;. \quad (18)$$

Closed loops of link variables are gauge invariant, and this is exploited in the construction of the action and of the lattice operators representing observables.

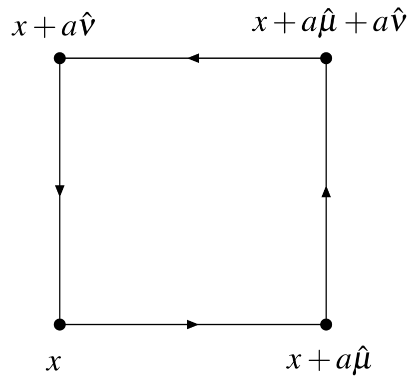


Figure 5: Pictorial representation of the Plaquette.

2.1 Wilson's Gauge Action

The action for the gluons is given in terms of the plaquette variable (illustrated in fig.5):

$$\mathscr{P}_{\mu\nu}(x) \equiv \mathrm{Tr}\,[U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\nu})U_\nu^\dagger(x)]\ . \tag{19}$$

The gluon action is then written as:

$$\begin{aligned} S_{\text{gluon}} &\equiv \beta \sum_{x,\mu,\nu} \left\{1 - \frac{1}{3}\mathrm{Re}[\mathscr{P}_{\mu\nu}(x)]\right\} \\ &= \int d^4x \frac{1}{4} F_{\mu\nu}^2(x) \ + \ O(a^2) \end{aligned} \tag{20}$$

where $\beta = 6/g_0^2$.

2.2 Lattice Fermions

Consider the free Dirac action $\bar{\psi}(m+\not{\partial})\psi$. Defining the derivative in terms of the difference between neighbouring points, the free propagator S is given by

$$S(q)^{-1}_{\text{free}} = m + \frac{i}{a}\sum_\mu \gamma_\mu \sin(aq_\mu)\ . \tag{21}$$

This has a pole at values of q such that $q^2+m^2=0$ as expected in Euclidean space. However, it also has additional poles when any component q_μ is replaced by $\pi/a - q_\mu$, leading to 16 states instead of one. This is the *Fermion Doubling* problem.

Wilson's solution to this problem was to add *irrelevant* operators to the action such that the mass of the doublers $\rightarrow \infty$ as $a \rightarrow 0$. In particular he added (for the free theory)

$$-\frac{ra}{2}\bar{\psi}\partial^2\psi = -ra^4\sum_x\left\{\frac{1}{2a}\sum_\mu\left[\bar{\psi}(x)\psi(x+a\hat{\mu}) + \bar{\psi}(x)\psi(x-a\hat{\mu}) - 2\bar{\psi}(x)\psi(x)\right]\right\}\ . \tag{22}$$

The free propagators now is modified to

$$S(q)_{\text{free}} = m + \frac{i}{a}\sum_\mu \gamma_\mu \sin(aq_\mu) + \frac{r}{a}\sum_\mu (1-\cos(aq_\mu)) \;, \tag{23}$$

so that the mass of the doublers $\to O(1/a)$.

The Wilson Action is given by:

$$\begin{aligned} S_W &= S_{\text{gluon}} + a^4\sum_x \Big\{ -\frac{\kappa}{a}\sum_\mu \Big[\bar\psi(x)(r-\gamma_\mu)U_\mu(x)\psi(x+a\hat\mu) \\ &\quad + \bar\psi(x+a\hat\mu)(r+\gamma_\mu)U_\mu^\dagger(x)\psi(x)\Big] + \bar\psi(x)\psi(x)\Big\} \;, \end{aligned} \tag{24}$$

where $\kappa = 1/(2m+8r)$ and the fields have been rescaled by a factor of 2κ. A new term with a corresponding value of κ is introduced for every quark flavour.

The Wilson term (i.e. the term proportional to the parameter r) breaks chiral symmetry! Chiral symmetry is restored for a special value of κ (=κ_c) or equivalently of the bare mass: $\kappa_c = 1/8r + O(\alpha_s)$. The Wilson action reduces to contrinuum *QCD* as $a \to 0$:

$$S_W = S_{\text{QCD}}^{\text{cont}} + O(a) \;, \tag{25}$$

where $O(a)$ means $O(pa)$, $O(ma)$ or $O(\Lambda_{\text{QCD}}a)$. There are other formulations of lattice fermions (staggered fermions, domain-wall fermions etc.), and in particular there is considerable activity in developing practical formulations of lattice fermions which preserve chiral symmetry. The use of *improved actions*, reduces the discretization errors from being linear in the lattice spacing to ones of $O(\alpha_s a)$ or even of $O(a^2)$.

3 Evaluation of Correlation Functions

3.1 Pure Gauge Theories:

We start by considering the evaluation of correlation functions in a pure gauge theory:

$$\langle 0|\, O(x_1,x_2,\cdots x_n)\,|0\rangle = \frac{1}{Z}\int [dU]\, e^{-S(U)}\, O(x_1,x_2,\cdots,x_n) \;. \tag{26}$$

The integral in eq.(26) is a well defined multi-dimensional one (for $SU(n)$ gauge theory it is a $4V(n^2-1)$ dimensional integral, where V is the number of space-time points in the lattice). It is estimated by Monte-Carlo sampling. Gluon configurations are generated (iteratively) on all the links of the lattice with probability $1/Z\exp(-S(U))$. The correlation function is then the average of the operator O evaluated on each of the independent configurations:

$$\langle 0|O|0\rangle = \frac{1}{N_c}\sum_{c=1}^{N_c} O(U_c) \;, \tag{27}$$

where c represents *configuration*. The *Statistical Error* is estimated from the variation of the result as configurations are added and subtracted.

From two-point correlation functions in the pure gauge theory one obtains the spectrum of glueball states, the potential between a static quark and antiquark, information about the topological properties of the theory, etc..

3.2 Including Fermions

We have seen that performing the integral over the fermion fields introduces the fermion determinant:

$$\begin{aligned}\int [d\psi][d\bar\psi]e^{-S_W} &= \det[\Delta(U)]e^{-S_{\rm gluon}}\\ \int [d\psi][d\bar\psi]\,\psi_i\bar\psi_j\,e^{-S_W} &= \Delta^{-1}_{ij}(U)\det[\Delta(U)]e^{-S_{\rm gluon}}\\ &\cdots\end{aligned}$$

where $\{i,j\}$ represent space-time coordinates, and colour and spin indices. $\Delta^{-1}(U)$ is the quark propagator in the gluon background field U. We therefore see that in the integral over the gluon fields:

$$S_{\rm gluon} \to S_{\rm gluon} - \ln\det[\Delta(U)]\ . \tag{28}$$

The second term in (28) represents the effects of closed quark-loops. It is non-local and expensive to evaluate. The expense of the repeated evaluation of the fermion determinant is the reason for the frequent use of the quenched approximation.

4 B-Physics From the Lattice – Introduction

Having discussed the basics of lattice simulations in the above sections, I now turn my attention to illustrating what lattice simulations can do for phenomenology. I will use the important field of B-physics for the purposes of this illustration. This will complement the lectures of Ben Grinstein at this school [6].

A huge amount of experimental data is currently being presented on the decays of the b-quark and much more will become available in the coming years. From this data we would like to extract fundamental information about the parameters of the standard model and CP-violation, and (by overconstraining the comparison with the standard model) to search for signals of new physics. A major difficulty (indeed probably *the* major difficulty) in achieving the above goals is our inability to quantify non-perturbative strong interaction effects. Lattice QCD provides the opportunity for evaluating these effects *ab initio*, and there is indeed a wide-ranging program of numerical simulations of B-physics. In the remaining 2 lectures I will briefly review some of the recent progress in lattice studies of B-physics, focusing on recent results and on prospects for future calculations.

In any simulation, computing resources are limited, and one therefore has to compromise when minimising competing systematic errors, those due to the granularity of the lattice (discretization

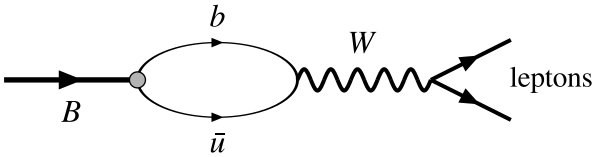


Figure 6: Diagrammatic representation of a leptonic decay of a B–meson.

effects) and finite-volume errors. In current simulations, this typically leads to a choice for the lattice spacing (a) in the range given by eq.(11), $a^{-1} \sim 2-4$ GeV, i.e. a is larger than the Compton wavelength of the b-quark. This means that we cannot study the propagation of a physical b-quark directly and either have to i) use effective theories, such as the Heavy Quark Effective Theory (HQET) or Non-Relativistic QCD (NRQCD) or ii) calculate physical quantities with the heavy-quark mass m_Q in the region of m_c (the mass of the charm-quark), and perform the extrapolation to $m_Q = m_b$.

5 Decay Constants, B–Parameters and Form-Factors

Leptonic decay constants f_B and f_{B_s}, the B–parameters of B–$\bar{B}$ mixing and the form-factors for semileptonic B-decays have been computed in lattice simulations for over ten years now. In this section I briefly discuss each of these quantities in turn.

5.1 $\mathbf{f_B}$

The strong interaction effects in leptonic decays of B–mesons (see fig.6) are contained in the matrix element

$$\langle 0 | A_\mu(0) | B(p) \rangle = i f_B p_\mu \ . \tag{29}$$

Lorentz and Parity Invariance imply that all the non-perturbative QCD effects are parametrized in terms of a single number, f_B, defined in eq.(29).

Claude Bernard, the reviewer at the Lattice 2000 symposium [7] summarised the current status of the results as [2]:

	Full Theory	$N_f = 0$
f_B	200 ± 30 MeV	175 ± 20 MeV
f_{B_s}/f_B	1.16 ± 0.04	1.15 ± 0.04

At last year's conference the reviewer S.Hashimoto stated that *all available data is consistent with the following estimates:* [8]

	$N_f = 2$	$N_f = 0$
f_B	210 ± 30 MeV	170 ± 20 MeV
f_{B_s}	245 ± 30 MeV	195 ± 20 MeV
f_{B_s}/f_B	1.16 ± 0.04	1.15 ± 0.04

[2]Ref. [7] contains a detailed critical analysis of all the lattice results for the quantities considered in this section.

These results have been largely stable for some time. The last time that I performed a compilation was in 1997 together with Jonathan Flynn [9] and reported (on what were mainly quenched calculations):

$$f_B = 170 \pm 35\,\mathrm{MeV},\;\; f_{B_s} = 195 \pm 35\,\mathrm{MeV},\;\; \frac{f_{B_s}}{f_B} = 1.14 \pm 0.08. \tag{30}$$

Although the results have been stable, the errors have been decreasing very slowly (if at all). The errors will not decrease significantly until we begin to get a serious control of quenching errors. In unquenched calculations, the value of the lattice spacing typically varies by 10% or so depending on the physical quantity which is used to set the scale. It is therefore not really possible to determine dimensional quantities such as f_B with a better precision than about 10%.

In the last two years we have began to study quenching effects, although it must be remembered that at present the masses of the sea-quarks are still relatively heavy (typically a little smaller that the strange quark mass). These effects can only be studied meaningfully if all other variables are kept constant. Using results from the MILC [10], CPPACS [11] and NRQCD [12] collaborations, C. Bernard estimates that there is about a 20 MeV increase in the value of f_B in going from the quenched approximation to the $N_f = 2$ case (using f_π to set the scale) [7]. The value of f_{B_s}/f_B appears to be the same in the two cases.

Although the errors have not been reduced substantially, there have been many systematic checks on the stability of the results. For example discretization errors have been studied by using improved actions and/or extrapolating the results to the continuum limit. It had been suggested that f_B may be significantly lower than the results presented above because of discretization errors (this suggestion was based on some extrapolations to the chiral limit), but this has not survived more careful analyses [7].

When lattice computations of the leptonic decay constants of heavy mesons were beginning there were no experimental data. In 1997 we compiled the (quenched) lattice predictions for the decays constants of charmed mesons as [9]:

$$f_D = 200 \pm 30\,\mathrm{MeV} \;\;\text{and}\;\; f_{D_s} = 220 \pm 30\,\mathrm{MeV}. \tag{31}$$

In view of this the experimental result

$$f_{D_s} = (251 \pm 30)\,\mathrm{MeV}$$

is very satisfying.

Finally let me mention that lattice computations have shown that there are large corrections to the HQET scaling law for the decay constants of heavy-light pseudoscalar mesons P

$$f_P \sim \frac{1}{\sqrt{m_P}}. \tag{32}$$

5.2 $\mathbf{B_B}$:

The relevant matrix element for B^0-$\bar{B}^0$ mixing is:

$$M(\mu) \equiv \langle \bar{B}^0 \,|\, \bar{b}\gamma_\mu(1-\gamma_5)q\; \bar{b}\gamma^\mu(1-\gamma_5)q \,|\, B^0 \rangle\,,$$

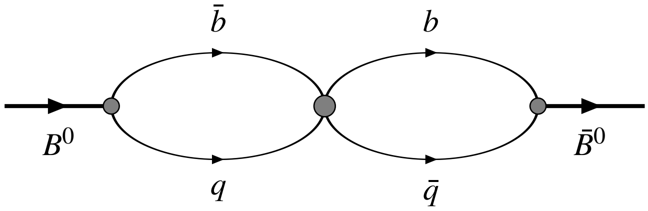


Figure 7: Diagramatic representation of B^0-$\bar{B}^0$ mixing.

where q represents d or s. This is illustrated in fig. 7.

Following the conventions introduced in kaon physics, B-parameters are defined by

$$M(\mu) = \frac{8}{3} f_B^2 m_B^2 B_B(\mu). \tag{33}$$

$B_B(\mu)$ is scheme and scale-dependent, so it is convenient to define a scheme-independent (up to NLO) quantity

$$\hat{B}_B^{\rm nlo} = \alpha_s(\mu)^{2/\beta_0} \left[1 + \frac{\alpha_s(\mu)}{4\pi} J_{n_f} \right] B_B(\mu). \tag{34}$$

where J_{n_f} is a known constant. Many groups have studied mixing in lattice simulations and C. Bernard, the reviewer at this year's lattice conference, summarised the results as:

$$\hat{B}_{B_d} = 1.30 \pm 0.12 \pm 0.13, \qquad \hat{B}_{B_s}/\hat{B}_{B_d} = 1.00 \pm 0.04 \tag{35}$$

$$f_B\sqrt{\hat{B}_{B_d}} = 230 \pm 40\,\mathrm{MeV}, \qquad \xi \equiv \frac{f_{B_s}\sqrt{\hat{B}_{B_s}}}{f_B\sqrt{\hat{B}_{B_d}}} = 1.16 \pm 0.05\,. \tag{36}$$

The second error for B_{B_d} is an estimate of the error due to quenching. ξ is a particularly important quantity for studies of the unitarity triangle. It should be noted that the error on ξ is not a small one from the lattice perspective, indeed given its central importance in phenomenological studies most groups tend to be cautious in determining this error. The reason that it should not be considered small is that the quantity which is being computed is $\xi - 1 = 0.16 \pm 0.05$, i.e. it has a quoted error of over 30%. In my judgement therefore the uncertainty in the value of ξ should not be further inflated in phenomenological studies. The results for ξ are stable, and there is no evidence that they change as sea-quarks are introduced.

For comparison with the results quoted above I present also those quoted in our 1997 review [9]:

$$\hat{B}_{B_d} = 1.4 \pm 0.1, \qquad \hat{B}_{B_d}/\hat{B}_{B_s} = 1.01 \pm 0.01 \tag{37}$$

$$f_B\sqrt{\hat{B}_{B_d}} = 201(42)\,\mathrm{MeV} \qquad \xi \equiv \frac{f_{B_s}\sqrt{\hat{B}_{B_s}}}{f_B\sqrt{\hat{B}_B}} = 1.14 \pm 0.08. \tag{38}$$

All the results presented above have been obtained with propagating heavy quarks (i.e by extrapolating results obtained with $m_Q \simeq m_c$). In the static approximation the perturbative corrections are very large and hence these have not given us much useful information to date.

5.3 Exclusive Semi-Leptonic B-Decays

Exclusive semileptonic decays of B-mesons are being used to determine the V_{cb} and V_{ub} CKM-matrix elements. Diagramatically the amplitudes can be represented by the diagram in fig. 8. Lorentz and parity invariance imply that the amplitudes can be expressed in terms of invariant

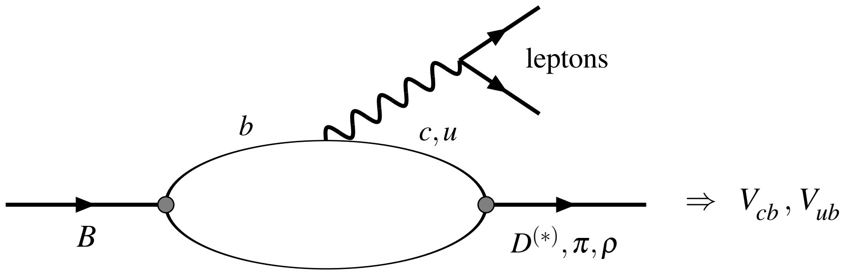


Figure 8: Diagramatic Representation of Semileptonic B-Decays.

form factors. For example for the decays into a pseudoscalar P ($B \to \pi$ or $B \to D$ decays), in the helicity basis,

$$\langle P(p_P)\,|V_\mu(0)\,|\,B(p_B)\,\rangle = f^0(q^2)\,\frac{M_B^2-M_P^2}{q^2}q_\mu + f^+(q^2)\left\{(p_B+p_P)_\mu - \frac{M_B^2-M_P^2}{q^2}q_\mu\right\}. \qquad (39)$$

As a result of parity invariance only the vector component (from the $V-A$ weak current) contributes in this case. When the final-state hadron is a vector meson (ρ or D^*) the amplitude is written in terms of four invariant form factors.

I will only make a few brief comments about semileptonic decays:

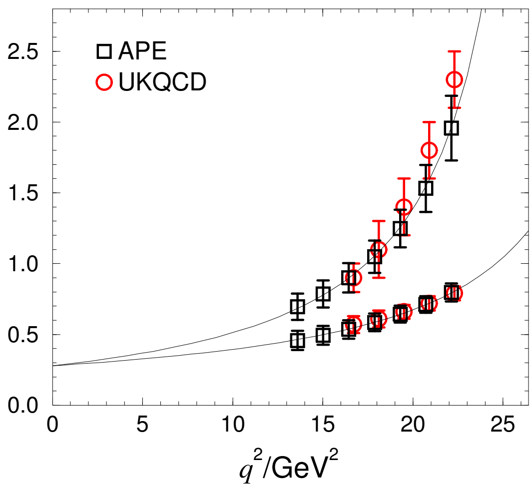
- Lattice simulations of $B \to \pi,\rho$ decays require the momentum of the final-state light meson to be small in order to avoid discretization errors. This means that from lattice simulations of $B \to \pi,\rho$ semileptonic decays we only obtain results at large values of q^2. This is illustrated by the first diagram in fig. 9, which contains recent data from the APE [13] and UKQCD collaborations [14] [3].

- Lattice computations of semileptonic form-factors have been performed for many years now. At the lattice 2000 symposium updated results were presented from the APE, UKQCD, JLQCD and Fermilab collaborations and "the results were fairly consistent in the region where all groups had direct calculations ($19\,\mathrm{GeV}^2 < q^2 < 23\,\mathrm{GeV}^2$)" [7]. As an example, the first diagram in fig. 9 shows the results for the form factors f^+ and f^0 for $B \to \pi$ decays.

- Much effort is being devoted to extrapolating the lattice results to smaller values of q^2, using as many theoretical constraints as possible (e.g. HQET scaling relations, unitarity and analyticity, kinematical constraints, soft pion relations etc.) [15]. The results of such extrapolations for f^0 and f^+ are also shown in fig. 9.

- For a direct application of lattice computations one should compare the lattice results to experimental distributions at large values of q^2. An example of such a comparison is shown

[3] The curves are APE collaborations fits to their points [13].



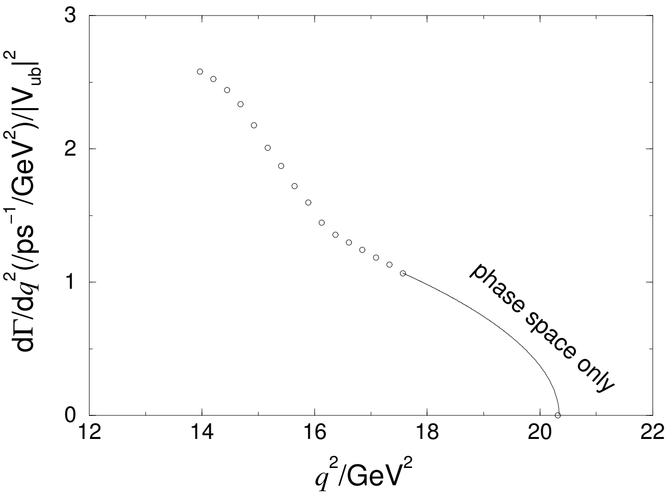


Figure 9: The left-hand diagram shows results for f^+ (upper points) and f^0 (lower points) from the APE [13] and UKQCD [14] collaborations. The curves are fits to the points from APE, showing an extrapolation to lower values of q^2. The right-hand diagram shows the UKQCD results for the distribution for $b \to \rho$ decays in the range $14\,\mathrm{GeV}^2 < q^2 < 20.3\,\mathrm{GeV}^2$, which corresponds to the high q^2 bin of CLEO. The region marked *phase space only* is inaccessible to the lattice calculations and the curve corresponds to taking the form-factors from the last lattice point and extrapolating using the phase-space only.

in the second diagram in fig.9, where the lattice results from the UKQCD collaboration for $B \to \rho$ decays [14] are compared with the high-q^2 bin from CLEO [16]. Integrating the results in this bin the comparison takes the form:

$$\frac{\Delta\Gamma(14 < q^2/\,\mathrm{GeV}^2 < 20.3)}{\mathrm{ps}^{-1}\,\mathrm{GeV}^{-2}} = 8.3\,|V_{ub}|^2 \qquad \text{UKQCD preliminary} \qquad (40)$$

$$= 7.1(2.4)\times 10^{-5} \qquad \text{CLEO} \qquad (41)$$

from which one obtains $V_{ub} = 2.9(0.5)\times 10^{-3}$.

6 Inclusive Decays and Lifetimes of Beauty Hadrons

In this section I will briefly mention some physical quantities for which the first lattice calculations were performed in the last few years. It will certainly be possible to improve on the precision of these first-generation computations.

6.1 Lifetimes of Beauty Hadrons:

The fact that the b-quark is heavy makes it possible to derive an operator product expansion for inclusive beauty decays [17], which results in an expansion in inverse powers of m_b. Specifically for widths of beauty hadrons:

$$\Gamma(H_b) = \frac{G_F^2\, m_b^5 |V_{cb}|^2}{192\pi^3} \left\{ c_3 \left(1 + \frac{\lambda_1 + 3\lambda_2}{2m_b^2} \right) + c_5 \frac{\lambda_2}{m_b^2} + O\left(\frac{1}{m_b^3} \right) \right\} \tag{42}$$

where c_3 and c_5 are calculable in perturbation theory and

$$-\lambda_1 = \frac{1}{2m_{H_b}} \langle H_b | \bar{h} \vec{D}^2 h | H_b \rangle \quad \text{and} \quad 3\lambda_2 = \frac{1}{2m_{H_b}} \langle H_b | \frac{g}{2} \bar{h} \sigma_{ij} G^{ij} h | H_b \rangle \tag{43}$$

defined in the rest frame of H_b (λ_1 and λ_2 are the matrix elements of the kinetic energy and chromomagnetic operators respectively). This leads to

$$\frac{\tau(B^-)}{\tau(B^0)} = 1 + O\left(\frac{1}{m_b^3} \right) \tag{44}$$

$$\frac{\tau(\Lambda_b)}{\tau(B^0)} = 1 - \frac{\lambda_1(\Lambda_b) - \lambda_1(B^0)}{2m_b^2} + 3\, c_G \frac{\lambda_2(\Lambda_b) - \lambda_2(B^0)}{m_b^2} + \cdots \tag{45}$$

$$= (0.98 \pm 0.01) + O\left(\frac{1}{m_b^3} \right), \tag{46}$$

to be compared to the experimental values [4]

$$\frac{\tau(B^-)}{\tau(B^0)} = 1.06 \pm 0.04 \ \text{ and } \ \frac{\tau(\Lambda_b)}{\tau(B^0)} = 0.79 \pm 0.05\ . \tag{47}$$

In view of the discrepancy between the theoretical prediction in (46) and the experimental results for the ratio $\tau(\Lambda_b)/\tau(B^0)$ it is important to evaluate the $O(1/m_b^3)$ corrections [5]. These may be significant because it is only at this order that the light-quark in B-meson is involved in the weak decay, and hence their contribution is enhanced by phase-space factors [19]. To evaluate these $O(1/m_b^3)$ corrections we need to compute the matrix elements of the following dimension-6 operators [19]:

$$\begin{gathered} O_1 = \bar{b}\gamma_\mu(1-\gamma^5)q\, \bar{q}\gamma^\mu(1-\gamma^5)b, \ \ O_2 = \bar{b}(1-\gamma^5)q\, \bar{q}(1+\gamma^5)b, \\ T_1 = \bar{b}\gamma_\mu(1-\gamma^5)T^a q\, \bar{q}\gamma^\mu(1-\gamma^5)T^a b \quad \text{and} \quad T_2 = \bar{b}(1-\gamma^5)T^a q\, \bar{q}(1+\gamma^5)T^a b, \end{gathered} \tag{48}$$

where T^a represents the colour matrix.

For mesons, the evaluation of these matrix elements is very similar to that of B_B and is totally straightforward. For the Λ_b baryon the calculation is a little less straightforward, but can be performed using standard techniques. Together with M. di Pierro we found [20]

$$\frac{\tau(B^-)}{\tau(B_d)} = 1.03 \pm 0.02 \pm 0.03\ , \tag{49}$$

in good agreement with the experimental result. The first error in (49) is the lattice one and the second is our estimate of the uncertainty due to the fact that the (perturbative) coefficient function

[4] See also ref[18] in which the result $\tau(B^-)/\tau(B^0) = 1.07(2)$ is quoted.

[5] The significance of the discrepancy is underlined when one notes that it is really $1 - \tau(\Lambda_b)/\tau(B^0)$ which we calculate.

is only known to one-loop. There is a remarkable factorization of the lattice operators, i.e. the lattice B-parameters are close to 1. For the Λ_b baryon we have only performed an exploratory calculation, in which we did not have sufficient control of the behaviour with the mass of the light-quark [21]. The spectator effects were found to be very significant but, at least for lights quarks corresponding to $m_\pi \simeq$1 GeV, the result of $\tau(\Lambda_b)/\tau(B_0) \simeq 0.91$ would imply that spectator effects would not be sufficiently large to account for the full discrepancy. It is clear however, that these calculations need to be improved (which will be relatively simple to do) and that this will make a considerable impact on our understanding of this important question.

6.2 B_s-Meson Lifetime Differences:

We now consider the difference of the widths of the B_s mesons. Using the operator product expansion we have [22]

$$\begin{aligned}\frac{\Delta\Gamma_{B_s}}{\Gamma_{B_s}} &= \frac{G_F^2 m_b^2}{12\pi m_{B_s}} \left|V_{cb}V_{cs}\right|^2 \tau_{B_s} \times \\ &\left\{G(z)\langle\overline{B}_s|Q_L(m_b)|B_s\rangle - G_S(z)\langle\overline{B}_s|Q_S(m_b)|B_s\rangle + \delta_{1/m}\sqrt{1-4z}\right\} \qquad (50)\end{aligned}$$

where $z = m_c/m_b$, the operators are $Q_L = \bar{b}\gamma_L^\mu s\, \bar{b}\gamma_L^\mu s$ and $Q_S = \bar{b}(1-\gamma_5)s\, \bar{b}(1-\gamma_5)s$, the $\delta_{1/m}$ term represents $O(1/m_b)$ corrections and $G(z)$ and $G_S(z)$ have been computed in perturbation theory up to NLO [23]. The evaluation of these hadronic matrix elements has recently began to be performed in lattice simulations by Hashimoto et al. [24] and APE collaborations [25] [6] The APE collaboration quote

$$\frac{\Delta\Gamma_{B_s}}{\Gamma_{B_s}} = (4.7 \pm 1.5 \pm 1.6)\% \quad \text{APE [25]}, \qquad (51)$$

where the second error represents an assumed 30% uncertainty on the $1/m_b$ corrections. The calculations were obtained with propagating heavy quarks. The result from Hashimoto et al. is

$$\frac{\Delta\Gamma_{B_s}}{\Gamma_{B_s}} = (10.7 \pm 2.6 \pm 1.4 \pm 1.7)\% \quad \text{Hashimoto et al. [24]}, \qquad (52)$$

where the first error is due to the uncertainty in our knowledge of f_{B_s}, the second is the lattice error in the matrix elements and the third is the estimate of the error due to the $O(1/m_b)$ corrections. These results were obtained by simulating NRQCD.

The central values in eqs. (51) and (52) are somewhat different (although they are compatible within errors), however it should be noted that this difference is not due to different lattice results for the matrix elements. Indeed the values of the matrix elements determined by the two groups are in good agreement. The difference is due to the different inputs: Hashimoto et al. use a lattice result for $f_{B_s} = 245 \pm 30\,\text{MeV}$ and the experimental value of the inclusive semileptonic branching ratio; APE use the lattice value of ξ and the experimental value of $\tau_{B_s}\Delta m_{B_d} m_{B_s}/m_{B_d}$.

[6] There are also some even more recent results in the static limit, both in the quenched approximation and with two flavours of sea quarks [26]. The results quoted in these papers are $5.1 \pm 1.9 \pm 1.7\,\%$ (quenched) and $4.3 \pm 2.0 \pm 1.9\,\%$ ($n_f = 2$), to be compared to those in eqs.(51) and (52).

7 The mass of the *b*-quark

The mass of the *b*-quark, m_b, is one of the parameters of the standard model. In this section I will review recent progress in evaluating m_b from simulations of effective theories, such as the HQET or NRQCD [7]. The discussion will be based on the evaluation of the two-point correlation function

$$C(t) = \sum_{\vec{x}} \langle 0|A_0(\vec{x},t)A_0(\vec{0},0)\,|0\rangle \tag{53}$$

in the HQET. A_μ is the axial current, $A_\mu = \bar{h}\,\gamma_\mu\gamma^5\,q$, and h (q) is the heavy-quark (light-quark) field. The evaluation of $C(t)$ is relatively straightforward and has been performed for many years now. The progress in recent years has been in the theoretical understanding of how one can determine m_b from $C(t)$, and in the perturbative calculations which have been performed up to three-loop order enabling the results to be obtained with good precision.

At large times t

$$C(t) \simeq Z^2 \exp(-\xi t) \tag{54}$$

and from the prefactor Z we obtain the value of the decay constant f_B in the static approximation [28]. It is for this reason that calculations of $C(t)$ have been performed for some years now. It was realised later that one can obtain m_b from the measured value of ξ (up to $O(\Lambda^2_{\rm QCD}/m_b)$ corrections) [29].

The relation between ξ and m_b is a delicate one:

$$M_B = m_b^{\rm pole} + \xi - \delta m\ , \tag{55}$$

where $M_B \simeq 5.28\,\mathrm{GeV}$ is the mass of the B-meson, $m_b^{\rm pole}$ is the pole-mass of the b-quark and ξ is determined from lattice computations using eq. (54). δm is the residual mass, generated in perturbation theory in the lattice formulation of the HQET,

$$\delta m = \frac{1}{a}\sum_n X_n\,\alpha_s^n(m_b)\ . \tag{56}$$

In other words even if the action in the HQET is written without a mass term, higher order terms in perturbation theory generate such a term, δm. Each term in the series for δm diverges linearly as $a \to 0$, and these terms (partially) cancel the divergence in ξ. In order for the cancellation to be sufficiently precise the lattice spacing a cannot be too small.

The perturbation series for δm diverges. Indeed it is not *Borel Summable*, leading to a *renormalon ambiguity*, which is an intrinsic ambiguity of $O(\Lambda_{\rm QCD})$. The pole mass of a quark is also not a physical quantity; it also contains a renormalon ambiguity [30, 31]. The renormalons in the pole mass and δm cancel (for a detailed discussion of the cancellation of such renormalons see ref. [32]). Let $\overline{m}_b$ be the b-quark mass defined in the $\overline{\rm MS}$ renormalization scheme defined at the scale $\overline{m}_b$ itself (I use $\overline{m}_b$ as an example of a "physical" quark mass). The pole mass and $\overline{m}_b$ can be related in perturbation theory:

$$\begin{aligned} \overline{m}_b &= [1+\sum_n D_n\,\alpha_s^n(m_b)]\,m_b^{\rm pole} && (57)\\ &= [1+\sum_n D_n\,\alpha_s^n(m_b)]\left\{M_B - \xi + \frac{1}{a}\sum_n X_n\,\alpha_s^n(m_b)\right\}\ . && (58)\end{aligned}$$

[7]For a detailed review of lattice determinations of quark masses in general, and m_b in particular, see ref. [27].

7.1 Perturbative Matching

All the perturbative coefficients necessary to compute $\overline{m}_b$ to N^3LO have recently been calculated.

- The coefficients D_2 and D_3 in the relation (57) between $\overline{m}_b$ and the pole mass have been calculated in refs. [33] and [34] respectively.
- The relation between the lattice (bare) coupling constant α_0 and the $\overline{\text{MS}}$ coupling takes the form
$$\alpha_s(\mu) = \left(1+\sum_n d_n(a\mu)\alpha_0^n\right)\alpha_0. \tag{59}$$
and the coefficient d_2 can be found in refs [35, 36].
- Finally we need to consider the perturbative expansion of the residual mass:
$$a\delta m = C_1\alpha_0 + C_2\alpha_0^2 + C_3\alpha_0^3 + \cdots \tag{60}$$
  - C_1 is well known, $C_1 = 2.1173$.
  - C_2 has been determined from the calculation of large Wilson loops
$$W(R,T) \sim \exp(-2\delta m(R+T)), \tag{61}$$
where R and T are the spatial and temporal lengths of the Wilson loop.
$C_2 = 11.152 + n_f(-0.282 + 0.035c_{SW} - 0.391c_{SW}^2)$ [37].
  - The Parma-Milan group has pioneered the numerical computation of perturbative coefficients at high orders using the stochastic formulation of theory. For the coefficients in eq. (60) they find [38]
$$C_1 = 2.09(4), \quad C_2 = 10.7(7), \qquad C_3 = 86.2(5)\ . \tag{62}$$
  - From a numerical Monte-Carlo study on very small and very fine-grained lattices Lepage at al. [39] find:
$$C_3 = 80(2) + 5.6(1.8)\ . \tag{63}$$

  For NRQCD only the first coefficient (C_1) is known.

7.2 Numerical results for $\overline{m}_b$

Before presenting the numerical results I repeat that the evaluation of ξ is relatively straightforward. The recent progress has been in the evaluation of the high order perturbative coefficients described in subsection 7.1. The results obtained using the method described in this section have been:

- $$\overline{m}_b = 4.15 \pm 0.05 \pm 0.20\,\text{GeV} \quad [40], \tag{64}$$
obtained in quenched QCD at NLO (i.e. with only the one-loop coefficient C_1). The second error is the estimate of the uncertainty due to higher order perturbation theory.

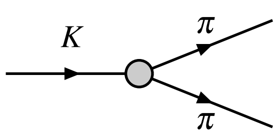
- $$\overline{m}_b = 4.30 \pm 0.05 \pm 0.10\,\mathrm{GeV} \quad [37], \tag{65}$$
  obtained in quenched QCD at N^{2}LO (with coefficients up to C_2).
- $$\overline{m}_b = 4.30 \pm 0.05 \pm 0.05\,\mathrm{GeV} \quad [41], \tag{66}$$
  obtained in quenched QCD at N^{3}LO (using all the known coefficients of section 7.1.
- $$\overline{m}_b = 4.34 \pm 0.03 \pm 0.06\,\mathrm{GeV} \quad [42], \tag{67}$$
  obtained from the static limit of quenched NRQCD at N^{3}LO.
- There is also an unquenched result:
  $$\overline{m}_b = 4.26 \pm 0.06 \pm 0.07\,\mathrm{GeV} \quad [43], \tag{68}$$
  obtained in unquenched QCD, with 2 flavours of sea-quarks at N^{2}LO.

We take the results in eqs.(66) (or (67)) and (68) as the current best estimates for $\overline{m}_b$.

Results obtained using NRQCD are in good agreement with those in the static theory, but we need to understand the errors due to higher orders of perturbation theory in that case.

8 Nonleptonic Decays

Up to now we have been unable to compute matrix elements with multi-hadron final states, in particular those relenant for the important class of two body decays such as $K \to \pi\pi$ and $B \to$ two-meson decays. The recent measurements of ε'/ε and the long-standing $\Delta I = 1/2$ rule problem (i.e. our inability to understand why amplitudes in which the isospin changes by 1/2 are enhanced by a factor of 22 or so with respect to those in which it changes by 3/2) make the control of non-perturbative QCD effects in $K \to \pi\pi$ decays particularly desirable. Exclusive two-body decays of B-mesons are one of the principal sources of information about the unitarity triangle and CP-violation. However, apart from the golden process $B \to J/\Psi K_s$, the determination of the fundamental parameters from two-body B-decays is severely restricted by our inability to quantify non-perturbative QCD effects. At present we are some way from understanding how to compute these effects in lattice simulations, however considerable effort is being devoted to developing the necessary theoretical framework for $K \to \pi\pi$ decay amplitudes

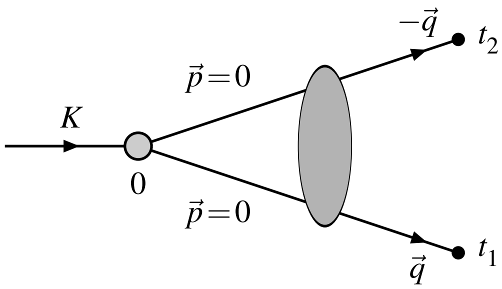


and I will briefly discuss recent recent ideas [44, 45, 46]. There are two important features which need to be understood:

1. Lattice calculations are performed in Euclidean space and hence yield real quantities. One can therefore ask how one can get the full decay amplitude including the phase due to the final state interactions? It is true that from each correlation function one obtains a real

quantity, however different correlators give different quantities and the full amplitude can (at least in principle) be reconstructed. For example, one can obtain the modulus of the amplitude from one correlator and the real part from another which is sufficient to determine the amplitude [46].

2. Although momentum is conserved in lattice calculations of correlation functions, energy is not.



   In this diagram the kaon decays at the origin producing two pions, which are annihilated at t_1 and t_2 respectively with momenta $\vec{q}$ and $-\vec{q}$. However the pions interact, and can rescatter (as represented by the grey oval). Whilst the total momentum remains zero, energy is not conserved and hence there is a contribution in which the two pions are each at rest (this is the lowest energy state). Correlation functions decay exponentially with time as $\exp(-Et)$, where E is the energy, and hence at large times the dominant contribution comes from the unphysical matrix element corresponding to a kaon at rest decaying into two pions, each of which is at rest. This is a major difficulty for lattice calculations.

   In a finite volume the energy levels of the two-pion states are discrete and Lüscher and Lellouch have proposed to exploit this fact to determine the decay amplitude directly [45]. One needs to have a state for which the energy of the two-pion state is equal to m_K, the mass of the kaon, and to be able to isolate the term corresponding to this energy. In practice, for the near future, this is likely to be the first excited state, which requires a lattice of about 6 fm, which is large but not hugely so [8].

The ideas discussed above need to be explored numerically and developed further. In particular we need to understand how to include the contributions from inelastic channels (i.e. contributions from intermediate states other than two-pion ones). The current numerical studies of $K \to \pi\pi$ decays are performed on lattices which are considerably smaller than 6 fm, and so correspond to unphysical kinematics. However, one can then use chiral perturbation theory to obtain the physical amplitudes. For B-physics of course chiral perturbation theory is inapplicable and the neglect of inelastic channels is not possible, so more theoretical developments are necessary.

9 Conclusions

Lattice simulations are a central tool in particle physics phenomenology. In order to gain the maximum return from the investment in future experimental facilities, it is crucial to gain a quantitative control of non-perturbative QCD effects. Lattice simulations provide the best prospect for the *ab*

[8] The finite volume corrections have also been discussed in considerable detail in refs. [45, 46].

initio computation of these effects. In these lectures I have presented an introduction to lattice phenomenology and illustrated the ideas by reviewing recent applications to B-physics.

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