Lattice Methods in Field Theory

Jonathan Flynn
University of Southampton

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University of Glasgow

Contents

1 Motivation

1.1 Theoretical

The lattice regularisation of quantum field theories

- is the only known nonperturbative regularisation
- admits controllable, quantitative nonperturbative calculations
- provides insight into how QFT's work and enables study of unsolved problems in QFT's

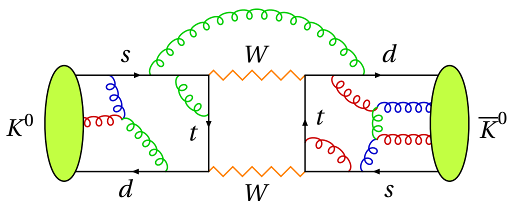
1.2 Applications of lattice field theories

- QED: 'triviality', fixed point structure, ...
- Higgs sector of the SM: bounds on Higgs mass, baryogenesis, ...
- Quantum gravity
- SUSY
- QCD: hadron spectrum, strong interaction effects in weak decays, confinement, chiral symmetry breaking, exotics, finite T and/or density, fundamental parameters (α_s, quark masses)

Why lattice QCD?

- evaluate non-perturbative strong interaction effects in physical amplitudes using large scale numerical simulations: observables found directly from QCD lagrangian
- long-distance QCD effects in weak processes are frequently the dominant source of uncertainty in extracting fundamental quantities from experiment

Example: K–$\overline{K}$ mixing and B_K



Since $m_t, m_W \gg \Lambda_{\rm QCD}$ can do perturbative analysis at high scales where QCD is weak and run by renormalisation group down to low scales. Left with:

$$\langle K^0 | C\Big(\alpha_s(\mu), \frac{m_t^2}{m_W^2}\Big) \frac{\overline{d}\gamma_\nu(1-\gamma_5)s\,\overline{d}\gamma^\nu(1-\gamma_5)s}{m_W^4} |\overline{K}^0\rangle_\mu + O\Big(\frac{1}{m_W^6}\Big)$$

- $C(\cdot)$: calculable perturbative coefficient (as long as $\mu/\Lambda_{\rm QCD}$ not too small)
- evaluate matrix element on a lattice with $\mu \sim 1/a$ (a is lattice spacing)
- match lattice result to continuum at scale $\mu \sim 1/a$

CKM matrix and unitarity triangle

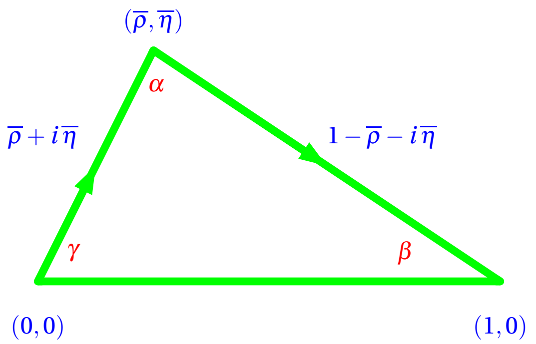
$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} =$$

$$\begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4)$$

Unitarity: $V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$

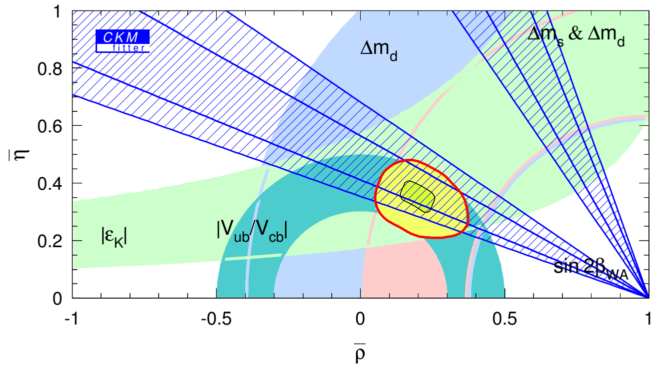
$$\begin{aligned} V_{ud}V_{ub}^* &= A\lambda^3(\overline{\rho} + i\overline{\eta}) + O(\lambda^7) \\ V_{cd}V_{cb}^* &= -A\lambda^3 + O(\lambda^7) \\ V_{td}V_{tb}^* &= A\lambda^3(1-\overline{\rho} - i\overline{\eta}) + O(\lambda^7) \end{aligned}$$

where $\overline{\rho} = \rho(1-\lambda^2/2)$ and $\overline{\eta} = \eta(1-\lambda^2/2)$.



Unitarity triangle

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$$



Measurement	$V_{\rm CKM} \times$ other	Constraint
$\dfrac{b\to u}{b\to c}$	$\left\lvert \dfrac{V_{ub}}{V_{cb}} \right\rvert^2$	$\overline{\rho}^2 + \overline{\eta}^2$
ΔM_d	$\lvert V_{td}\rvert^2 f_{B_d}^2 B_{B_d} f(m_t)$	$(1-\overline{\rho})^2 + \overline{\eta}^2$
$\dfrac{\Delta M_d}{\Delta M_s}$	$\left\lvert \dfrac{V_{td}}{V_{ts}} \right\rvert^2 \dfrac{f_{B_d}^2 B_{B_d}}{f_{B_s}^2 B_{B_s}}$	$(1-\overline{\rho})^2 + \overline{\eta}^2$
ε_K	$f(A,\overline{\eta},\overline{\rho},B_K)$	$\propto \overline{\eta}(1-\overline{\rho})$

(CKMfitter Spring 2002: H Höcker et al, hep-ph/0104062; http://ckmfitter.in2p3.fr/)

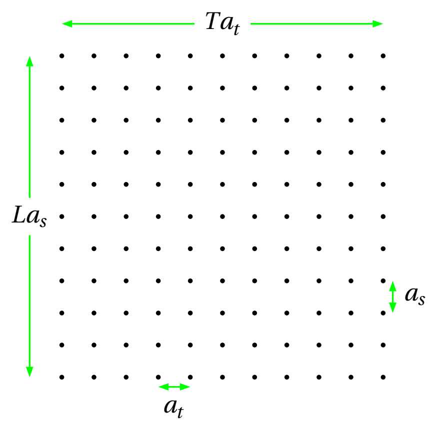
...with $\sin 2\beta$ from BaBar and Belle, Standard Model is in good shape. Errors in the nonperturbative parameters are now the limiting factor in more precise testing to look for effects from New Physics.

There is also a rich upcoming experimental programme in the next few years which will need or test lattice results:

- B-factories: constraining unitarity triangle, rare decays
- Tevatron Run II: ΔM_{B_s}, $\Delta\Gamma_{B_s}$, b-hadron lifetimes, ...
- CLEOc: leptonic and semileptonic D decays, masses of quarkonia, hybrids, glueballs
- LHC: ...

2 Basics: Euclidean quantisation

Lattice embedded in d-dimensional Euclidean spacetime



a_s, a_t	lattice spacings
La_s	length in spatial dimension(s)
Ta_t	length in temporal dimension

Matter fields live on lattice sites x. Example: scalar field

$$\phi(x) \quad \text{with} \quad \begin{array}{ll} x_j = na_s, & n = 0,\ldots,L-1 \\ x_0 = ma_t, & m =,\ldots,T-1 \end{array}$$

2.1 Lattice as a regulator

Fourier transform of a lattice scalar field in one dimension: $x = na$, $n = 0, \ldots, L-1$, with periodic boundary conditions:

$$\begin{aligned} \tilde{\phi}(p) &= a \sum_{n=0}^{L-1} \mathrm{e}^{-ipna} \phi(x) \\ \phi(x + La) &= \phi(x) \end{aligned}$$

- discretisation implies
  - $\tilde{\phi}(p)$ periodic with period $2\pi/a$
  - momenta lie in first Brillouin zone
  $$-\frac{\pi}{a} < p \leq \frac{\pi}{a}$$
  - have introduced a momentum cutoff;
  $$\Lambda = \frac{\pi}{a}$$
- spatial periodicity implies momentum p quantised in units of $2\pi/La$
- gauge invariance and gauge fields, fermions: later

Lattice provides both UV and IR cutoffs. Ultimately want infinite volume ($L, T \to \infty$) and continuum ($a \to 0$) limits. Most effort devoted to continuum limit.

2.2 Euclidean quantisation on the lattice

Path integral well-defined in Euclidean space

Minkowski $\longleftrightarrow$ (Wick rotation) Euclidean

$i\varepsilon$ prescription avoids poles

Procedure

1. Continuum classical Euclidean field theory
2. Discretisation $\longrightarrow$ lattice action
3. Quantisation $\longrightarrow$ functional integral

Step 1

Euclidean fields $\phi(x)$ obtained formally from analytic continuation

$$t \to -ix^0, \qquad \phi(\mathbf{x}, t) \to \phi(x)$$

Action:

$$S_E[\phi] = \int d^4x \left\{ \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi) \right\}$$

where $\mu = 0, 1, 2, 3$ and

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4$$

Minkowski	$\longleftrightarrow$	Euclidean
Lorentz symmetry		$O(4)$ symmetry
$t^2 - \mathbf{x}^2$ invariant		$(x^0)^2 + \mathbf{x}^2$ invariant
$\begin{pmatrix} + & & & \\ & - & & \\ & & - & \\ & & & - \end{pmatrix}$		$\begin{pmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{pmatrix}$

Step 2: Discretisation

Introduce a hypercubic lattice Λ_E with $a_t = a_s = a$.

$$\Lambda_E = \left\{ x \in a\mathbb{Z}^4 \middle| \frac{x^0}{a} = 0, \ldots, T-1; \quad \frac{x^{1,2,3}}{a} = 0, \ldots, L-1 \right\}$$

- $L^3 T$ lattice sites
- finite volume
- finite number d.o.f.

Lattice action:

$$S_E[\phi] = a^4 \sum_{x \in \Lambda_E} \left\{ \frac{1}{2} \nabla_\mu \phi(x) \nabla_\mu \phi(x) + V(\phi) \right\}$$

with forward and backward lattice derivatives

$$\begin{aligned} \nabla_\mu \phi(x) &\equiv \frac{1}{a}\big(\phi(x+a\hat{\mu}) - \phi(x)\big) \\ \nabla^*_\mu \phi(x) &\equiv \frac{1}{a}\big(\phi(x) - \phi(x-a\hat{\mu})\big) \end{aligned}$$

Lattice Laplacian:

$$\begin{aligned} \Delta\phi(x) &\equiv \sum_{\mu=0}^{3} (\nabla^*_\mu \nabla_\mu)\phi(x) \\ &= \frac{1}{a^2} \sum_{\mu=0}^{3} \big(\phi(x+a\hat{\mu}) + \phi(x-a\hat{\mu}) - 2\phi(x)\big) \end{aligned}$$

Lattice action for a free scalar field:

$$S_E[\phi] = a^4 \sum_{x\in\Lambda_E} \left\{ -\frac{1}{2}\phi(x)\Delta\phi(x) + V(\phi) \right\}$$

Remarks

- Discretisation is not unique. Can use different definitions for $\nabla_\mu^{(*)}$ and/or $V(\phi)$ as long as they become the same in the naive continuum limit, $a \to 0$.
  - Universality: discretisations fall into classes, each member of which has the same continuum limit
  - Improvement: optimise choice of lattice action for a faster approach to the continuum limit
- $O(4)$ (eventually Lorentz symmetry) is not preserved. Have cubic symmetry instead; recover $O(4)$ symmetry as $a \to 0$.

Step 3: Quantisation—functional integral

$$Z_E \equiv \int D[\phi] e^{-S_E[\phi]}$$

$D[\phi]$ is the measure, eg:

$$D[\phi] = \prod_{x\in\Lambda_E} d\phi(x)$$

- finite number of integrations

Correlation functions

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle \equiv \frac{1}{Z_E}\int D[\phi]\phi(x_1)\cdots\phi(x_n)e^{-S_E[\phi]}$$

- $\langle\cdot\rangle$ is shorthand for $\langle 0|\mathrm{T}\cdot|0\rangle$, time-ordered vacuum expectation value
- well-defined if $S_E[\phi] > 0$
- particle spectrum implicitly determined by correlation functions
- analytically continue to Minkowski space and get S-matrix elements (= physics) via LSZ

2.3 Generating functional

Scalar product on space $\mathscr{F}$ of fields ϕ over Λ_E:

$$(\phi_1,\phi_2) = a^4 \sum_{x\in\Lambda_E} \phi_1(x)\phi_2(x)$$

Action for free scalar field:

$$S_E[\phi] = \frac{1}{2}(\phi, K\phi), \qquad K = -\nabla_\mu^*\nabla_\mu + m^2$$

K is a linear operator on $\mathscr{F}$.

Let $J(x)$ be an external field (source) on Λ_E, $J\in\mathscr{F}$, and define the generating functional $W[J]$ through,

$$\begin{aligned} e^{W[J]} &\equiv \langle e^{(J,\phi)}\rangle \\ &= \frac{1}{Z_E}\int \prod_{x\in\Lambda_E} d\phi(x) e^{-S_E[\phi]} e^{(J,\phi)} \end{aligned}$$

Correlation functions found by differentiating w.r.t. $J(x)$:

$$\begin{aligned} \frac{\partial}{\partial J(x)} e^{W[J]} &= a^4\langle\phi(x)e^{(J,\phi)}\rangle \\ \left.\frac{\partial^2}{\partial J(x_1)\partial J(x_2)} e^{W[J]}\right|_{J=0} &= (a^4)^2\langle\phi(x_1)\phi(x_2)\rangle \end{aligned}$$

Lattice propagator

Relation between $W[J]$ and K:

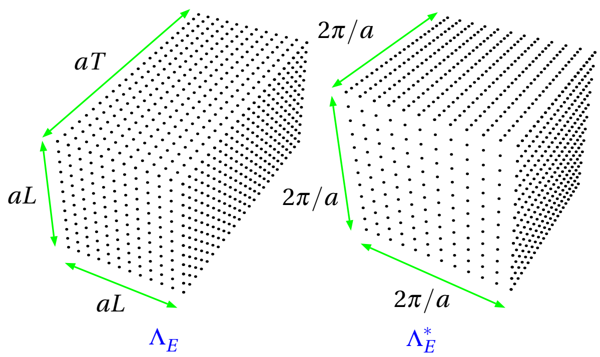
$$e^{W[J]} = \frac{1}{Z_E}\int \prod_{x\in\Lambda_E} d\phi(x) e^{-S_E[\phi]} e^{(J,\phi)} = e^{\frac{1}{2}(J,K^{-1}J)}$$

Diagonalise K through Fourier transform:

$$\tilde{J}(p) = a^4 \sum_{y\in\Lambda_E} e^{-ip\cdot y} J(y), \qquad J(x) = \frac{1}{a^4 L^3 T}\sum_{p\in\Lambda_E^*} e^{ip\cdot x}\tilde{J}(p)$$

Λ_E^* is the dual lattice (or set of momentum points in the Brillouin zone):

$$\begin{aligned} \Lambda_E^* = & \left\{ p \,\middle|\, p^0 = \frac{2\pi}{Ta} n_0, \quad p^{1,2,3} = \frac{2\pi}{La} n_{1,2,3}; \right. \\ & \left. n_0 = 0,\ldots,T-1, \quad n_{1,2,3} = 0,\ldots,L-1 \right\} \end{aligned}$$



Propagator (continued)

Find:

$$(K^{-1}J)(x) = a^4 \sum_{y\in\Lambda_E} G(x-y)J(y)$$

$G(x-y)$ is the Green function for K:

$$G(x-y) = \frac{1}{a^4L^3T}\sum_{p\in\Lambda_E^*}\frac{e^{ip\cdot(x-y)}}{\hat{p}^2+m^2}$$

with

$$\hat{p}^2 = \sum_{\mu=0}^{3}\hat{p}_\mu\hat{p}_\mu, \qquad \hat{p}_\mu = \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right)$$

$$e^{W[J]} = \exp\left\{\frac{1}{2}a^8\sum_{x,y\in\Lambda_E}J(x)G(x-y)J(y)\right\}$$

Therefore the propagator is:

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{a^8}\frac{\partial^2}{\partial J(x)\partial J(y)}e^{W[J]}\bigg|_{J=0} = G(x-y)$$

Remarks

- As $a\to 0$ (and $L,T\to\infty$), $G(x-y)$ becomes the Euclidean Feynman propagator:

$$G(x-y) \xrightarrow{a\to 0} \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{p^2+m^2}$$

  using $\hat{p}^2 = p^2 + O(a^2)$.

- Particle masses defined through poles of the propagator, here poles of $(\hat{p}^2+m^2)^{-1}$, which is periodic in each component of p with period $2\pi/a$.

$$\hat{p}^2 = \frac{4}{a^2}\sum_{\mu=0}^{3}\sin^2\left(\frac{ap_\mu}{2}\right)$$

  Unique mass spectrum inside first Brillouin zone, $p_\mu \in (-\pi/a, \pi/a]$.

Free Scalar Two-point Correlator in Position Space

$$C(x_0) = a^3\sum_{\mathbf{x}}\langle\phi(x_0,\mathbf{x})\phi(0)\rangle = a^3\sum_{\mathbf{x}}G(x_0,\mathbf{x})$$

- Create a particle at the origin; propagate it to any spatial point at time x_0
- $\sum_{\mathbf{x}}\cdot$ projects onto zero 3-momentum ($\sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}}\cdot$ would project on momentum $\mathbf{p}$).
- Can evaluate explicitly for free scalar field (exercise)

$$C(x_0) = \frac{e^{-\overline{m}|x_0|}}{2m(1+m^2a^2/4)^{1/2}}$$

  with 'physical mass' (position of pole in propagator for $\mathbf{p}=0$) $\overline{m}$, satisfying

$$\sinh\left(\frac{a\overline{m}}{2}\right) = \frac{am}{2}$$

- Fitting exponential decay of a lattice 2-point function lets you extract masses. Works more generally, see later.

QCD Lore

Confinement:

- gluons confine quarks and antiquarks into colour singlets

$\overline{q}q$	qqq
mesons	baryons

- gluons confine gluons into glueballs
- no free quarks or gluons as asymptotic states

Lattice QCD: a non-perturbative regulator which preserves gauge symmetry $\longrightarrow$ allows us to study these questions

3 Nonabelian lattice gauge fields

Perform steps 1–3 for Yang-Mills

Step 1

Classical continuum Euclidean Yang-Mills action

$$S_E[A] = -\frac{1}{2g_0^2}\int d^4x \,\mathrm{Tr}\left(F_{\mu\nu}(x)F_{\mu\nu}(x)\right)$$

g_0 is the bare gauge coupling

$SU(N)$ gauge fields in $\mathbb{R}^4$ (N colours):

$A_\mu(x)$	$x \in \mathbb{R}^4$, $\mu = 0,\ldots,3$
$A_\mu = A_\mu^a T^a$	$A_\mu \in su(N)$ (Lie algebra)
T^a	$a = 1,\ldots,N^2-1$. Generators
A_μ^a	real vector field
$[T^a, T^b] = f^{abc}T^c$	Structure constants f^{abc}
$A_\mu^\dagger = -A_\mu$	(antihermitian)
$D_\mu = \partial_\mu + A_\mu$	

Field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Gauge transformations:

$$\begin{aligned} A_\mu(x) &\to g(x)A_\mu(x)g^{-1}(x) + g(x)\partial_\mu g^{-1}(x) \\ F_{\mu\nu} &\to g(x)F_{\mu\nu}g^{-1}(x) \end{aligned}$$

Step 2: Discretisation

For the discretised gauge field, $A_\mu(x)$, $x \in \Lambda_E$, the transformation law

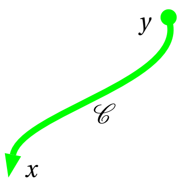
$$A_\mu(x) \to g(x)A_\mu(x)g^{-1}(x) + g(x)\nabla_\mu g^{-1}(x)$$

is inconsistent with group multiplication for nonabelian groups (Exercise).

- naive discretisation of classical Y-M action fails
- need a different concept to discretise pure gauge theory
- use parallel transport on lattice

3.1 Defining lattice gauge fields

Continuum



Curve $\mathscr{C}$ from y to x parametrised by:

$$z(t), \quad 0 \le t \le 1$$
$$z(0) = y, \quad z(1) = x$$

Parallel transport along $\mathscr{C}$:

$$\left\{\frac{d}{dt} + \dot{z}^\mu A_\mu(z)\right\} v(t) = 0$$

Solution

$$v(1) = \mathrm{PO}\exp\left\{-\int_y^x dz^\mu A_\mu(z)\right\} v(0)$$

Parallel transporter from y to x along $\mathscr{C}$ is,

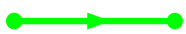
$$U^{\mathscr{C}}(x,y) = \mathrm{PO}\exp\left\{-\int_y^x dz^\mu A_\mu(z)\right\}$$

Lattice

Choose $\mathscr{C}$'s connecting neighbouring lattice sites.

$U(x, x + a\hat{\mu}) \equiv U_\mu(x) \in SU(N)$

x $\qquad$ $x + a\hat{\mu}$



$U(x + a\hat{\mu}, x) = U^{-1}(x, x + a\hat{\mu}) = U_\mu^{-1}(x)$

Definition

A lattice gauge field is a set of $SU(N)$ matrices $U_\mu(x)$, $x \in \Lambda_E$, $\mu = 0,1,2,3$. $U_\mu(x)$ is called a link variable.

Remarks

- Where is the gauge potential $A_\mu(x)$? Can define a lattice potential A_μ via

  $$U_\mu(x) = \mathrm{e}^{aA_\mu(x)}$$

  but this is not unique. If $A_\mu^{\mathrm{ctm}}(x)$ is a given continuum gauge potential, one can use a link variable to approximate it for small a:

  $$\lim_{a\to 0}\frac{1}{a}\left(U_\mu(x) - 1\right) = A_\mu^{\mathrm{ctm}}(x)$$

- Gauge transformations on the lattice. Let $g(x) \in SU(N)$ for $x \in \Lambda_E$.

  $$U_\mu(x) \to g(x)U_\mu(x)g^{-1}(x + a\hat{\mu})$$

  By inspection, if $\mathscr{C}$ is a closed loop of link variables then

  $$W(\mathscr{C}) = \mathrm{Tr}\, U^{\mathscr{C}}(x,x)$$

  is gauge-invariant. This is called a Wilson loop.

- Approximate locally gauge invariant continuum fields by gauge invariant combinations of link variables (see following example …).

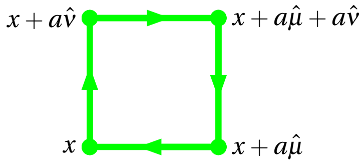
Exercise

$\mathrm{Tr}\left(F^{\mathrm{ctm}}_{\mu\nu}(x)F^{\mathrm{ctm}}_{\mu\nu}(x)\right)$

$F^{\mathrm{ctm}}_{\mu\nu}$ is a field strength defined in terms of a given continuum gauge potential A^{ctm}_{μ}.

Consider the plaquette:

$P_{\mu\nu}(x) \equiv$



Show that

$$\begin{aligned}\mathrm{Tr}\,P_{\mu\nu}(x) &= \mathrm{Tr}\left\{U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^{-1}(x+a\hat{\nu})U_\nu^{-1}(x)\right\} \\ &\stackrel{a\to 0}{=} N + \frac{a^4}{2}\mathrm{Tr}\left(F^{\mathrm{ctm}}_{\mu\nu}(x)F^{\mathrm{ctm}}_{\mu\nu}(x)\right) + O(a^5)\end{aligned}$$

3.2 Wilson plaquette action

Return to Step 2: discretise the continuum action:

$$S_E[A] = -\frac{1}{2g_0^2}\int d^4x\,\mathrm{Tr}\left(F_{\mu\nu}(x)F_{\mu\nu}(x)\right)$$

Consider

$$\begin{aligned}S_E[U] &= \frac{1}{g_0^2}\sum_{x\in\Lambda_E}\sum_{\mu,\nu}\mathrm{Tr}\left(1 - P_{\mu\nu}(x)\right) \\ &= \frac{1}{g_0^2}\sum_{x\in\Lambda_E}\sum_{\mu,\nu}\left(-\frac{a^4}{2}\mathrm{Tr}\left(F_{\mu\nu}(x)F_{\mu\nu}(x)\right) + O(a^5)\right) \\ &\stackrel{a\to 0}{\to} -\frac{1}{2g_0^2}\int d^4x\,\mathrm{Tr}\left(F_{\mu\nu}(x)F_{\mu\nu}(x)\right)\end{aligned}$$

Rewrite as

$$\begin{aligned}S_E[U] &= \frac{N}{g_0^2}\sum_{x\in\Lambda_E}\sum_{\substack{\mu,\nu\\ \mu<\nu}}\left(2 - \frac{1}{N}\mathrm{Tr}(P_{\mu\nu} + P^{\dagger}_{\mu\nu})\right) \\ &= \frac{2N}{g_0^2}\sum_{x\in\Lambda_E}\sum_{\substack{\mu,\nu\\ \mu<\nu}}\left(1 - \frac{1}{N}\mathrm{Re\,Tr}\,P_{\mu\nu}(x)\right) \\ &= \beta\sum_{\Box}\left(1 - \frac{1}{N}\mathrm{Re\,Tr}\,P_{\Box}\right)\end{aligned}$$

- $\sum_{\Box}$ is sum over all *oriented* plaquettes
- $\beta \equiv \frac{2N}{g_0^2}$ is the lattice coupling
- Last line is the Wilson plaquette action

$$\begin{aligned}S_E[U] &= \frac{N}{g_0^2}\sum_{x\in\Lambda_E}\sum_{\substack{\mu,\nu\\ \mu<\nu}}\left(2 - \frac{1}{N}\mathrm{Tr}(P_{\mu\nu} + P^{\dagger}_{\mu\nu})\right) \\ &= \frac{2N}{g_0^2}\sum_{x\in\Lambda_E}\sum_{\substack{\mu,\nu\\ \mu<\nu}}\left(1 - \frac{1}{N}\mathrm{Re\,Tr}\,P_{\mu\nu}(x)\right) \\ &= \beta\sum_{\Box}\left(1 - \frac{1}{N}\mathrm{Re\,Tr}\,P_{\Box}\right)\end{aligned}$$

- $\sum_{\Box}$ is sum over all *oriented* plaquettes
- no A_μ fields: degrees of freedom are $SU(3)$ matrices
- $\beta \equiv \frac{2N}{g_0^2}$ is the lattice coupling
- Last line is the Wilson plaquette action
- not obligatory to use simple plaquette: all traces of closed Wilson loops are proportional to $F{\cdot}F$ as $a \to 0$, allowing other choices for lattice gauge action

Step 3: Quantisation

Define a functional integral:

$$Z = \int D[U]\mathrm{e}^{-S_E[U]} = \int\prod_{x\in\Lambda_E}\prod_{\mu=0}^{3}dU_\mu(x)\mathrm{e}^{-S_E[U]}$$

$dU_\mu(x)$: invariant group measure for compact Lie group, eg $SU(N)$

$$U_\mu(x) \to U^g_\mu(x) = g(x)U_\mu(x)g^{-1}(x + a\hat{\mu})$$

$$dU^g_\mu(x) = dU_\mu(x) \quad \text{so that} \quad D[U^g] = D[U]$$

Measure can be normalised, since $SU(N)$ compact:

$$\int_{SU(N)} dU = 1$$

Not true for $\int dA_\mu$, $A_\mu \in su(N)$

- Functional integral well-defined: finite number of variables integrated over compact domain
- No gauge fixing required in lattice gauge theory (in general: but becomes necessary if you want to do a perturbative evaluation of the integral because of zero modes in the quadratic part of the action)

3.3 Strong coupling expansion

Expectation values in lattice Yang-Mills theory:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int D[U] \mathcal{O} e^{-S_E[U]}$$

$$S_E[U] = \beta \sum_{\square} \left(1 - \frac{1}{N} \operatorname{Re} \operatorname{Tr} P_{\square}\right) = -\frac{\beta}{2N} \sum_{\square} \operatorname{Tr}(P_{\square} + P_{\square}^{\dagger}) + \text{const}$$

- $\beta = 2N/g_0^2$ is a small parameter for large g_0^2
- evaluate $\langle \mathcal{O} \rangle$ by expanding $\exp(-S_E[U])$ in powers of β
- strong coupling expansion (high T, $\beta = 1/T$)
- evaluate integrals in group space order by order in β

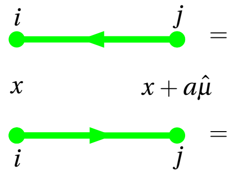
$$\exp\left\{\frac{\beta}{2N} \sum_{\square} \operatorname{Tr}(P_{\square} + P_{\square}^{\dagger})\right\} = \prod_{\square} \left\{1 + \frac{\beta}{2N} \operatorname{Tr}(P_{\square} + P_{\square}^{\dagger}) + O(\beta^2)\right\}$$

Group integration (compact Lie groups)

Consider link variable $U_\mu(x) \equiv U \in SU(N)$

U is a complex $N \times N$ matrix, $\det U = 1$

Write U_{ij}, with matrix (colour) indices i, j



$= U_{ij}$

$= U_{ij}^{-1}$

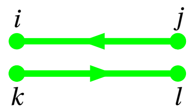
Group integrals:

$$\int dU = 1$$

$$\int dU\, U_{ij} = 0$$

$$\int dU\, U_{ij} U_{kl}^{-1} = \frac{1}{N} \delta_{ik} \delta_{jl}$$

$$\int dU\, U_{i_1 j_1} \cdots U_{i_N j_N} = \frac{1}{N!} \varepsilon_{i_1 \cdots i_N} \varepsilon_{j_1 \cdots j_N}$$



$= \frac{1}{N} \delta_{ik} \delta_{jl}$

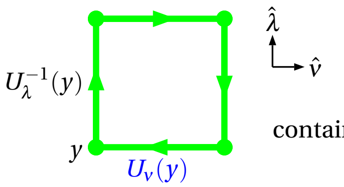
To Show $\langle U_\nu(y) \rangle = 0$

$$\langle U_\nu(y) \rangle = \frac{1}{Z} \int \prod_{x \in \Lambda_E} \prod_{\mu=0}^{3} dU_\mu(x) U_\nu(y) e^{-S_E[U]}$$

with

$$S_E[U] = -\frac{\beta}{2N} \sum_{\square} \operatorname{Tr}(P_{\square} + P_{\square}^{\dagger})$$

Pick out plaquettes involving $U_\nu(y)$



contains $\operatorname{Tr}(\cdots U_\lambda^{-1}(y) U_\nu(y) \cdots)$

Change variables on *other* links starting/ending at y.

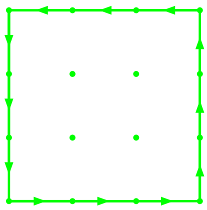
$$U_\lambda(y) \to U_\nu(y) U_\lambda(y)$$

- makes S_E independent of $U_\nu(y)$
- doesn't change measure
- leaves factor $\int dU_\nu(y)\, U_\nu(y) = 0$

An example of Elitzur's theorem: all gauge *non*-invariant combinations of U's have vanishing expectation values.

Area Law

Let $\mathcal{O}$ be a Wilson loop, ie a rectangle of size $R \times T$.



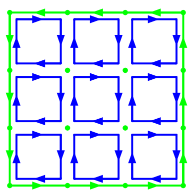
$$\equiv W(R, T) = \frac{1}{N} \operatorname{Tr} U^{R \times T}$$

Expectation value of $W(R, T)$

$$\langle W(R,T) \rangle = \frac{1}{Z} \int D[U] \frac{1}{N} \operatorname{Tr} \prod_{U \in \mathscr{C}} U \times \prod_{\square} \left\{1 + \frac{\beta}{2N}(P_{\square} + P_{\square}^{\dagger}) + O(\beta^2)\right\}$$

List the contributions to $\langle W(R,T) \rangle$ order by order in β

- Order β^0: only group integrals of type
$$\int dU U = 0$$
- Order β: consider all plaquettes inside $W(R, T)$, so that each link of $W(R, T)$ pairs up with a link in the opposite direction. This is tiling the Wilson loop with plaquettes.



Order β contributions

1. Each plaquette inside $W(R,T)$ contributes

$$\frac{\beta}{2N} \quad \text{leading to a factor} \quad \left(\frac{\beta}{2N}\right)^{RT}$$

2. Each group integration contributes

$$\frac{1}{N} \quad \text{leading to a factor} \quad \left(\frac{1}{N}\right)^{(R+1)T+(T+1)R}$$

3. Each site contributes a factor N
   - colour indices of all links meeting at each site must be the same (group integration plus trace)
   - N possibilities to choose the colour at each site

$$N \quad \text{leading to a factor} \quad N^{(R+1)(T+1)}$$

4. All integrations outside $W(R,T)$ give 1

The total contribution is:

$$\frac{1}{N}\left(\frac{\beta}{2N}\right)^{RT}\left(\frac{1}{N}\right)^{2RT+R+T-RT-R-T-1} = \left(\frac{\beta}{2N^2}\right)^{RT}$$

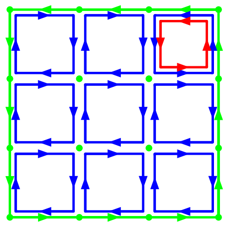
Therefore

$$\langle W(R,T)\rangle = \left(\frac{\beta}{2N^2}\right)^{RT} + \text{higher orders}$$

But $RT = A$, area of the Wilson loop

Exercise

Work out the $O(\beta^2)$ contribution in the strong coupling expansion of the Wilson loop



leads to

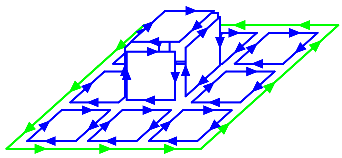
$$\left(\frac{\beta}{2N^2}\right)^{RT} RT \left(\frac{\beta}{4N}\right)$$

So that

$$\langle W(R,T)\rangle = \left(\frac{\beta}{2N^2}\right)^{RT}\left(1 + RT\left(\frac{\beta}{4N}\right) + O(\beta^2)\right)$$

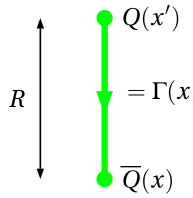
Still higher orders involve evaluation of *non-planar* graphs

...but all successive terms depend on the area RT



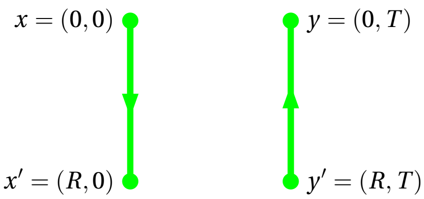
3.4 Area law and linear confinement

Physical interpretation of area law: consider a static quark-antiquark pair separated by distance R:



$$= \Gamma(x,x') = \overline{Q}(x)U(x,x')Q(x')$$

Static quarks: propagate only in (Euclidean) time



Correlation function:

$$\begin{aligned} C(R,T) &= \langle 0|\Gamma^\dagger(y,y')\Gamma(x,x')|0\rangle \\ &= \sum_n \langle 0|\Gamma^\dagger(y,y')|n\rangle\langle n|\Gamma(x,x')|0\rangle \\ &= \sum_n \left|\langle 0|\Gamma|n\rangle\right|^2 e^{-E_nT} \\ &\overset{T\to\infty}{\propto} e^{-E(R)T} \end{aligned}$$

$E(R)$: energy of a $Q\overline{Q}$ pair separated by distance R

Relation of $C(R,T)$ with Wilson loop?

$$\begin{aligned} C(R,T) &= \langle 0|\overline{Q}(y')U(y',y)Q(y)\overline{Q}(x)U(x,x')Q(x')|0\rangle \\ &\sim \left\langle \mathrm{Tr}\{S_Q(x',y')U(y',y)S_Q(y,x)U(x,x')\}\right\rangle \end{aligned}$$

- Tr is over colour and spin

Solution for static quark propagator S_Q:

$$\begin{aligned} S_Q(y,x) &= \delta^3(\mathbf{y}-\mathbf{x})U(y,x)\frac{1+\gamma_0}{2}e^{-m_Q(y_0-x_0)}, \quad y_0 > x_0 \\ S_Q(x',y') &= \gamma_5\left[S_Q(y',x')\right]^\dagger\gamma_5, \quad y_0' > x_0' \end{aligned}$$

Substituting:

$$\begin{aligned} C(R,T) &\sim \left\langle \mathrm{Tr}\{U(x',y')U(y',y)U(y,x)U(x,x')\}\right\rangle \\ &\quad \times \operatorname*{Tr}_{\text{spin}}\left\{\left(\frac{1+\gamma_0}{2}\right)^2\right\} e^{-2m_QT} \\ &\propto \langle W(R,T)\rangle e^{-2m_QT} \end{aligned}$$

So finally:

$$\langle W(R,T)\rangle \sim e^{-(E(R)-2m_Q)T} = e^{-V(R)T}$$

$V(R)$ is static quark potential, potential of a $Q\overline{Q}$ pair separated by R

Linear confinement

Use strong coupling expansion for $\langle W(R,T)\rangle$ to compute $V(R)$:

$$\langle W(R,T)\rangle = \left(\frac{\beta}{2N^2}\right)^{RT} = \mathrm{e}^{\ln(\beta/2N^2)RT}$$

Write $r = Ra$, $t = Ta$

$$\langle W(R,T)\rangle = \mathrm{e}^{a^{-2}\ln(\beta/2N^2)rt} = \mathrm{e}^{-V(r)t}$$

$$V(r) = -a^{-2}\ln(\beta/2N^2)r \equiv \sigma r$$

- area law implies linearly rising potential $V(r)$
- need infinite energy to separate Q and $\overline{Q}$ entirely
- linear confinement
- σ is called the string tension

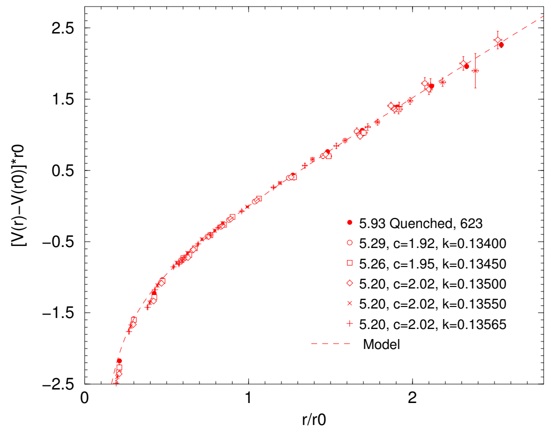
Result suggestive: strong coupling is *opposite* of continuum limit. Should supplement result with numerical studies extrapolated to continuum limit to confirm. Nonetheless, see a characteristic behaviour of strong-coupling gauge theories.

Static quark potential

- Strong coupling expansion yields $V(r) \sim \sigma r$
- Expect to see Coulomb part, $V(r) \sim 1/r$, for small r
- General functional form of $V(r)$:

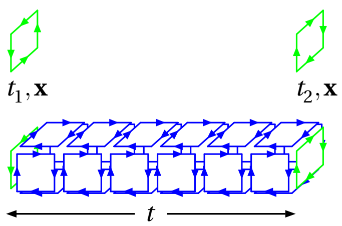
$$\begin{array}{rcl} V(r) & = & V_0 + \sigma r - \dfrac{e}{r} \\ \sigma & & \text{string tension} \\ e & & \text{'charge'} \end{array}$$

- Determine $V(r)$ via numerical simulation by 'measuring' Wilson loops (UKQCD hep-lat/0107021)
- $e = \pi/12$ in bosonic string model (Lüscher 1981): confirmed numerically (Lüscher and Weisz, hep-lat/0207003)



3.5 Plaquette-plaquette correlation

Correlator of two plaquettes at same spatial position, different times. Smallest linking surface is a 1×1 tube joining the plaquettes



$$\langle \mathrm{Tr}(U_1)\,\mathrm{Tr}(U_2)\rangle \sim \mathrm{e}^{-mt}$$

with

$$m = -4\ln\beta + \cdots$$

Dynamical mass generation in pure Yang-Mills (glueball mass)

4 Lattice Fermi fields

Step 1:

Classical continuum Euclidean action for free fermions:

$$S[\overline{\psi},\psi] = \int d^4x\, \overline{\psi}(x)(\gamma_\mu \partial_\mu + m_0)\psi(x)$$

$\overline{\psi}$, ψ Grassmann valued

Recall in Minkowski spacetime: $\{\gamma_\mu^{\mathrm{M}}, \gamma_\nu^{\mathrm{M}}\} = 2g_{\mu\nu}$. Now define Euclidean Hermitian γ-matrices by:

$$\left.\begin{array}{rcl} \gamma_0 & = & \gamma_0^{\mathrm{M}} \\ \gamma_j & = & -i\gamma_j^{\mathrm{M}} \end{array}\right\} \quad \text{so} \quad \{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}, \qquad \gamma_\mu^\dagger = \gamma_\mu$$

Step 2: discretisation

$$\begin{array}{rcl} S[\overline{\psi},\psi] & = & a^4 \displaystyle\sum_{x\in\Lambda_E} \overline{\psi}(x)\big(\gamma_\mu \frac{1}{2}(\nabla_\mu + \nabla_\mu^*) + m_0\big)\psi(x) \\ & = & a^4 \displaystyle\sum_{x\in\Lambda_E} \overline{\psi}(x)Q\psi(x) \end{array}$$

where

$$Q = \frac{1}{2}(\nabla_\mu + \nabla_\mu^*)\gamma_\mu + m_0$$

is the 'naive' lattice Dirac operator

Step 3: Quantisation

$$Z \equiv \int D[\overline{\psi},\psi] e^{-S[\overline{\psi},\psi]}$$

Correlation functions:

$$\langle \psi(x)\overline{\psi}(y)\rangle = \frac{1}{Z}\int D[\overline{\psi},\psi]\,\psi(x)\overline{\psi}(y) e^{-S[\overline{\psi},\psi]}$$

Add Grassmann sources $\overline{\eta}$, ξ to get generating functional

$$e^{W[\overline{\eta},\xi]} = \left\langle e^{(\overline{\eta},\psi)+(\overline{\psi},\xi)}\right\rangle = e^{(\overline{\eta},Q^{-1}\xi)}$$

Diagonalise via Fourier transform:

$$\begin{aligned}(Q\xi)(x) &= \frac{1}{a^4L^3T}\sum_{p\in\Lambda_E^*} Q e^{ip\cdot x}\tilde{\xi}(p)\\ &= \frac{1}{a^4L^3T}\sum_{p\in\Lambda_E^*}(i\gamma_\mu\overline{p}_\mu + m_0)e^{ip\cdot x}\tilde{\xi}(p)\end{aligned}$$

- have defined $\overline{p}_\mu \equiv \frac{1}{a}\sin(ap_\mu)$
- Q acts by multiplication with $i\gamma_\mu\overline{p}_\mu + m_0$
- now easy to invert …

$$\begin{aligned}(Q^{-1}\xi)(x) &= \frac{1}{a^4L^3T}\sum_{p\in\Lambda_E^*}\frac{e^{ip\cdot x}}{i\overline{\not p}_\mu + m_0}\tilde{\xi}(p)\\ &= a^4\sum_{y\in\Lambda_E}\left(\frac{1}{a^4L^3T}\sum_{p\in\Lambda_E^*}\frac{e^{ip\cdot(x-y)}}{i\overline{\not p}_\mu + m_0}\right)\xi(y)\\ &\equiv a^4\sum_{y\in\Lambda_E} S_F(x-y)\xi(y)\end{aligned}$$

Generating functional:

$$e^{W[\overline{\eta},\xi]} = \exp\left\{a^4\sum_{x,y\in\Lambda_E}\overline{\eta}(x)S_F(x-y)\xi(y)\right\}$$

Two point function:

$$\begin{aligned}\langle\psi(x)\overline{\psi}(y)\rangle &= \frac{1}{a^8}\frac{\partial^2}{\partial\overline{\eta}(x)\partial\xi(y)}e^{W[\overline{\eta},\xi]}\Big|_{\overline{\eta},\xi=0}\\ &= S_F(x-y)\\ &\xrightarrow{a\to 0}\int\frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{i\not p + m_0} + O(a^2)\end{aligned}$$

Problems with naive discretisation

1. $\overline{\not p} = \not p + O(a^2)$
2. Particle masses are defined through poles of the propagator. Here, poles of $(i\overline{\not p} + m_0)^{-1}$ for $m_0 \ll p_\mu$ are near:
   $$\overline{\not p} = 0 \quad\text{or}\quad \frac{1}{a}\sin(ap_\mu) = 0$$
   - satisfied for $p_\mu = 0, \pi/a$
   - corners of Brillouin zone yield additional poles
   - in $D = 4$ there are $2^D = 16$ poles and hence a 16-fold degeneracy in the spectrum

   This is the fermion doubling problem

   In interacting theory, momenta of order π/a can flip you between different doublers: spurious 'flavour-changing' interactions
3. How to deal with fermion doubling?
   - ignore it: quarks come in sixteen different flavours ✗
   - staggered fermions (Kogut-Susskind): partial lifting of degeneracy, $16 \to 4$.
   - Wilson fermions: complete lifting of degeneracy but explicit chiral symmetry breaking at finite a.

4.1 Wilson fermions

Add extra term to the naive lattice Dirac operator which formally vanishes as $a \to 0$:

$$\begin{aligned}S_W[\overline{\psi},\psi] &= a^4\sum_{x\in\Lambda_E}\overline{\psi}(x)\left(\gamma_\mu\frac{1}{2}(\nabla_\mu+\nabla_\mu^*)+m_0\right)\psi(x)\\ &\quad -\frac{ra^5}{2}\sum_{x\in\Lambda_E}\overline{\psi}(x)\nabla_\mu^*\nabla_\mu\psi(x)\\ &= a^4\sum_{x\in\Lambda_E}\overline{\psi}(x)\left[Q_W\psi\right](x)\end{aligned}$$

Have defined the Wilson-Dirac operator

$$Q_W \equiv \frac{1}{2}\gamma_\mu(\nabla_\mu+\nabla_\mu^*) + m_0 - \frac{ra}{2}\nabla_\mu^*\nabla_\mu$$

where r is the Wilson parameter, $r = O(1)$ (and usually set to 1)

Q_W acts by multiplication with

$$i\overline{\not p} + m_0 + \frac{ra}{2}\hat{p}^2$$

Wilson propagator:

$$S_W(x-y) = \frac{1}{a^4L^3T}\sum_{p\in\Lambda_E^*}\frac{e^{ip\cdot(x-y)}}{i\overline{\not p} + m_0 + \frac{ra}{2}\hat{p}^2}$$

Adding the Wilson term, $-(ra/2)\overline{\psi}(x)\Delta\psi(x)$, modifies the dispersion relation:

$$m_0 \longrightarrow m_0 + \frac{ra}{2}\hat{p}^2$$

Term proportional to the Wilson parameter r vanishes in the classical continuum limit $a \to 0$ and we recover the continuum Euclidean fermion propagator.

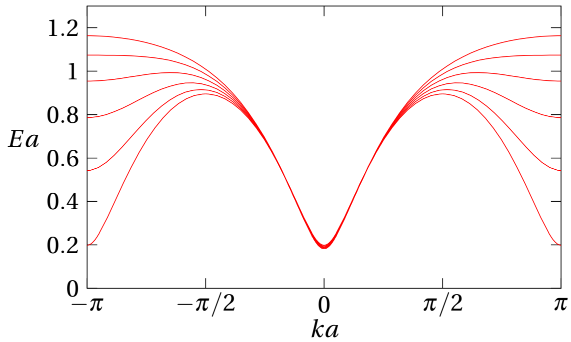
After adding the Wilson term, mass terms near corners of BZ are:

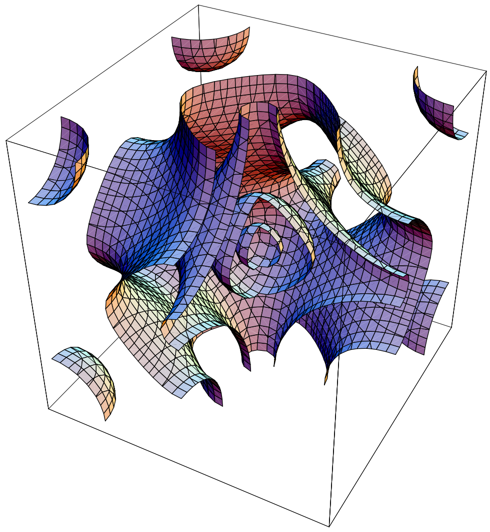
p_μ	mass	multiplicity
$(0,0,0,0)$	m_0	1
$(\frac{\pi}{a},0,0,0)$	$m_0 + 2\frac{r}{a}$	4
$(\frac{\pi}{a},\frac{\pi}{a},0,0)$	$m_0 + 4\frac{r}{a}$	6
$(\frac{\pi}{a},\frac{\pi}{a},\frac{\pi}{a},0)$	$m_0 + 6\frac{r}{a}$	4
$(\frac{\pi}{a},\frac{\pi}{a},\frac{\pi}{a},\frac{\pi}{a})$	$m_0 + 8\frac{r}{a}$	1

Choose $r = 1$: states associated with corners of BZ receive masses of order $1/a$, ie of order the cutoff scale

- these states are removed from the spectrum
- one fermion species survives in the continuum limit

Wilson fermion dispersion relation for momentum $(k,0,0)$ with $-\pi < ka \leq \pi$, $ma = 0.2$ and $r = 0, 0.2, 0.4, 0.6, 0.8, 1$.





Explicit form of the Wilson action:

$$\begin{aligned} S_W[\overline{\psi},\psi] &= a^4 \sum_{x\in\Lambda_E} \Big\{ \frac{1}{2a}\sum_\mu \big[\overline{\psi}(x)(\gamma_\mu - r)\psi(x+a\hat{\mu}) \\ &\qquad - \overline{\psi}(x+a\hat{\mu})(\gamma_\mu + r)\psi(x)\big] \\ &\qquad + \big(m_0 + \frac{4r}{a}\big)\overline{\psi}(x)\psi(x) \Big\} \end{aligned}$$

Set $r = 1$: 'project out' components of Dirac spinor through appearance of $\frac{1}{2}(1 \pm \gamma_\mu)$ to lift the degeneracy.

Problem: for $m_0 = 0$, $S_W[\overline{\psi},\psi]$ is no longer invariant under chiral transformations

$$\psi(x) \longrightarrow e^{i\alpha\gamma_5}\psi(x)$$

- chiral symmetry is broken explicitly by the regularisation procedure
  - only restored as $a \to 0$: chiral and continuum limits are bound together for Wilson fermions
  - lack of chiral symmetry makes operator mixing more complicated in lattice case than in continuum
  - possible to show that explicit chiral symmetry breaking by Wilson term appears in chiral Ward identities and becomes the anomaly term as $a \to 0$

4.2 Chiral symmetry on the lattice

Consider massless free fermions on the lattice with a lattice Dirac operator $Q = Q(x-y)$

$$S[\overline{\psi},\psi] = a^4 \sum_{x,y\in\Lambda_E} \overline{\psi}(x)Q(x-y)\psi(y)$$

Desirable properties of Q:

1. $Q(x-y)$ is local
2. $\tilde{Q}(p) = i\gamma_\mu p_\mu + O(ap^2)$
3. $\tilde{Q}(p)$ is invertible for $p \neq 0$
4. $\gamma_5 Q + Q\gamma_5 = 0$

Nielsen-Ninomiya no-go theorem (1981): 1–4 do not hold simultaneously

$\longrightarrow$ either left with doublers or chiral symmetry is explicitly broken

Ginsparg-Wilson relation

You *can* realise exact chiral symmetry on the lattice by replacing 4 with

$$\gamma_5 Q + Q\gamma_5 = aQ\gamma_5 Q$$

(P Ginsparg and KG Wilson PRD 25 (1982) 2649, M Lüscher hep-lat/9802011, 1998)

More on N-N conditions

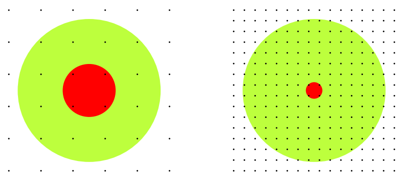
1. Locality Needed for renormalisability and universality of the continuum limit

Range over which fields are coupled in the action is infinitely smaller than any physical distance: compare

$$Q(x,y) \overset{|x-y|\gg 1}{\sim} \mathrm{e}^{-\gamma|x-y|} = \mathrm{e}^{-\frac{\gamma}{a}a|x-y|}$$

$$\text{correlation function} \sim \mathrm{e}^{-ma|x-y|}$$

where $\gamma = O(1)$ and m is a physical mass. As $a \to 0$ the former is exponentially suppressed with respect to the latter.



Cannot have long-range (non-universal) couplings in the action which would compete with the physical signals arising from universal collective behaviour.

2. $\tilde{Q}(p) = i\gamma_\mu p_\mu + O(ap^2)$ Want correct continuum limit

3. $\tilde{Q}(p)$ **invertible for** $p \neq 0$ No extra poles at non-zero momentum: no doublers

4. $\{Q, \gamma_5\} = 0$ Chiral symmetry

Wilson fermions give up entirely on chiral symmetry. Recent breakthrough: modify 4 to get chiral symmetry without doublers.

More on the GW relation

$$\gamma_5 Q + Q\gamma_5 = aQ\gamma_5 Q$$

or

$$Q^{-1}\gamma_5 + \gamma_5 Q^{-1} = a\gamma_5$$

Q^{-1} is highly non-local, but $\{Q^{-1}, \gamma_5\}$ should be local: the GW relation is highly non-trivial

GW relation is expected to imply 'physical' chiral symmetry on the lattice. Look at Ward identity for $\psi(x)\overline{\psi}(y)$ with $|x-y|$ a long distance, using usual chiral (γ_5) transformation. Get extra term from variation of the action:

$$\langle \overline{\psi(x)\overline{\psi}(z)}(aQ\gamma_5 Q)_{zz'}\overline{\psi(z)\overline{\psi}(y)}\rangle \sim$$
$$(Q^{-1})_{xz}(aQ\gamma_5 Q)_{zz'}(Q^{-1})_{z'y} \sim a\gamma_{5\,xy}$$

$\to$ this is local so negligible at long distances

In fact there's an exact chiral symmetry (Lüscher) (see later)

History

- 1982: GW wrote down the relation but no solution was found in the interacting case—it was forgotten
- 1997
  - realised that the Fixed Point Dirac operator of 'classically perfect' action satisfies GW
  - followed by observation that Dirac operators for Domain Wall Fermions (Kaplan, Shamir) and overlap formalism (Neuberger) also satisfy GW
- 1998: Lüscher demonstrated the chiral symmetry

Led to an explosion of interest. DWF and overlap already used in some numerical studies.

Exact chiral symmetry on the lattice

GW relation implies that $\overline{\psi}Q\psi$ is invariant under flavour singlet chiral transformations:

$$\psi \to \psi + i\varepsilon\gamma_5(1 - \frac{a}{2}Q)\psi$$
$$\overline{\psi} \to \overline{\psi} + i\varepsilon\overline{\psi}(1 - \frac{a}{2}Q)\gamma_5$$

and non-singlet chiral transformations:

$$\psi \to \psi + i\varepsilon T\gamma_5(1 - \frac{a}{2}Q)\psi$$
$$\overline{\psi} \to \overline{\psi} + i\varepsilon\overline{\psi}(1 - \frac{a}{2}Q)\gamma_5 T$$

where T is an $SU(N_f)$ generator

Slightly smeared version of usual chiral transformation.

Looks too good? In fact, singlet chiral transformation alters the measure

$$\delta D[\overline{\psi},\psi] = -\mathrm{Tr}(\gamma_5 Q)D[\overline{\psi},\psi]$$

$\to$ gives the correct anomalous Ward identity (just like Fujikawa in the continuum).

No anomaly in non-singlet case since $\mathrm{Tr}\, T = 0$

Anomaly in LGT with GW relation

Expectation value of some fermion operator

$$\langle \mathscr{O} \rangle = \frac{1}{Z} \int D[\overline{\psi}, \psi] \mathscr{O} e^{-S}$$

Apply chiral transformation as change of variable, remembering that S is invariant:

$$\delta\psi = \varepsilon\gamma_5(1 - \frac{1}{2}aQ)\psi \qquad \delta\overline{\psi} = \varepsilon\overline{\psi}(1 - \frac{1}{2}aQ)\gamma_5$$

$$\langle \mathscr{O} \rangle = \frac{1}{Z} \int D[\overline{\psi}', \psi'] \mathscr{O}' e^{-S} = \frac{1}{Z} \int D[\overline{\psi}, \psi] J(\mathscr{O} + \varepsilon\delta\mathscr{O}) e^{-S}$$

with Jacobian factor $J = \left| \frac{\partial(\psi', \overline{\psi}')}{\partial(\psi, \overline{\psi})} \right|$.

$$\begin{aligned} \frac{\partial \psi'_x}{\partial \psi_y} &= \delta_{xy} + \varepsilon\gamma_5(1 - \frac{1}{2}aQ_{xy}) \\ \frac{\partial \overline{\psi}'_x}{\partial \overline{\psi}_y} &= \delta_{xy} + \varepsilon(1 - \frac{1}{2}aQ_{xy})\gamma_5 \end{aligned}$$

$$\begin{aligned} J &= \det \begin{pmatrix} 1 + \varepsilon\gamma_5(1 - \frac{1}{2}aQ) & 0 \\ 0 & 1 + \varepsilon(1 - \frac{1}{2}aQ)\gamma_5 \end{pmatrix} \\ &= \det(1 + \varepsilon X) \det(1 + \varepsilon Y) \\ &= 1 + \varepsilon \operatorname{tr}(X + Y) = 1 - \varepsilon a \operatorname{tr}(\gamma_5 Q) \end{aligned}$$

where $X = \gamma_5(1 - \frac{1}{2}aQ)$, $Y = (1 - \frac{1}{2}aQ)\gamma_5)$ and used $\det = \exp \operatorname{tr} \ln$, $\operatorname{tr} \gamma_5 = 0$.

Combining:

$$\langle \mathscr{O} \rangle = \frac{1}{Z} \int D[\overline{\psi}, \psi] \big(1 - \varepsilon a \operatorname{tr}(\gamma_5 Q)\big)(\mathscr{O} + \varepsilon\delta\mathscr{O}) e^{-S}$$

To order ε

$$\int D[\overline{\psi}, \psi] \big(\delta\mathscr{O} - a \operatorname{tr}(\gamma_5 Q)\big) e^{-S} = 0$$

... giving the correct anomalous Ward identity for a global flavour-singlet axial transformation.

(Note: $\operatorname{tr}(\gamma_5 Q)$ vanishes in the free case, but it's non-zero in the presence of gauge fields.)

LH and RH chiral fermions

If have chiral symmetry expect to decompose

$$\overline{\psi} Q \psi = \overline{\psi}_+ Q \psi_+ + \overline{\psi}_- Q \psi_-$$

It's really possible:

$$\begin{aligned} \psi_- &= \hat{P}_- \psi \qquad & \psi_+ &= \hat{P}_+ \psi \\ \overline{\psi}_- &= \overline{\psi} P_+ \qquad & \overline{\psi}_+ &= \overline{\psi} P_- \end{aligned}$$

where $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ as usual and

$$\hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5)$$

$\hat{\gamma}_5$ is a 'smeared' γ_5:

$$\begin{aligned} \hat{\gamma}_5 &= \gamma_5(1 - aQ) \\ \hat{\gamma}_5 \hat{\gamma}_5 &= 1 \\ \gamma_5 Q &= -Q \hat{\gamma}_5 \end{aligned}$$

'Left' and 'right' become gauge-dependent ideas

Neuberger's operator

An operator Q satisfying the GW relation can be defined as follows. Let

$$Q_W = \frac{1}{2}\big(\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - a\nabla^*_\mu \nabla_\mu\big)$$

be the massless free Wilson-Dirac operator. Neuberger's operator is defined (in its simplest form) as:

$$Q_N = \frac{1}{a}\big(1 - A(A^\dagger A)^{-1/2}\big)$$

where

$$A = 1 - aQ_W$$

Exercise

Show that Q_N satisfies the GW relation

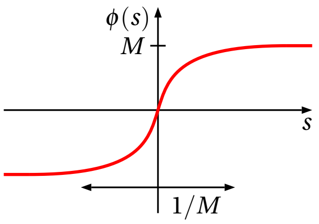
4.3 Domain Wall Fermions

Dirac fermion in 5 dimensions

$$D_5 = \gamma_\mu \partial_\mu + \gamma_5 \partial_s - \phi(s)$$

$\gamma_5 = -\gamma_0\gamma_1\gamma_2\gamma_3, \qquad \mu = 0, 1, 2, 3$
s: extra spatial coordinate

ϕ is a given potential representing a domain wall with height and width set by a scale M, e.g. $\phi(s) = M \tanh(Ms)$, but exact form not needed.



Planewave solutions

$$D_5\chi(x,s) = 0 \qquad \text{with } \chi(x,s) = e^{ip\cdot x}u(s)$$
$p = (iE, \mathbf{p})$ physical 4-momentum
$m^2 = E^2 - \mathbf{p}^2$ mass of the mode

Allowed m^2 determined from:

$$\left[\gamma_5\partial_s - \phi(s)\right]u(s) = -i\gamma_\mu p_\mu u(s)$$

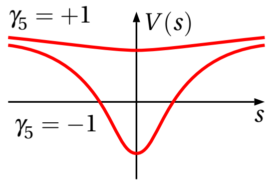
Multiply on left by $i\gamma_\mu p_\mu$

$$\left[-\partial_s^2 + V(s)\right]u(s) = m^2 u(s)$$

with $V(s) = \gamma_5\partial_s\phi(s) + \phi^2(s)$.

$$\left[-\partial_s^2 + V(s)\right]u(s) = m^2 u(s)$$

Assume eigenfunctions have definite chirality since $-\partial_s^2 + V(s)$ commutes with γ_5. Three cases:



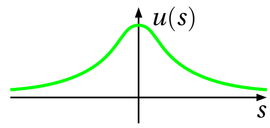
1. Continuous spectrum
   $V(s) \stackrel{|s|\to\infty}{\longrightarrow} M^2$ leads to eigenvalues with $m^2 \geq M^2$
2. Discrete spectrum
   eigenfunctions with $m^2 < M^2$ decay exponentially $\longrightarrow$ discrete spectrum. All non-zero masses are of order M (only scale). No negative masses since $-\partial_s^2 + V(s) = (-\gamma_5\partial_s + \phi)^\dagger(-\gamma_5\partial_s + \phi)$.
3. Massless modes

$$(-\gamma_5\partial_s + \phi)u(s) = 0, \qquad \gamma_\mu p_\mu u(s) = 0$$

with solutions

$$u(s) = \exp\left\{\pm\int_0^s \phi(t)dt\right\}v, \qquad \begin{cases} P_\pm v = v \\ \gamma_\mu p_\mu v = 0 \end{cases}$$

Only LH solution is normalisable. Massless mode bound to the wall



Summary

- all but one mode have mass $m \geq O(M)$
- massless mode: left-handed and bound to domain wall
- at energies $E \ll M$, theory describes a left-handed fermion in 4-dimensions

Domain Wall Fermions

Mechanism is stable against changes in setup:

- domain wall $\longrightarrow$ Dirichlet boundary condition
- Dirac fields $\chi(x,s)$ in $s \geq 0$ with

$$D_5 = D_4 + \gamma_5\partial_s - M$$

  satisfying

$$D_5\chi(x,s) = 0, \qquad P_+\chi(x,s)|_{s=0} = 0$$

- $\longrightarrow$ massless mode as before
- 5-dim fermion propagator satisfies

$$\begin{aligned} D_5 G(x,s;y,t)\big|_{s,t\geq 0} &= \delta(x-y)\delta(s-t) \\ P_+ G(x,s;y,t)\big|_{s=0} &= 0 \end{aligned}$$

- on the boundary you find:

$$G(x,0;y,0) = 2MP_- S(x,y)P_+$$

where $S(x,y)$ is the 4-dimensional propagator of the operator

$$\begin{aligned} D &\equiv M + (D_4 - M)\left[1 - (D_4/M)^2\right]^{-1/2} \\ &= D_4(1 - D_4/2M + \cdots) \end{aligned}$$

D describes a massless 4-dim fermion, reduces to D_4 as $M \to \infty$.

- D satisfies a Ginsparg-Wilson relation

$$\gamma_5 D + D\gamma_5 = \frac{1}{M}D\gamma_5 D$$

- (Kaplan 1992) The construction also works
  * in the presence of gauge fields (no s-dependence)
  * and on the lattice: $M \to 1/a$, $D_4 \to Q_W$ (massless Wilson-Dirac)

$$\begin{aligned} D &= \frac{1}{a}\left(1 - (1 - aQ_W)\left[(1-aQ_W)^\dagger(1-aQ_W)\right]^{-1/2}\right) \\ &= \frac{1}{a}\left(1 - A(A^\dagger A)^{-1/2}\right) \end{aligned}$$

  where $A = 1 - aQ_W$

- use a finite 5th-dimension: can have one chirality exponentially bound to one wall, other chirality on other wall

5 Lattice QCD

Formulate a lattice theory of quarks and gluons.

Lattice action:

$$S_{QCD}[U,\overline{\psi},\psi] = S_G[U] + S_F[U,\overline{\psi},\psi]$$

$S_G[U]$	Wilson plaquette action
$S_F[U,\overline{\psi},\psi]$	Wilson fermion action

Define a covariant derivative:

$$D_\mu\psi(x) = \frac{1}{a}\big(U_\mu(x)\psi(x+a\hat{\mu}) - \psi(x)\big)$$
$$D_\mu^*\psi(x) = \frac{1}{a}\big(\psi(x) - U_\mu^\dagger(x-a\hat{\mu})\psi(x-a\hat{\mu})\big)$$

For the Wilson-Dirac operator:

$$\frac{1}{2}\gamma_\mu(\nabla_\mu+\nabla_\mu^*) + m_0 - \frac{ra}{2}\nabla_\mu^*\nabla_\mu$$
$$\to \frac{1}{2}\gamma_\mu(D_\mu+D_\mu^*) + m_0 - \frac{ra}{2}D_\mu^* D_\mu$$

Set:
$r=1$
$a=1$ express all quantities in units of a

5.1 Fermion action in LQCD

$$S_F[U,\overline{\psi},\psi] = \sum_{x\in\Lambda_E}\Big\{ -\frac{1}{2}\sum_{\mu=0}^{3}\big[\overline{\psi}(x)(1-\gamma_\mu)U_\mu(x)\psi(x+\hat{\mu}) + \overline{\psi}(x+\hat{\mu})(1+\gamma_\mu)U_\mu^\dagger\psi(x)\big] + \overline{\psi}(x)\big(m_0+4\big)\psi(x)\Big\}$$

Rescale ψ and $\overline{\psi}$ by

$$\psi(x)\to\sqrt{2\kappa}\,\psi(x), \qquad \overline{\psi}(x)\to\overline{\psi}(x)\sqrt{2\kappa}$$

and fix κ by requiring $(m_0+4)2\kappa=1$

Lattice action for QCD with Wilson fermions becomes:

$$S_{\rm QCD}[U,\overline{\psi},\psi] = \beta\sum_{\Box}\big(1-\frac{1}{3}\,\mathrm{Re\,Tr}\,P_\Box\big) + \sum_{x\in\Lambda_E}\Big\{ -\kappa\sum_{\mu=0}^{3}\big[\overline{\psi}(x)(1-\gamma_\mu)U_\mu(x)\psi(x+\hat{\mu}) + \overline{\psi}(x+\hat{\mu})(1+\gamma_\mu)U_\mu^\dagger\psi(x)\big] + \overline{\psi}(x)\psi(x)\Big\}$$

We have traded parameters: $(g_0, m_0)\mapsto(\beta,\kappa)$, with:

$$\beta = \frac{6}{g_0^2}, \qquad \kappa = \frac{1}{2m_0+8} \qquad \text{(hopping parameter)}$$

5.2 Effective gauge action

Rewrite fermionic piece of LQCD action:

$$S_F[U,\overline{\psi},\psi] = \sum_{x\in\Lambda_E}\overline{\psi}(x)[Q_W\psi](x) \equiv \sum_{x,y\in\Lambda_E}\overline{\psi}(x)Q_{xy}\psi(y)$$

Q_{xy} is Wilson-Dirac operator in matrix notation ('quark matrix')

$$Q_{xy} = \delta_{xy} - \kappa\sum_{\mu=0}^{3}\delta_{y,x+\hat{\mu}}(1-\gamma_\mu)U_\mu(x) + \delta_{y,x-\hat{\mu}}(1+\gamma_\mu)U_\mu^\dagger(x)$$

Functional integral:

$$Z = \int D[U,\overline{\psi},\psi]\mathrm{e}^{-S_G[U]-S_F[U,\overline{\psi},\psi]}$$

Integrate out fermions:

$$Z = \int D[U]\mathrm{e}^{-S_G[U]}\det Q[U]$$

Exercise

Show that $\det Q[U]$ is real.

Introduce the effective gauge action, using

$$\det X = \mathrm{e}^{\log\det X} = \mathrm{e}^{\mathrm{Tr}\log X}$$

so that

$$Z = \int D[U]\mathrm{e}^{-S_{\rm eff}[U]}, \qquad S_{\rm eff}[U] \equiv S_G[U] - \mathrm{Tr}\log Q[U]$$

Quark propagator:

$$\langle\psi(y)\overline{\psi}(x)\rangle = \frac{1}{Z}\int D[U]Q_{yx}^{-1}[U]\mathrm{e}^{-S_{\rm eff}[U]}$$

Now examine the fermionic contribution to $S_{\rm eff}[U]$ in greater detail. Split:

$$Q[U] = Q^{(0)} - V[U]$$

$Q^{(0)}$ describes free Wilson fermions:

$$Q_{xy}^{(0)} = \delta_{xy} - \kappa\sum_{\mu=0}^{3}\big[\delta_{y,x+\hat{\mu}}(1-\gamma_\mu) + \delta_{y,x-\hat{\mu}}(1+\gamma_\mu)\big]$$

$$Q^{(0)\,-1} \equiv S_W^{(0)} \qquad \text{(free Wilson propagator)}$$

while V is the interaction term:

$$V_{xy}[U] = \kappa\sum_{\mu=0}^{3}\Big[\delta_{y,x+\hat{\mu}}(1-\gamma_\mu)\big(U_\mu(x)-1\big) + \delta_{y,x-\hat{\mu}}(1+\gamma_\mu)\big(U_\mu^\dagger(y)-1\big)\Big]$$

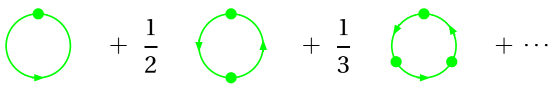
Now write

$$Q[U] = Q^{(0)} Q^{(0)^{-1}} Q[U] = Q^{(0)} \left(1 - Q^{(0)^{-1}} V[U]\right)$$

The effective gauge action becomes

$$\begin{aligned} S_{\text{eff}}[U] &= S_G[U] - \log\det Q[U] \\ &= S_G[U] - \operatorname{Tr}\log\left(1 - Q^{(0)^{-1}} V[U]\right) + \text{const} \\ &= S_G[U] + \sum_{j=1}^{\infty} \frac{1}{j} \operatorname{Tr}\left(S_W^{(0)} V[U]\right)^j \end{aligned}$$

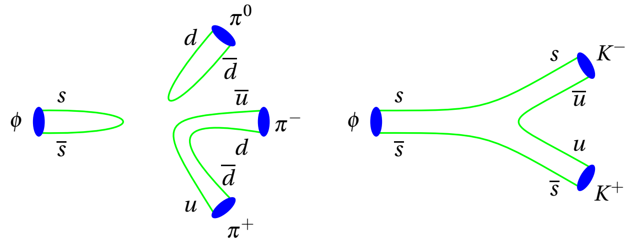
- Trace here is over *all* quark indices: Dirac, colour, site
- each term is a closed loop of j free quark propagators and j vertices
- the sum contributes closed quark loops to the effective action



5.3 The quenched approximation

There is phenomenological evidence that quark loops have only small effects on hadronic physics.

Zweig's (OZI) rule: $\phi \to 3\pi$ is suppressed relative to $\phi \to K^+K^-$



This motivates the quenched approximation which corresponds to setting

$$\det Q[U] = 1, \quad \text{ie} \quad S_{\text{eff}}[U] = S_G[U]$$

- $\det Q[U] = 1$ corresponds to setting $\kappa = 0$ for *internal* quarks (in loops)

$$\kappa = 0 \quad \Leftrightarrow \quad m_q = \infty$$

$\longrightarrow$ infinitely heavy quarks in loops contributing to the effective gluon interaction
- quenching is an *enormous* simplification for numerical simulations:

$$\frac{\text{cost of full QCD}}{\text{cost of quenched QCD}} > 10\,000$$

6 Numerical simulations

Return to the problem of computing observables in QCD (restrict to $SU(3)$ gauge group)

$$\begin{aligned} \langle \mathscr{O} \rangle &= \frac{1}{Z} \int D[U] \mathscr{O} e^{-S_{\text{eff}}[U]} \\ &= \frac{1}{Z} \int \prod_{x \in \Lambda_E} \prod_{\mu=0}^{3} dU_\mu(x) \mathscr{O} e^{-S_{\text{eff}}[U]} \end{aligned}$$

- strong coupling expansions have a small radius of convergence
- weak coupling expansion is asymptotic
- . . . and the two don't overlap
- *exact* evaluation of $\langle \mathscr{O} \rangle$ or Z on a computer is not practical (although possible in principle)
- instead use stochastic methods to evaluate $\langle \mathscr{O} \rangle$ or Z
- Monte Carlo integration: evaluate the observable on a finite number of 'typical' field configurations

Field configuration

Assignment of an $SU(3)$ matrix $U_\mu(x)$ to every link (x,μ) on the lattice:

$$\mathscr{C} = \left\{ U_\mu(x) \middle| x \in \Lambda_E, \mu = 0,1,2,3 \right\}, \qquad \mathscr{C} = \{U\}$$

6.1 Monte Carlo integration

Integrand is *strongly* peaked around configurations $\mathscr{C}$ with *large* values of

$$W(\mathscr{C}) \equiv e^{-S_{\text{eff}}[\mathscr{C}]} = e^{-S_{\text{eff}}[U]}$$

$W(\mathscr{C})$: Boltzmann factor or statistical weight of configuration $\mathscr{C}$

Monte Carlo procedure

- generate a sample or ensemble of gauge field configurations, $\mathscr{C}_i$, $i = 1, \ldots, N_{\text{cfg}}$, with statistical weights $W(\mathscr{C}_i)$
- sample comprises predominantly configurations with large $W(\mathscr{C}_i)$
- importance sampling: design an algorithm which generates a configuration $\mathscr{C}$ with likelihood $W(\mathscr{C})$
- common algorithms
  - Metropolis
  - heat bath (for $SU(N)$ gauge theory, scalar field theories, spin systems)
  - cluster algorithms (Swendsen-Wang, Wolff) (for spin systems, $O(N)$ models, *not* gauge theories)
  - hybrid Monte Carlo (HMC) or multiboson algorithms (for QCD with dynamical fermions)

- quenched QCD: use $W(\mathscr{C}) = \mathrm{e}^{-S_G[\mathscr{C}]} = \mathrm{e}^{-S_G[U]}$ as probability measure
- 'full' QCD: use $W(\mathscr{C}) = \det Q[U]\mathrm{e}^{-S_G[U]}$ as measure
  - $\det Q[U]$ is real but not positive definite
  - use $\det(Q^\dagger Q)\mathrm{e}^{-S_G}$, corresponding to two flavours of dynamical quark
  - hard to simulate odd numbers of fermions
- evaluate observables on each configuration in the ensemble, $\mathscr{O}[\mathscr{C}_i]$, $i = 1,\ldots,N_{\text{cfg}}$, giving N_{cfg} 'measurements'
  - sample average of observable

$$\overline{\mathscr{O}} = \frac{1}{N_{\text{cfg}}}\sum_{i=1}^{N_{\text{cfg}}} \mathscr{O}[\mathscr{C}_i]$$

  - expectation value

$$\langle \mathscr{O}\rangle = \lim_{N_{\text{cfg}}\to\infty} \overline{\mathscr{O}}$$

- results from Monte Carlo integration have statistical error $\propto 1/\sqrt{N_{\text{cfg}}}$

6.2 Hadronic correlation functions

Recall spectral decomposition of two-point function:

$$\langle A(x)B(y)\rangle = \sum_{n,\mathbf{p}_n} \langle 0|A(0,\mathbf{x})|n\rangle \mathrm{e}^{-(E_n-E_0)(x_0-y_0)}\langle n|B(0,\mathbf{y})|0\rangle$$

Now consider the pion two-point function:

$$C_\pi(t) \equiv \sum_{\mathbf{x}} \langle 0|P(t,\mathbf{x})P^\dagger(0)|0\rangle$$

- $P(x) = \overline{\psi}(x)\gamma_5\psi(x)$ is an interpolating operator between the pion state and the vacuum. $P^\dagger = -P$
- $\sum_{\mathbf{x}}$ projects onto zero momentum: states $|n\rangle$ at rest
- states $|n\rangle$ in sum have same quantum numbers as pion, $J^P = 0^-$

$$\begin{aligned} C_\pi(t) &= \sum_n \frac{\langle 0|P(0)|n_{\mathbf{p}=0}\rangle\langle n_{\mathbf{p}=0}|P^\dagger(0)|0\rangle}{2M_n^{(\pi)}}\mathrm{e}^{-M_n^{(\pi)}t} \\ &= \sum_n \frac{\left|\langle 0|P|n\rangle\right|^2}{2M_n^{(\pi)}}\mathrm{e}^{-M_n^{(\pi)}t} \end{aligned}$$

- For large Euclidean times t the state with the *lowest* mass dominates, call it M_π

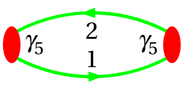
Numerical calculation of 2pt correlator

Let

$$\begin{aligned} P(x) &= \overline{\psi}_1(x)\gamma_5\psi_2(x) \\ P^\dagger(x) &= -\overline{\psi}_2(x)\gamma_5\psi_1(x) \end{aligned}$$

$1,2$ label distinct quark flavours: limits the contractions appearing; different choices of $1,2$ let you study $m_\pi, m_K,\ldots$ The correlator:

$$\begin{aligned} C_\pi(t) &= \sum_{\mathbf{x}} \langle 0|\mathrm{T}\,P(t,\mathbf{x})P^\dagger(0)|0\rangle \\ &= \sum_{\mathbf{x}} -\langle 0|\mathrm{T}\,\overline{\psi}_1(x)\gamma_5\psi_2(x)\overline{\psi}_2(0)\gamma_5\psi_1(0)|0\rangle \\ &= \sum_{\mathbf{x}} \frac{1}{Z}\int \prod_{\substack{x\in\Lambda_E \\ \mu=0,\ldots,3}} dU_\mu(x)\mathrm{e}^{-S_{\text{eff}}[U]}\,\mathrm{Tr}\left(\gamma_5 Q_2^{-1}[U]_{x0}\gamma_5 Q_1^{-1}[U]_{0x}\right) \\ &= \sum_{\mathbf{x}} \left\langle \mathrm{Tr}\left(\gamma_5 Q_2^{-1}[U]_{x0}\gamma_5 Q_1^{-1}[U]_{0x}\right)\right\rangle \end{aligned}$$

$= \sum_{\mathbf{x}}$ $t,\mathbf{x}$ 

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Sample average

$$\overline{C_\pi(t)} = \frac{1}{N_{\text{cfg}}}\sum_{i=1}^{N_{\text{cfg}}}\sum_{\mathbf{x}} \mathrm{Tr}\left(\gamma_5 S_{W,2}^{\mathscr{C}_i}(x,0)\gamma_5 S_{W,1}^{\mathscr{C}_i}(0,x)\right)$$

where $S_{W,j}^{\mathscr{C}_i}(y,x)$ is propagator for quark type j from x to y on the ith configuration $\mathscr{C}_i$.

Calculating propagators

- $S_W(x,y)_{ab,\mu\nu}$ has site, spin and colour indices and depends on the gauge field and the quark mass (κ)
- on a given configuration the propagator for quark type j (with mass fixed by κ_j) solves

$$Q_{zx}^{\mathscr{C}_i} S_{W,j}^{\mathscr{C}_i}(x,y) = \delta_{x,y}$$

  suppressing colour and spin indices
- impractical to solve for the whole matrix: instead, fix $y = 0$:

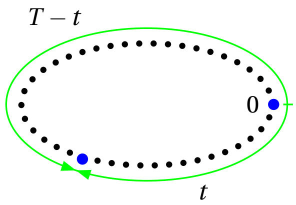
$$Q_{zx}^{\mathscr{C}_i} S_{W,j}^{\mathscr{C}_i}(x,0) = \delta_{x,0}$$

  - ∗ solve matrix equation $Q\cdot X = b$ for vector X
  - ∗ repeat for each configuration i

  $\longrightarrow$ gives propagator from 0 to any x
- correlation function also contains $S_W(0,x)$.
  - ∗ since $Q = \gamma_5 Q^\dagger\gamma_5$, then $S_W = \gamma_5 S_W^\dagger\gamma_5$
  - ∗ $\longrightarrow$ get $S_W(0,x)$ from $S_W(x,0)$

  $\longrightarrow$ have all propagators needed
- now just evaluate the trace with γ_5's using propagators evaluated on each gauge configuration

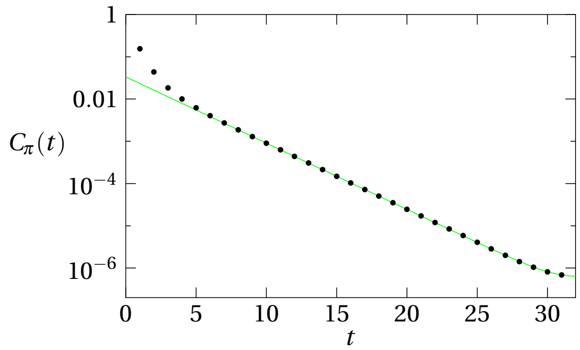
- On a finite lattice with periodic temporal boundary conditions $C_\pi(t)$ is symmetric for $t \to T-t$



$$C_\pi(t) \overset{0 \ll t \ll T}{\longrightarrow} \frac{1}{2M_\pi}\left|\langle 0|P|\pi\rangle\right|^2 (e^{-M_\pi t} + e^{-M_\pi (T-t)})$$

$$= \frac{1}{M_\pi}\left|\langle 0|P|\pi\rangle\right|^2 e^{-M_\pi T/2} \cosh\left(M_\pi (T/2 - t)\right)$$

- Obtain M_π and the matrix element $Z = \langle 0|P|\pi\rangle \propto f_\pi$ by fitting $C_\pi(t)$ to the above cosh formula

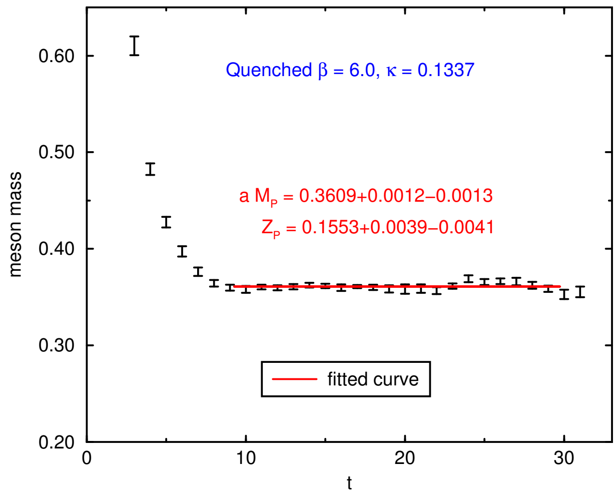


Example: Quenched, $\beta = 6$, $\kappa = 0.1337$, $32^3 \times 64$ lattice. Fitted curve has $aM_\pi = 0.3609^{+12}_{-13}$, $Z = 0.1553^{+39}_{-41}$. (D Lin, APE data)

Effective mass plot

Plot

$$M_\pi^{\text{eff}}(t) = \ln\left[\frac{C_\pi(t) + \sqrt{C_\pi^2(t) - C_\pi^2(T/2)}}{C_\pi(t+1) + \sqrt{C_\pi^2(t+1) - C_\pi^2(T/2)}}\right] \overset{0 \ll t \ll T}{\approx} M_\pi$$



(D Lin, APE data)

Simpler: if $T \to \infty$, then $C_\pi(t) \propto e^{-M_\pi t}$ and plot

$$\ln\left(C_\pi(t)/C_\pi(t+1)\right) \approx M_\pi$$

Differs only at right hand end of above plot

- By choosing appropriate interpolating operators for:

vector mesons	$\rho, K^*, \phi, \ldots$
octet baryons	$N, \Sigma, \Lambda, \Xi, \ldots$
decuplet baryons	$\Delta, \Sigma^*, \Xi^*, \Omega$

one can extract the hadron spectrum from fits to the correlation functions

6.3 Elimination of bare parameters

Hadron masses obtained from correlation functions depend implicitly on the input bare parameters, β and κ. Moreover, you determine dimensionless quantities, like aM_π and have to fix *a afterwards*.

Eliminate bare parameters by matching lattice hadron masses to experiment.

Can study quark mass dependence of hadrons on the lattice by computing aM_{had} for several values of κ at fixed β. From leading order chiral perturbation theory:

$$\begin{aligned} M_\pi^2 &= B(m_u + m_d) \\ M_{K^\pm}^2 &= B(m_u + m_s) \\ M_\rho &= A + C(m_u + m_d) \\ M_{K^*} &= A + C(m_u + m_s) \end{aligned}$$

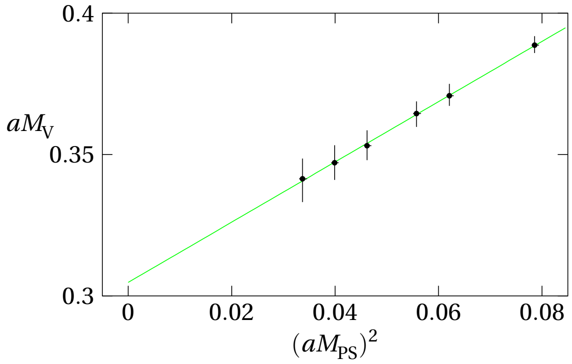
$\longrightarrow$ information on quark mass dependence resides in parameters A, B and C

Motivates ansatz for quark mass dependence of lattice data:

$$\begin{aligned} (aM_{\text{PS}})^2 &= (aB)(am_{q_1} + am_{q_2}) \\ aM_{\text{V}} &= (aA) + C(am_{q_1} + am_{q_2}) \\ &= (aA) + \frac{C}{(aB)}(aM_{\text{PS}})^2 \end{aligned}$$

To first approximation, assume $m_u = m_d = 0$, so that one expects $M_\pi^2 = 0$ (in real life: $M_\pi^2 = 0.018\,\text{GeV}^2$ compared to $M_\rho^2 = 0.59\,\text{GeV}^2$).

Compute aM_ρ by plotting aM_V versus $(aM_\text{PS})^2$ and extrapolating to $(aM_\text{PS})^2 = 0$.



Example: Quenched, $\beta = 6.2$ (UKQCD PRD 62 054506,2000)

Then use experimental value to 'calibrate' lattice spacing:

$$a^{-1} = \frac{M_{\rho,\text{phys}}}{(aM_\rho)_\text{latt}}$$

- fix all other masses in terms of M_ρ
- have traded a hadronic quantity, M_ρ, for a bare parameter, β
- could use other physical (dimensionful) quantities, such as f_π, to fix a

To compute masses of strange hadrons, one has to determine the value of κ which corresponds to the strange quark mass: κ_s

Fix κ_s at the point where

$$\frac{(aM_\text{PS})^2}{(aM_\rho)^2} = \frac{M_{K^\pm}^2}{M_\rho^2} = \frac{(494\,\text{MeV})^2}{(770\,\text{MeV})^2} = 0.4116$$

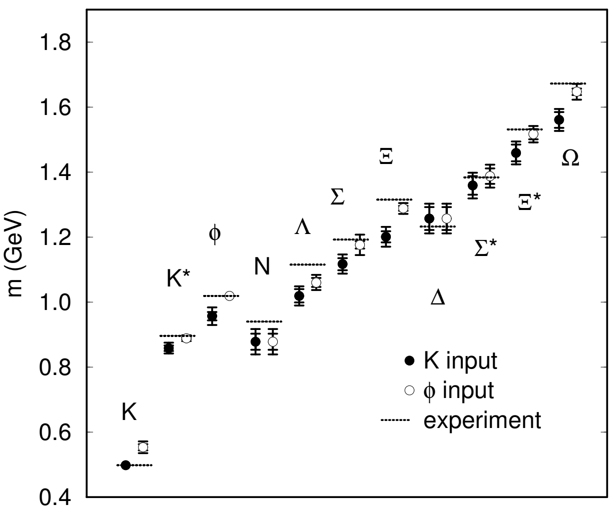
Use similar procedure for κ_c, κ_b

Summary

parameter	fixed through
$\kappa_u = \kappa_d$	$(aM_\text{PS})^2 = 0$
a	$aM_\text{V} = aM_\rho$ at $\kappa = \kappa_u$
κ_s	$(aM_\text{PS})^2/(aM_\rho)^2 = 0.4116$
⋮	⋮

M_π, M_ρ and M_K are used to eliminate β, κ_u, κ_d. This is called a hadronic renormalisation scheme. The dependence of lattice estimates on β and κ has been eliminated by matching to the observed hadron spectrum.

Light hadron spectrum in quenched LQCD

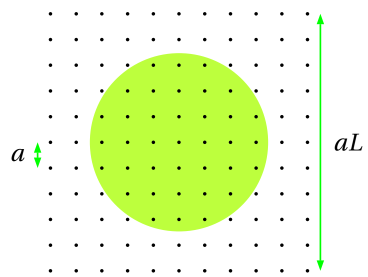


Errors shown are statistical and sum of statistical and systematic.

(CP-PACS collaboration hep-lat/0206009)

6.4 Systematic errors

Lattice computations are truly first principles. Errors can be systematically reduced.



We want

$$aL \gg 1\,\text{fm} \qquad \text{and} \qquad a^{-1} \gg \Lambda_\text{QCD}$$

Computer power limits the number of lattice points which can be used and hence the precision of the calculation. Typically, full QCD simulations use about 24 points in each spatial direction ($O(50)$ in quenched simulations) so compromises have to be made.

Statistical errors Functional integral is evaluated by importance sampling. Statistical error estimated from fluctuations of computed quantities within different clusters of configurations

Discretisation errors Current simulations typically have

$$a \sim 0.05 \text{ to } 0.1\,\text{fm}$$

Errors with Wilson fermions are $O(a\Lambda_{\text{QCD}})$, $O(am_q)$ and can be particularly severe in heavy quark physics, although we are helped by:

- guidance from heavy quark symmetry
- use of discretised effective theories

Efforts to reduce discretisation errors:

- Use several lattice spacings a and extrapolate $a \to 0$
- Improvement (Symanzik) Adjust the discretisation so that errors are formally reduced. Simple eg:

$$f'(x) = \frac{f(x+a) - f(x)}{a} + O(a)$$

compared to

$$f'(x) = \frac{f(x+a) - f(x-a)}{2a} + O(a^2)$$

Relatively easy to reduce errors from $O(a)$ to $O(\alpha_s a)$. Also possible, though more involved, to use nonperturbative improvement to get to $O(a^2)$.
- Perfect actions: apply renormalisation group to continuum action to construct (classical) action with no discretisation errors. Truncations are necessary in practice: not used in large-scale QCD simulations to date

Finite volume effects Pion is light (pseudo Goldstone boson of chiral symmetry breaking): it can propagate over large distances. Simulations are performed with heavier pions (ie using quarks around the strange mass) and results are extrapolated to the chiral limit.

Typically impose $m_\pi aL > 4$

Quenching Repeated evaluation of fermion determinant to generate unquenched gauge configurations *very* expensive. More and more simulations now use dynamical quarks, although typically have two flavours of degenerate sea-quarks a bit below the strange mass.

Renormalisation Need to relate bare lattice operators to standard renormalised ones (eg $\overline{\text{MS}}$): introduces uncertainties.

6.5 Continuum Limit

Consider calculating a physical mass M_{phys}

- lattice gives dimensionless $m = M_{\text{phys}}a$
- M_{phys} should not depend on a (at least as $a \to 0$), hence m depends on $g_0(a)$:

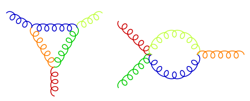
$$\frac{dM_{\text{phys}}}{da} = 0$$

$$m + B(g_0)\frac{\partial m}{\partial g_0} = 0$$

Dependence of bare coupling g_0 on cutoff a

$$B(g_0) = -a\frac{\partial g_0}{\partial a} = -\beta_0 g_0^3 - \beta_1 g_0^5 + \cdots$$

- find $g_0 \to 0$ as $a \to 0$
- calculate $B(g_0)$ in lattice PT

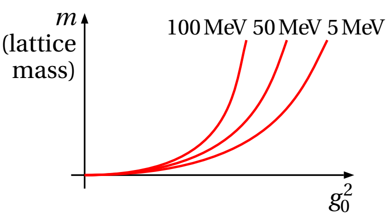


- ...or nonperturbatively (ALPHA)

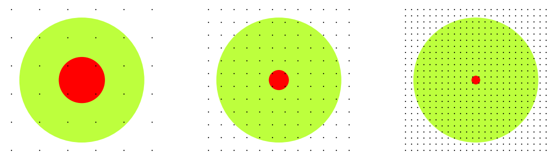
Solve to find

$$m = C\exp\left(\frac{-1}{2\beta_0 g_0^2}\right)$$

with a different C for each physical mass: finding C is the hard part (where all the 'physics' lies).



- lattice mass vanishes in the continuum limit



  - ∗ corresponding correlation length $\xi = 1/m$ *diverges* (in lattice units)
  - ∗ continuum limit is a critical point
  - ∗ once $\xi \gg a$ the system 'forgets' the fine details of the original lattice $\longrightarrow$ universality
- mass ratios should be pure numbers, independent of g_0, a:

$$M_{\text{phys}_i} = C_i \Lambda_{\text{latt}}$$

  - ∗ Λ_{latt} says how 'strong' the strong interaction is
  - ∗ it's strongly-dependent on the details of regularisation: $\Lambda_{\overline{\text{MS}}}/\Lambda_{\text{latt}} = 28.8$ for $SU(3)$ YM

Scaling

- calibrate a from m_ρ, f_π, σ, ...
- further calculations yield mass ratios m_i/m_0
- if close enough to ctm limit, m_i/m_0 is constant as $\beta \nearrow$
- this is scaling

Asymptotic Scaling

- PT in g_0^2 should work for large enough $\beta = 2N/g_0^2$
- observe scaling according to the β-function (1-loop)

$$M_{\text{phys}}\, a \propto \exp\left(\frac{-1}{2\beta_0 g_0^2}\right)$$

- this is asymptotic scaling

6.6 Renormalisation of Lattice Operators

Typically (ignoring operator mixing):

$$\mathcal{O}^{\text{ren}}(\mu) = Z_{\mathcal{O}}(\mu a, g)\,\mathcal{O}^{\text{latt}}(a)$$

- if $a^{-1} \gg \Lambda_{\text{QCD}}$ and $\mu \gg \Lambda_{\text{QCD}}$ can use PT to relate
- $Z_{\mathcal{O}}$ depends on short-distance physics
- IR physics common to matrix elements of $\mathcal{O}^{\text{ren,latt}}$

Example: axial vector current in Wilson LQCD

$$A_\mu^{\text{latt}} = \overline{\psi}(x)\gamma_\mu\gamma_5\psi(x)$$

Use this in a 2-point correlation function:

$$\begin{aligned} C(t) &= \sum_{\mathbf{x}} \langle 0|\mathrm{T}\, A_0^{\text{latt}}(\mathbf{x},t) A_0^{\text{latt}\dagger}(0)|0\rangle \\ &\overset{\text{large } t>0}{=} \frac{\left|\langle \pi(\mathbf{p}=0)|A_0^{\text{latt}\dagger}(0)|0\rangle\right|^2}{2m_\pi} \mathrm{e}^{-m_\pi t} \end{aligned}$$

But

$$A_\mu^{\text{ren}} = Z_A A_\mu^{\text{latt}} \qquad \text{and} \qquad \langle \pi(\mathbf{p}=0)|A_0^{\text{ren}\dagger}(0)|0\rangle = f_\pi m_\pi$$

so that

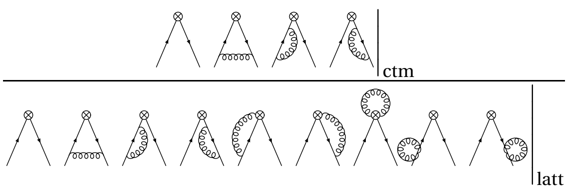
$$f_\pi = \frac{Z_A \left|\langle \pi|A_0^{\text{latt}\dagger}|0\rangle\right|}{m_\pi}$$

...you need Z_A to get the physical f_π.

Perturbative renormalisation

Calculate matrix element of quark bilinear $\mathcal{O}$ between, say, the same quark states and fix $Z_{\mathcal{O}}$ by demanding agreement ($Z_{\mathcal{O}}$ is a property of $\mathcal{O}$ so use any convenient states).

$Z_{\mathcal{O}} =$



$$Z_{\mathcal{O}} = 1 + \frac{\alpha_s}{4\pi}\left(\gamma \ln(\mu a) + c\right) + \cdots$$

For axial current with $\mu a = 1$

$$Z_A = 1 - 15.8\frac{\alpha_s}{4\pi} C_F$$

15.8 is a large coefficient...

- $\alpha_s^{\overline{\text{MS}}}/\alpha_s^{\text{latt}} \approx 2.7$: α_s^{latt} is a poor expansion parameter
- related to tadpoles: extra vertices in lattice PT from expanding $\exp\left(aA_\mu(x)\right)$
- turn to nonperturbative renormalisation...

Nonperturbative renormalisation

Impose a physical condition to fix $Z_{\mathcal{O}}$

- Example 1: local Wilson vector current

  $$V_\mu = \overline{\psi}(x)\gamma_\mu\psi(x)$$

  not conserved $\longrightarrow Z_V \neq 1$
  Possible to define a *conserved* lattice vector current V_μ^{C}, which has $Z = 1$. Hence, fix Z_V using

  $$Z_V = \frac{\langle \pi(\mathbf{p})|V_\mu^{\text{C}}(0)|\pi(\mathbf{p})\rangle}{\langle \pi(\mathbf{p})|V_\mu(0)|\pi(\mathbf{p})\rangle}$$

- Example 2: Use Ward Identities to relate Z's of different operators. For example, impose continuum axial current WID

  $$\langle \partial_\mu A_\mu \mathcal{O}\rangle = 2m\langle P\mathcal{O}\rangle$$

  with $\mathcal{O}$ arbitrary operator
  m renormalised quark mass
  P pseudoscalar density