Lecture 8

- Fractals and Strange Attractors
  - What is a fractal?
    - Countable and uncountable sets, examples
    - The Cantor set
  - Fractal dimensions
    - Similarity dimension
    - Box dimension
    - Pointwise and correlation dimension
  - Why should strange attractors be fractal?
    - Stretching and folding
    - The pastry map

Introduction

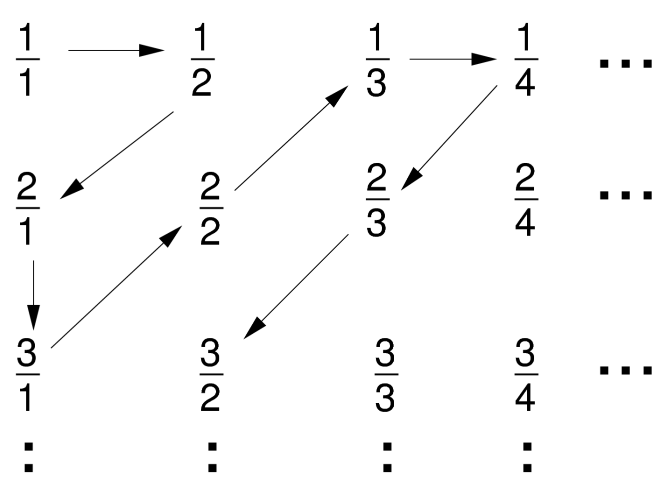
- So far: Claimed that the Lorenz attractor is a fractal, but no definition yet
- Roughly speaking: Fractals are complex geometric shapes with fine structure at arbitrarily small scales
  - Usually they exhibit a degree of self-similarity (magnify tiny part of the whole it has properties that are reminiscent of the whole; often “statistical”)
  - Examples: Clouds, Coastlines, blood vessel networks, broccoli

Countable and Uncountable Sets

- Are some infinities larger than others?
- Cantor: sets X and Y have same cardinality if there exists and invertible mapping from X to Y
- Natural numbers N={1,2,3,...} provides basis for comparisons
  - If X has same cardinality as N X is **countable**, otherwise **uncountable**
- Example: The set E={2,4,6,...} of even numbers is countable
  - Proof: use mapping e(n)=2n
  - Exactly as many numbers as natural numbers!

Countable and Uncountable Sets (1)

- Alternative definition: X is countable if it can be written as a list $\{x_1, x_2, x_3, \ldots\}$ such that for any x there is an n with $x_n = x$.
- Example: the set of integers is countable
  - {0,1,-1,2,-2,3,-3, ... } -> any particular integer appears, so set is countable
- Example: the set of positive rational numbers is countable



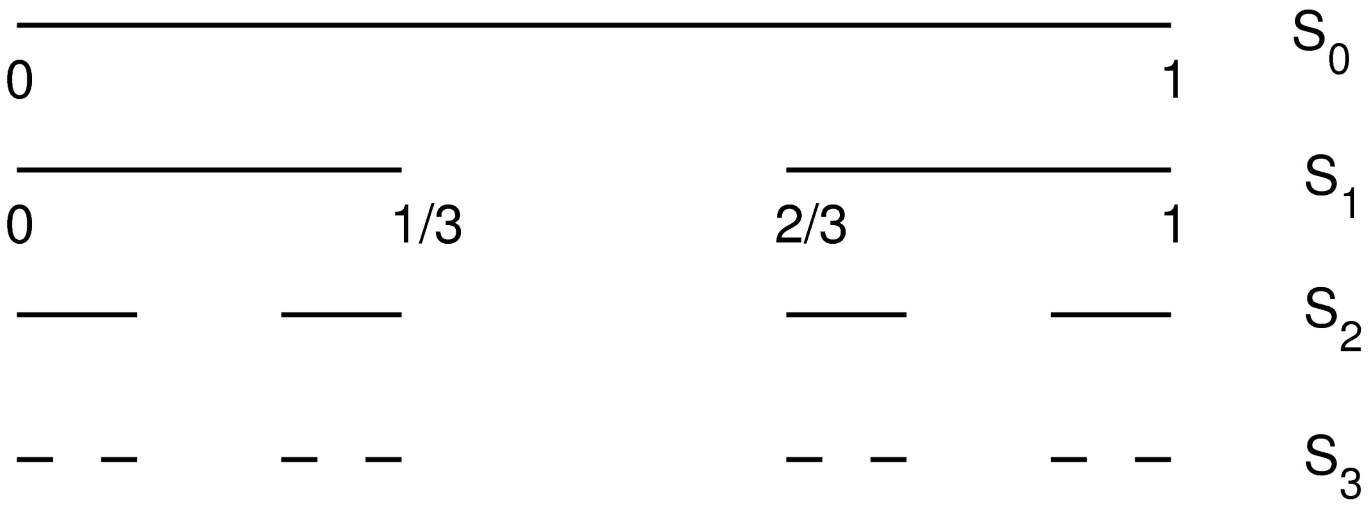
Countable and Uncountable Sets (2)

- Example: The set X of all real numbers between 0 and 1 is uncountable
  - If X were countable we could write it as $X=\{x_1,x_2,x_3,\ldots\}$
  - Write numbers in decimal form

$$x_1 = 0.x_{11}\, x_{12}\, x_{13}\, x_{14} \ldots$$
$$x_2 = 0.x_{21}\, x_{22}\, x_{23}\, x_{24} \ldots$$
$$x_3 = 0.x_{31}\, x_{32}\, x_{33}\, x_{34} \ldots$$

  - Show that number $\bar{x}$ between 0 and 1 is not in the list
    - 1st digit: anything other than x_{11} → $\overline{x_{11}} \neq x_{11}$
    - 2nd digit: anything other than x_{22} → $\overline{x_{22}} \neq x_{22}$
    - And so on. $\bar{x} = \overline{x_{11}}\,\overline{x_{22}}\,\overline{x_{33}} \ldots$ is not in the list!
    - "**Diagonal argument**"

Cantor Set



- Start with $S_0=[0,1]$
- Remove the open middle (1/3,2/3) to obtain S_1
- Remove open middle of remaining intervals in S_1 -> S_2
- Repeat ... the limiting set $C=S_\infty$ is the Cantor set

Cantor Set (1)

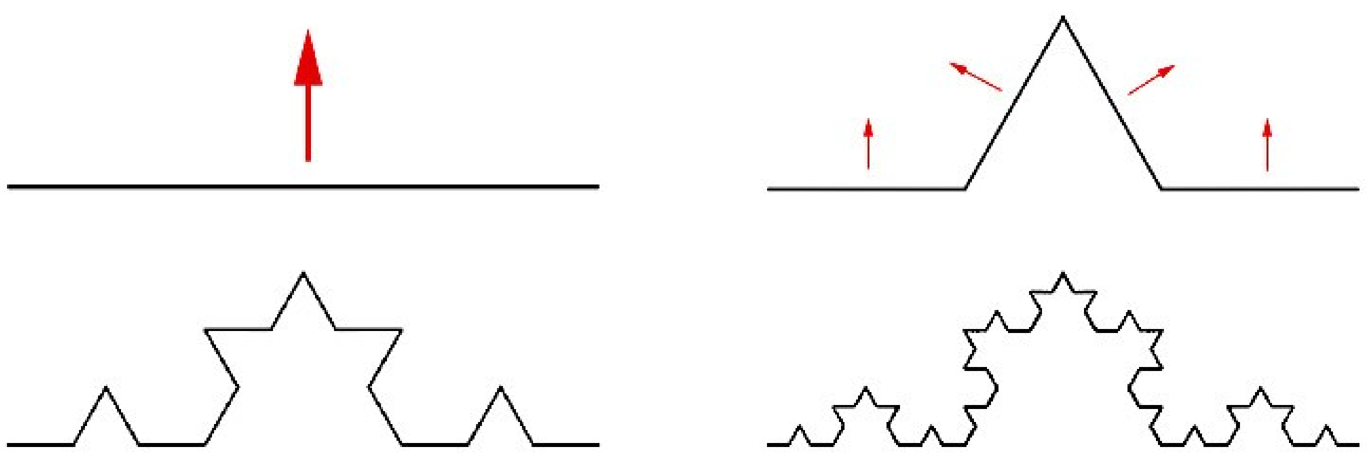
- Has some properties typical of fractals more generally
  - Structure at arbitrarily fine scales
  - C is self-similar (more generally fractals are only approximately self-similar)
  - C has a non-integer dimension ln2/ln3=0.63...
- C has measure zero
  - C is covered by S_n. $L_n=(2/3)^n$
- C is uncountable
  - n base 3 expansion C is set of numbers without a “1”
  - Then use diagonal argument

General Cantor Sets

- A closed set is a topological Cantor set if:
  - S is "totally disconnected", i.e. S contains no connected subsets (or no intervals in 1d)
  - S contains no "isolated points", i.e. every point in S has a neighbour arbitrarily close by
- Cantor sets are spread apart and packed together!
- Cross section of strange attractors are often topological Cantor sets but not necessarily self-similar

Dimensions of Self-Similar Fractals

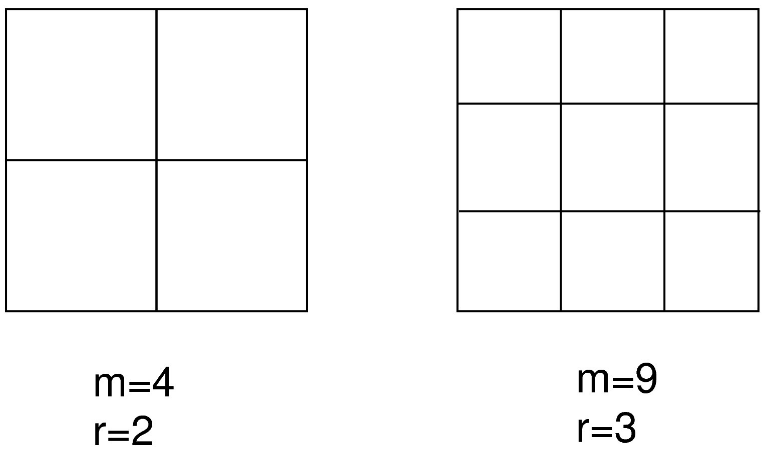
- “Classically”: minimum number of coordinates needed to describe every point in a set
  - -> Paradoxes, like with von Koch curve, which has infinite length and every point is infinitely far away from every other point!



  - K has dimension larger than 1, but no area, so 1<d<2?

Similarity Dimension

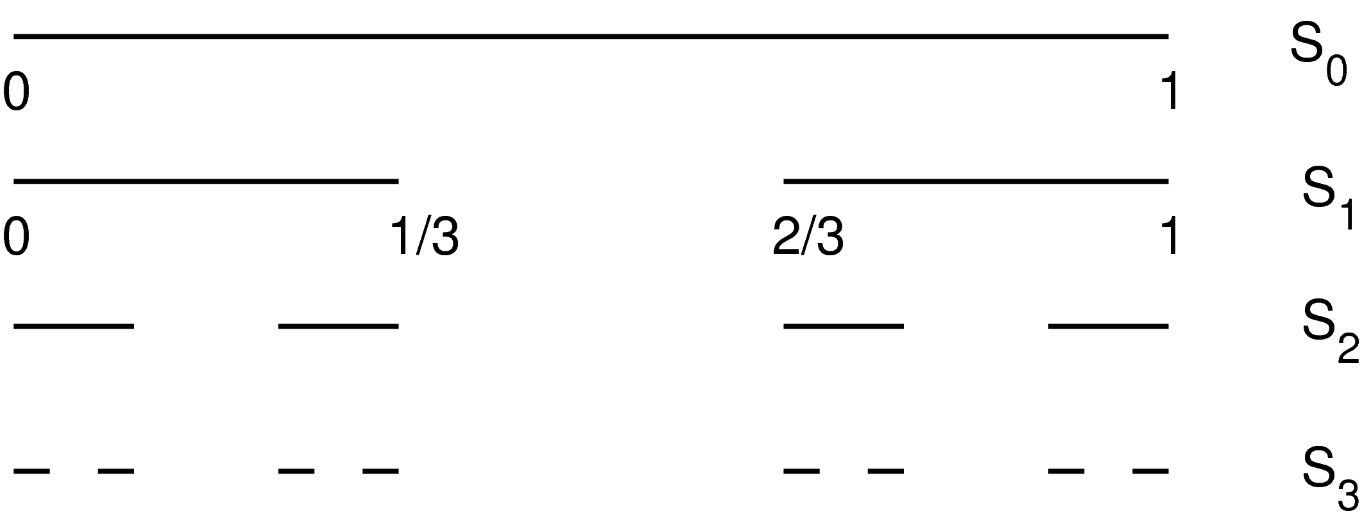
- In 2d: If we scale linear dimension of objects by r it takes $m=r^2$ scaled objects to cover the original object



- In 3d: -> need $m=r^3$ scaled objects ...
- Suppose a self-similar object is composed of m copies of itself scaled down by a factor r -> $m=r^d$
- **Similarity dimension** d= ln m/ln r

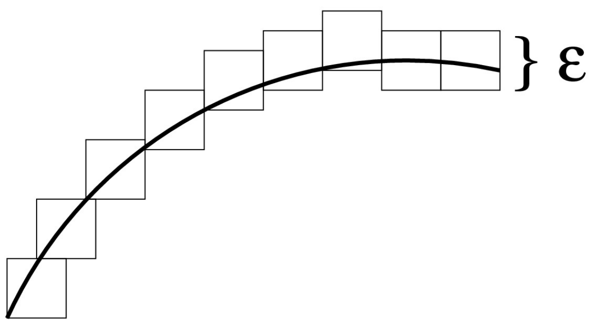
Similarity Dimension (1)

- What is the similarity dimension of the Cantor set?
  - Need m=2 objects scaled down by a factor of r=3 to reproduce the original
  - d=ln 2/ ln 3!

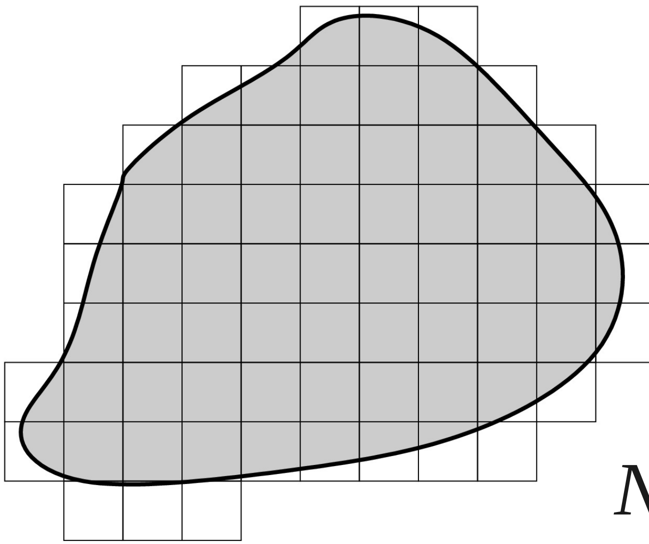


Box Dimension

- Idea: “Measure a set at scale ε” and investigate how measurements vary for ε to 0.



$N(\epsilon) \propto L/\epsilon$



$N(\epsilon) \propto A/\epsilon^2$

- N ... minimum number of D-dimensional cubes of side ε needed to cover S
- Key observation: $N(\epsilon) \propto 1/\epsilon^d$
- Box dimension $d = \lim_{\epsilon \to 0} \ln N(\epsilon) / \ln(1/\epsilon)$ (if it exists)

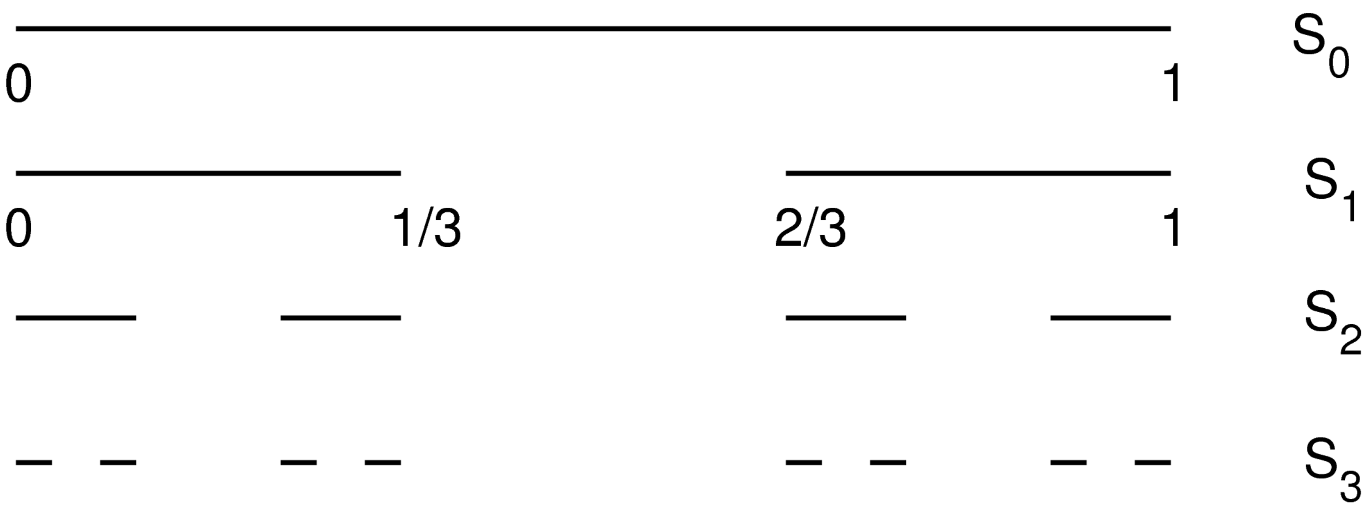
Box Dimension of the Cantor Set

- Set S_n consists of 2^n intervals of size $(1/3)^n$
- Pick $\varepsilon=(1/3)^n$, need 2^n of these intervals to cover S^n

$$N(\epsilon)=2^n \quad \text{for } \epsilon=(1/3)^n$$

- Hence:

$$d=\lim_{\epsilon\to 0} \ln N(\epsilon)/\ln(1/\epsilon)=\ln 2^n/\ln 3^n=\ln 2/\ln 3$$

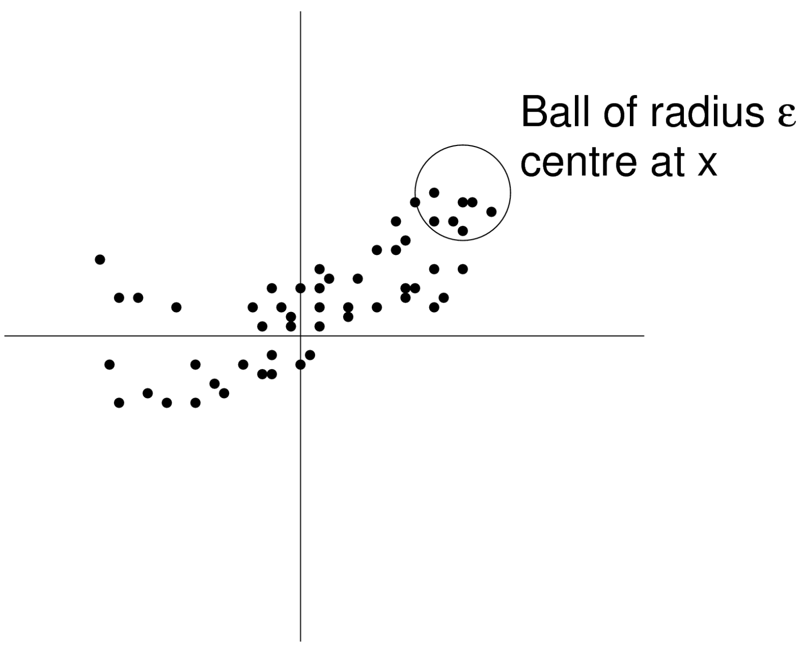


Box Dimension

- Problems:
  - Not always easy to find minimal covers
    - Alternatively: use mesh of side ε and count number of occupied boxes $N(\varepsilon)$ in mesh
    - Rarely used in practice: -> storage space and computing time
  - Mathematical problems ...
    - Box dimension of rational numbers is 1 (!)
    - -> Hausdorff dimension

Pairwise and Correlation Dimensions

- Suppose we have a chaotic system that settles on a strange attractor. How to measure its dimension?
- E.g.: sample many points and calculate box dim.
- Better: Grassberger and Procaccia



- $N_x(\varepsilon)$ ... number of points in ε-environment of x
- Pointwise dimension at x:

$$N_x(\epsilon) \propto \epsilon^d$$

- Correlation dimension:

$$\langle N_x(\epsilon) \rangle_x \propto \epsilon^d$$

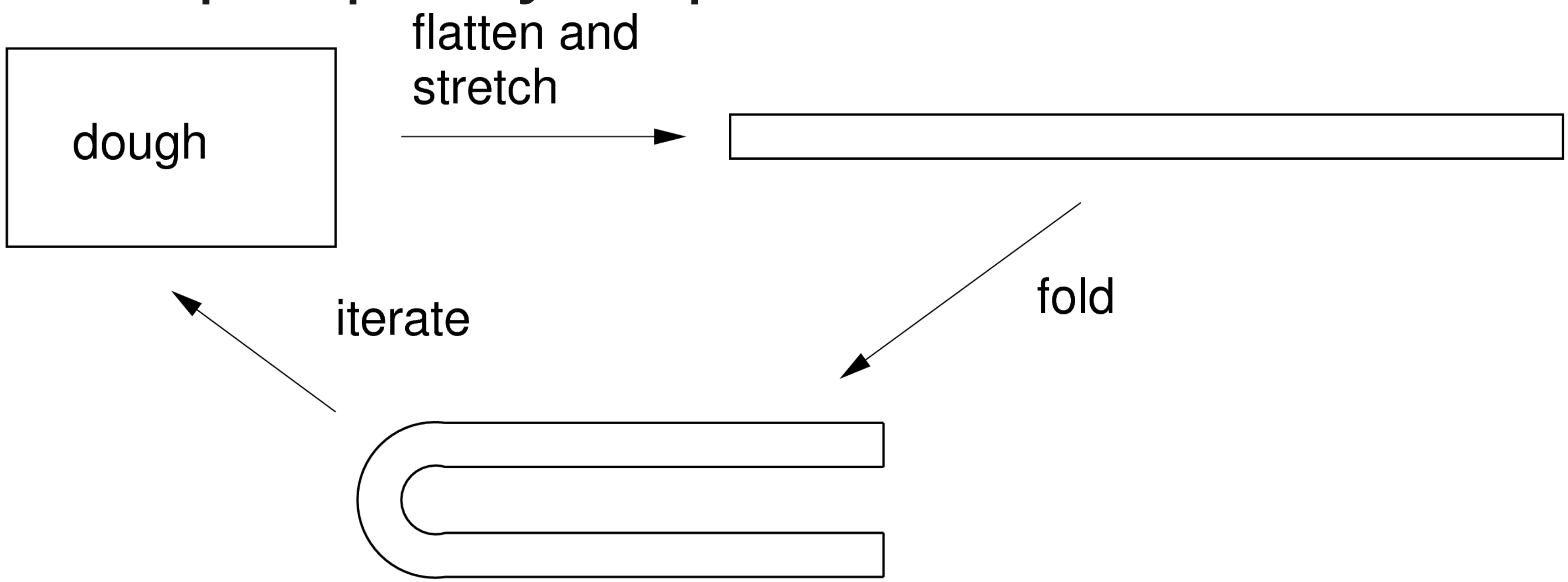
- Generally: $d_{correlation} \leq d_{box}$ (but not much difference)

Strange Attractors and Cantor Sets

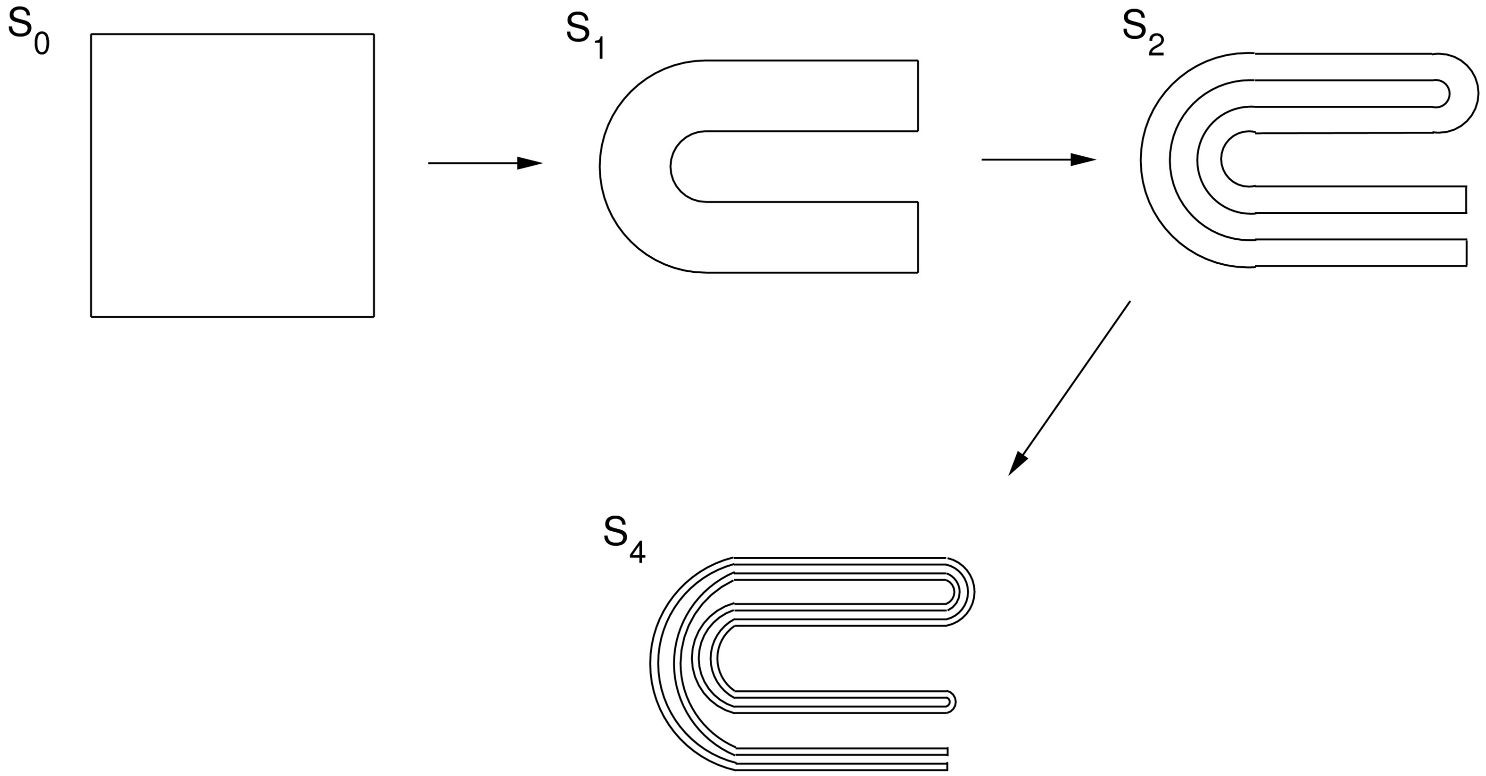
- So far:
  - we know what happens, but not why it happens.
  - E.g.: Why can a differential equation generate a fractal attractor?
- Strange attractors have two properties that seem hard to reconcile
  - Confined to a bounded region in phase space
  - Trajectories separate exponentially fast from neighbours
- How is both possible?!

Stretching and Folding!

- Consider small blob of ICs in phase space
  - Flow often contracts blob in some direction (dissipation)
  - And stretches it in the other (exponential separation)
  - Cannot stretch forever (bounded region), so it must fold back on itself
- Example: pastry map



Hoerseshoes ...



- Limiting set consists of infinitely many smooth layers separated by gaps of varying sizes ...
- ... eventually becomes a Cantor set!

Summary

- Fractals
  - What is it?
  - Countable vs uncountable sets
  - Cantor set construction
  - Fractal dimensions:
    - Similarity dimension
    - Box dimension
    - Pointwise+correlation dimensions
  - Stretching and folding