1 P1

$$\begin{aligned} 6x + y + z &= 11 \\ 2x + 3y - z &= 5 \\ x - y - 2z &= -7 \end{aligned}$$

$$A = \begin{pmatrix} 6 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & -2 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 11 \\ 5 \\ -7 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 6 & 1 & 1 & 11 \\ 2 & 3 & -1 & 5 \\ 1 & -1 & -2 & -7 \end{array}\right) \begin{array}{l} \\ \mid \cdot(-3) \\ \mid \cdot(-6) \end{array}$$

$$\left(\begin{array}{ccc|c} 6 & 1 & 1 & 11 \\ 0 & -8 & 4 & -4 \\ 0 & 7 & 13 & 53 \end{array}\right) \begin{array}{l} \\ \mid \cdot \frac{7}{8} \\ \\ \end{array}$$

$$\frac{7}{2} + 13 = \frac{7+26}{2} = \frac{33}{2}$$

$$-\frac{7}{2} + \frac{106}{2} = \frac{99}{2}$$

$$\left(\begin{array}{ccc|c} 6 & 1 & 1 & 11 \\ 0 & -8 & 4 & -4 \\ 0 & 0 & \frac{33}{2} & \frac{99}{2} \end{array}\right)$$

$\Rightarrow z = 3$

$-8y + 4 \cdot z = -8y + 12 = -4 \Rightarrow -8y = -16 \Rightarrow y = 2$

$6x + 2 + 3 = 11 \Rightarrow 6x = 6 \Rightarrow x = 1.$

<u>Qs</u>:
- How to continue if one wanted to invert the matrix A?
- Suppose, our system has <u>infinitely</u> many solutions. How would we see this in Gaussian elimination?
- Suppose, our system has <u>no</u> solution. How would this show up in GE?

- GE vs. LU-factorisation?
- How does Doolittle work?

Iterative methods

Jacobi: $A = D + R$

$$Ax = (D+R)x = b$$

$$\rightarrow Dx = b - Rx \qquad x_{n+1} = D^{-1}(b - Rx_n)$$

Gauss-Seidel $A = L + U$

$$Ax = (L+U)x = b$$

$$\rightarrow Lx = b - Ux \rightarrow x_{n+1} = L^{-1}(b - Ux_n)$$

P2

$$5x + y = 1$$
$$2x + 3y = 1 \qquad A = \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solve exactly? $y = 1 - 5x$

$$2x + 3 - 15x = 1 \rightarrow 13x = 2, \quad \underline{x = \frac{2}{13}}, \quad \underline{y} = 1 - \frac{10}{13} = \underline{\frac{3}{13}}$$

Jacobi? $D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \quad D^{-1} = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/3 \end{pmatrix}$

$R = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

$$x_{n+1} = \begin{pmatrix} 1/5 \\ 1/3 \end{pmatrix} - \begin{pmatrix} 0 & 1/5 \\ 2/3 & 0 \end{pmatrix} x_n$$

$$x_{n+1} = D^{-1}(b - Rx_n)$$

$$D^{-1}b = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 1/3 \end{pmatrix}$$

$$D^{-1}R = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/5 \\ 2/3 & 0 \end{pmatrix}$$

③

say: $\vec{x}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1/5 \\ 1/3 \end{pmatrix} - \begin{pmatrix} 0 & 1/5 \\ 2/3 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1/5 \\ 1/3 \end{pmatrix} - \begin{pmatrix} y_n/5 \\ 2/3\, x_n \end{pmatrix}$$

$$x_{n+1} = 1/5 - y_n/5 = 1/5 - \frac{1}{15} + \frac{2}{15} x_{n-1}$$

$$y_n = 1/3 - 2/3\, x_{n-1}$$

i.e.:

$$x_n = \frac{2}{15} + \frac{2}{15} x_{n-2}$$

$$= \frac{2}{15} + \frac{2}{15}\left(\frac{2}{15} + \frac{2}{15} x_{n-4} \right)$$

$$= \frac{2}{15} + \left(\frac{2}{15}\right)^2 + \left(\frac{2}{15}\right)^2 x_{n-4}$$

$$= \frac{2}{15} + \left(\frac{2}{15}\right)^2 + \left(\frac{2}{15}\right)^3 + \left(\frac{2}{15}\right)^3 x_{n-6}$$

$\vdots$

$n = 2k$

$$x_{2k} = \sum_{i=1}^{k} \left(\frac{2}{15}\right)^i + \left(\frac{2}{15}\right)^k \cdot x_0$$

$$= \underbrace{\sum_{i=0}^{k} \left(\frac{2}{15}\right)^i}_{\downarrow} - 1 + \left(\frac{2}{15}\right)^k \cdot x_0$$

$$\frac{\left(\frac{2}{15}\right)^{i+1} - 1}{\underbrace{\frac{2}{15} - 1}_{-\frac{13}{15}}} - 1 \to \frac{15}{13} - 1 = \frac{2}{13}$$

- A is diagonally dominant, but here we have explicitly seen convergence. Convergence is exponential.
- Result independent of the starting point