TAYLOR SERIES

(P1) $f(x) = \exp(x)$, $f(x) = \sin(x)$, $x_0 = 0$

$$f(x+h) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} h^k$$

a)

$$\frac{d^k}{dx^k} \exp(x) = \exp(x) \; ; \quad \exp(0) = 1$$

$$\rightarrow \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

b)

$f(x) = \sin(x)$ — for $x_0 = 0$

$\sin'(x) = \cos(x)$	period 4	1
$\cos'(x) = -\sin(x)$		0
$-\sin' x\,(x) = -\cos x$		-1
$-\cos' x\,(x) = \sin(x)$		0

$$\Rightarrow \sin(x) = \underbrace{\sin(0)}_{0} + \frac{\cos(0)}{1!} x - \underbrace{\frac{\sin(0)}{2!} x^2}_{0} - \frac{\cos(0)}{3!} x^3 + \frac{\sin(0)}{4!} x^4 + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{k!} (-1)^k$$

c) $\frac{1}{5+x}$ for $x \gg 1$

geometric series ..

$$\frac{1}{5+x} = \frac{1}{x} \cdot \left(\frac{1}{1 + 5/x} \right) = \frac{1}{x} \cdot \sum_{k=0}^{\infty} \left(-\frac{5}{x} \right)^k$$

$$= \frac{1}{x} - \frac{5}{x^2} + \frac{25}{x^3} - \ldots$$

(P2) multivariate Taylor

$$f(\vec{x}+\vec{h}) = \sum_{|\alpha|\le k} \frac{\vec{h}^{\alpha}}{|\alpha|!} (D^{\alpha} f)(\vec{x});$$

Example:

$f(x,y) = x^2+y^2$ (How do you expect the 2nd order expansion around $(0,0)$ to look like?)

$$f(x,y) = f(0,0) + \underbrace{\frac{x}{1!} f_x(0,0) + \frac{y}{1!} f_y(0,0)}_{|\alpha|=1} +$$

$$\underbrace{\frac{x^2}{2!} f_{xx}(0,0) + \frac{y^2}{2!} f_{yy}(0,0) + \frac{1}{2!}\left(xy\, f_{xy}(0,0) + yx\, f_{yx}(0,0)\right)}_{|\alpha|=2} + \dots$$

$$= 0 + 0 + 0 + \tfrac{1}{2}x^2\cdot 2 + \tfrac{1}{2}y^2\cdot 2 + 0 + 0 = x^2+y^2.$$

$f(x,y) = \exp(x+y)$

$$= 1 + x + y + \frac{x^2}{2} + \frac{y^2}{2} + xy + \dots$$

same

$$f(z) = 1 + z + \frac{z^2}{2!} + \dots$$

$$= 1 + (x+y) + \frac{(x+y)^2}{2} + \dots = 1 + x + y + \frac{x^2}{2} + \frac{y^2}{2} + xy + \dots$$

(P3) Evaluate $\frac{df}{dx}$ for $f(x) = \exp(x)$ using

forward, backwd., and central diff scheme.

ideas:

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \frac{h^4}{4!} f^{(4)}(x) + \ldots$$

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \frac{h^4}{4!} f^{(4)}(x) + \ldots$$

fwd: $f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(x) \ldots$

bwd $f'(x) = \frac{f(x) - f(x-h)}{h} + \frac{h}{2} f''(x) \ldots$

ctrl. $f'(x) = \frac{1}{2h}\Big(f(x+h) - f(x-h)\Big) + \frac{h^2}{3} f'''(x)$

for $\exp(x)$ at $x=0$; $h=0.1$; real: $\frac{d}{dx} e^x = e^x$; $e^0 = 1$

$$\frac{\exp(0.1) - \exp(0)}{0.1} \approx 1.052$$

$$\frac{\exp(0) - \exp(-0.1)}{0.1} \approx 0.952$$

$$\frac{\exp(0.1) - \exp(-0.1)}{0.2} \approx 1.002 \quad \Leftarrow \text{much closer!}$$

P4

$$f(x+2h) = f(x) + 2h f' + 2h^2 f'' + \frac{8}{6} h^3 f''' + \frac{16}{4!} h^4 f^{(4)}$$

$$f(x+h) = f(x) + h f' + \frac{h^2}{2} f'' + \frac{h^3}{6} f''' + \frac{h^4}{4!} f^{(4)}$$

$$f(x-h) = f(x) - h f' + \frac{h^2}{2} f'' - \frac{h^3}{6} f''' + \frac{h^4}{4!} f^{(4)}$$

$$f(x-2h) = f(x) - 2h f' + 2h^2 f'' - \frac{8}{6} h^3 f''' + \frac{16}{4!} f^{(4)}$$

$$f(x+h) - f(x-h) = 2h f' + \frac{2}{6} h^3 f''' + O(h^5)$$

$$f(x+2h) - f(x-2h) = 4h f' + \frac{16}{6} h^3 f''' + O(h^5)$$

$$f' = \frac{f(x+h) - f(x-h)}{2h} - \frac{1}{6} h^2 f''' + O(h^4) \qquad (1)$$

$$f' = \frac{f(x+2h) - f(x-2h)}{4h} - \frac{4}{6} h^2 f''' + O(h^4) \qquad (2)$$

Can combine (1) and (2), e.g. by using (2) - 4(1):

$$-3f' = \frac{f(x+2h) - f(x-2h)}{4h} - 2\,\frac{f(x+h) - f(x-h)}{h} \cancel{- \frac{4}{6} h^2 f'''} \cancel{+ \frac{4}{6} h^2 f'''} + O(h^4)$$

$$\Rightarrow \boxed{f' = \frac{2f(x+h) - 2f(x-h) + \frac{f(x-2h) - f(x+2h)}{4}}{3h} + O(h^4)}$$