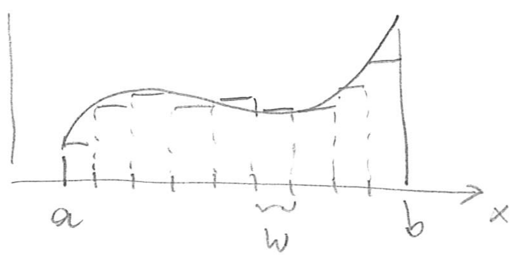
$\int_0^1 f(x)\,dx$, $f(x) = x$, dependence of numerical errors on h.

<u>general idea:</u>



$$\int_a^b f(x)\,dx \simeq \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i) = h \sum_{i=0}^{n-1} f(x_i)$$

for our case:

$$\int_0^1 x\,dx \simeq h \sum_{i=0}^{n-1} x_i \qquad x_i = i \cdot h \;;\; n = \frac{1}{h} \text{ (suppose } n \text{ is integer...)}$$

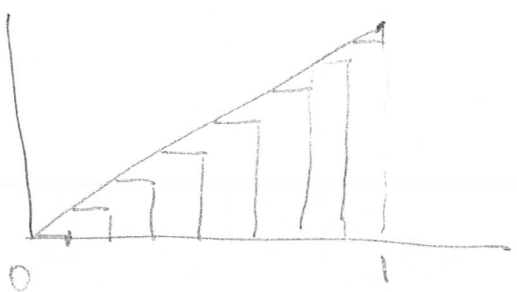
$$= h^2 \sum_{i=0}^{n-1} i$$

$$= h^2 \cdot \frac{n(n-1)}{2} = \frac{h^2}{2} \frac{1}{h}\left(\frac{1}{h} - 1\right) = \frac{1}{2} - \frac{h}{2}$$

The real value is: $\int_0^1 x\,dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$

Error $= -\frac{h}{2}$ → linear in h. First order.

How would this change of rectangle → trapezoid?



rectangle method

trapezoid.

→ for trapezoid our result would be exact for linear functions!

Similarly, rectangle is exact if functions are constant.

The idea of Richardson extrapolation

- Calculate trapezoid for increasingly refined step size

$T(f;h)$, $T(f;h/2)$ etc.

- One can show:

$$E(f;h) = I(f;h) - T(f;h) = a_2 h^2 + a_4 h^4 + a_6 h^6 + \dots$$

$$E\left(f;\frac{h}{2}\right) = I(f;h) - T\left(f;\frac{h}{2}\right) = a_2\left(\frac{h}{2}\right)^2 + a_4\left(\frac{h}{2}\right)^4 + a_6\left(\frac{h}{2}\right)^6 + \dots$$

$$I = T(f;h) - a_2 h^2 - a_4 h^4 - a_6 h^6 - \dots \qquad (1)$$

$$I = T\left(f;\frac{h}{2}\right) - a_2 \frac{h^2}{4} - \frac{a_4}{16} h^4 - \frac{a_6}{32} h^6 - \dots \qquad (2)$$

Combine both of these! $(1) - 4 \times (2)$:

$$-3I = T(f;h) - 4T\left(f;\frac{h}{2}\right) + a_4' h^4 + \dots$$

$$\rightarrow \quad I = \underbrace{\frac{4T\left(f;\frac{h}{2}\right) - T(f;h)}{3}}_{\text{Richardson extrapolation}} + a_4'' h^4 + a_6'' h^6 + \dots$$

Richardson extrapolation

Romberg method ↙