Partial Functions in B

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Partial Functions

Special kind of relation:
Each domain element has at most one range element associated with it. To declare f as a partial function:

$$f \in \boxed{X \pfun Y}$$

This says that f is a many-to-one relation
Each domain element is mapped to exactly one range element:

$$x \in dom(f) \quad \Rightarrow \quad card(\ f[\{x\}]\ ) = 1$$

More usually formalised as

$$x \mapsto y_1 \in f \ \land \ x \mapsto y_2 \in f \quad \Rightarrow \quad y_1 = y_2$$

Function Application

We can use function application for partial functions.

If $x \in \text{dom}(f)$, then we write $\boxed{f(x)}$ for the unique range element associated with x in f.

If $x \notin \text{dom}(f)$, then $f(x)$ is undefined.

If $card(\ f[\{x\}]\ ) > 1$, then $f(x)$ is undefined.

$$dir1 \;=\; \{\; mary \mapsto 398620,\; jim \mapsto 493028,\; jane \mapsto 493028 \;\}$$

$$dir2 \;=\; \{\; mary \mapsto 287573,\; mary \mapsto 398620,\; jane \mapsto 493028 \;\}$$

$$dir1 \;\in\; Person \pfun Phone$$

$$dir1(jim) \;=\; 493028$$

$dir1(sarah)$ is undefined

$dir2$ is not a partial function

$dir2(mary)$ is undefined

Function Operators

All the relational operators can be used on partial functions (restriction, subtraction, image, composition, etc).

Be careful with some operators!

- Set Union: $f \cup g$ is a partial function provided

  $$x \in \mathrm{dom}(f) \ \wedge \ x \in \mathrm{dom}(g) \quad \Rightarrow \quad f(x) = g(x)$$

  Why?

- Inverse: f^{-1} is not always a partial function. Why not?

Function Overriding

Override f by g $\boxed{f \lhd\!\!+ g}$

f and g must be partial functions of the same type

Replace an existing mappings with new ones

$$f \lhd\!\!+ \{a \mapsto b\} = (\{a\} \lhd f) \cup \{a \mapsto b\}$$
$$f \lhd\!\!+ g = (\mathrm{dom}(g) \lhd f) \cup g$$

$$dir1 = \{\ mary \mapsto 398620,\ john \mapsto 829483,$$
$$jim \mapsto 493028,\ jane \mapsto 493028\ \}$$

N.B. $\lhd\!\!+$ is sometimes written $\oplus$

Birthday Book Spec

Birthday book relates people to their birthday.
Each person can have at most one birthday.
People can share birthdays.

SETS $Person$; $Date$

VARIABLES bb

INVARIANT $bb \in Person \pfun Date$

INITIALISATION $bb := \{\}$

Add an entry to the directory:

$$
\begin{array}{lcl}
AddEntry(p, d) & \widehat{=} & \mathsf{PRE} \\
 & & \quad p \in Person \ \wedge \\
 & & \quad d \in Date \\
 & & \mathsf{THEN} \\
 & & \quad bb \ := \ bb \lhd\!\!+ \{p \mapsto d\} \\
 & & \mathsf{END}
\end{array}
$$

Syntactic shorthand:

$$
\begin{array}{lcl}
AddEntry(p, d) & \widehat{=} & \mathsf{PRE} \\
 & & \quad p \in Person \ \wedge \\
 & & \quad d \in Date \\
 & & \mathsf{THEN} \\
 & & \quad bb(p) \ := \ d \\
 & & \mathsf{END}
\end{array}
$$

Check a person's birthday:

$$
\begin{array}{lcl}
d \longleftarrow Check(p) & \widehat{=} & \mathsf{PRE} \\
 & & \quad p \in Person \ \wedge \\
 & & \quad p \in dom(bb) \\
 & & \mathsf{THEN} \\
 & & \quad d \ := \ bb(p) \\
 & & \mathsf{END}
\end{array}
$$

Check birthdays on a particular date:

$$pp \longleftarrow Who(d) \quad \widehat{=} \quad \begin{array}{l} \mathsf{PRE} \\ \quad d \in Date \\ \mathsf{THEN} \\ \quad pp := bb^{-1}[\{d\}] \\ \mathsf{END} \end{array}$$

$$bb^{-1} \quad \in \quad Date \leftrightarrow Person$$

Alternative:

$$pp \longleftarrow Who(d) \quad \widehat{=} \quad \begin{array}{l} \mathsf{PRE} \\ \quad d \in Date \\ \mathsf{THEN} \\ \quad pp := dom(bb \rhd \{d\}) \\ \mathsf{END} \end{array}$$