Relations in B

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Cartesian Products

An ordered pair is written: (x, y) or $x \mapsto y$

Cartesian Product is the type for ordered pairs: $S \times T$

Given $x \in S$, $y \in T$, we have $\boxed{x \mapsto y \in S \times T}$

$$4 \mapsto 7 \in$$

$$\{5, 6, 3\} \mapsto 4 \in$$

$$\{\ 4 \mapsto 8,\ 3 \mapsto 0,\ 2 \mapsto 9\ \} \in$$

Cartesian Product is a constructed type.

Sets of Order Pairs

A database can be modelled as a set of ordered pairs:

$$\begin{aligned} directory \;=\; \{\; & mary \mapsto 287573, \\ & mary \mapsto 398620, \\ & john \mapsto 829483, \\ & jim \mapsto 398620 \;\} \end{aligned}$$

$directory$ has type

$$directory \quad \in \quad \mathbb{P}(Person \times Number)$$

This is also known as a *relation*.

Relation is a constructed type:

$$\boxed{T \leftrightarrow S} \quad = \quad \mathbb{P}(T \times S)$$

So we write:

$$directory \quad \in \quad Person \leftrightarrow Number$$

Note:
$\leftrightarrow$ combines 2 types to form a type.
$\mapsto$ combines 2 elements to form an ordered pair.

Domain and Range

Given a pair $x \mapsto y$, we say that

x is the first component of the pair and

y is the second component.

Domain is the set of first components of all the pairs in a relation.

Range is the set of second components of all the pairs.

$$\text{dom}(directory) = \{mary, john, jim\}$$

$$\text{ran}(directory) = \{287573, 398620, 829483\}$$

Phone Directory Spec

Phone directory relates people to their phone numbers.
Each person can have zero or more numbers.
People can share numbers.

SETS $Person$; $PhoneNum$

VARIABLES dir

INVARIANT $dir \in Person \leftrightarrow PhoneNum$

INITIALISATION $dir := \{\}$

Add an entry to the directory:

$$AddEntry(p, n) \;\widehat{=}\; \begin{array}{l} \text{PRE} \\ \quad p \in Person \;\wedge \\ \quad n \in PhoneNum \\ \text{THEN} \\ \quad dir \;:=\; dir \cup \{p \mapsto n\} \\ \text{END} \end{array}$$

Check if a person is known in the directory:

$$res \longleftarrow IsKnown(p) \;\widehat{=}\; \begin{array}{l} \text{PRE} \\ \quad p \in Person \\ \text{THEN} \\ \quad res \;:=\; bool(\; p \in \mathsf{dom}(dir)\;) \\ \text{END} \end{array}$$

Relational Image

Given $R \in S \leftrightarrow T$ and $A \subseteq S$

Relational image of set A under relation R is $\boxed{R[A]}$

$$R[A] = \{ y \mid x \in A \wedge x \mapsto y \in R \}$$

Example:

$$\begin{aligned} directory = \{ & mary \mapsto 287573, \\ & mary \mapsto 398620, \\ & john \mapsto 829483, \\ & jim \mapsto 398620 \} \end{aligned}$$

$$directory[\ \{mary\}\ ] = \{ 287573,\ 398620 \}$$

Return all the numbers associated with a person in the directory:

$$nums \longleftarrow GetNumbers(p) \quad \widehat{=} \quad \begin{array}{l} \text{PRE} \\ \quad p \in Person \\ \text{THEN} \\ \quad nums \ := \ dir[\ \{p\}\ ] \\ \text{END} \end{array}$$

Return all the numbers associated with a set of people:

$$nums \longleftarrow GetMultiNumbers(ps) \quad \widehat{=} \quad \begin{array}{l} \text{PRE} \\ \quad ps \subseteq Person \\ \text{THEN} \\ \quad nums \ := \ dir[\ ps\ ] \\ \text{END} \end{array}$$

Relational Inverse

Given $R \in S \leftrightarrow T$

Relational inverse of R is $\boxed{R^{-1}}$

$$R^{-1} = \{ y \mapsto x \mid x \mapsto y \in R \}$$

Example:

$$\begin{aligned} directory^{-1} = \{ & 287573 \mapsto mary, \\ & 398620 \mapsto mary, \\ & 829483 \mapsto john, \\ & 398620 \mapsto jim \} \end{aligned}$$

$$directory^{-1}[\ \{398620\}\ ] = \{ mary,\ jim \}$$

Return all the people associated with a number in the directory:

$$peo \longleftarrow GetNames(n) \quad \widehat{=} \quad \begin{array}{l} \text{PRE} \\ \quad n \in PhoneNum \\ \text{THEN} \\ \quad peo \ := \ dir^{-1}[\ \{n\}\ ] \\ \text{END} \end{array}$$

Return all the people associated with a set of numbers:

$$peo \longleftarrow GetMultiNames(ns) \quad \widehat{=} \quad \begin{array}{l} \text{PRE} \\ \quad ns \subseteq PhoneNum \\ \text{THEN} \\ \quad peo \ := \ dir^{-1}[\ ns\ ] \\ \text{END} \end{array}$$

Domain Restriction

Given $R \in S \leftrightarrow T$ and $A \subseteq S$

The domain restriction of R by A is $\boxed{A \lhd R}$

Restrict R so that it only contains pairs whose first component is in the set A.

Example:

$$directory = \{\ mary \mapsto 287573,\quad mary \mapsto 398620,\quad john \mapsto 829483,\quad jim \mapsto 398620\ \}$$

$$\{john, jim, jane\} \lhd directory = \{\ john \mapsto 829483,\quad jim \mapsto 398620\ \}$$

Domain Subtraction

Given $R \in S \leftrightarrow T$ and $A \subseteq S$

The domain subtraction of R by A is $\boxed{A \lhd R}$

Remove those pairs from R whose first component is in A.

Example:

$$\{john, jim, jane\} \lhd directory = \{\ mary \mapsto 287573,\ mary \mapsto 398620\ \}$$

Range Restriction

Given $R \in S \leftrightarrow T$ and $B \subseteq T$

The range restriction of R by B is $\boxed{R \rhd B}$

Restrict R so that it only contains pairs whose second component is in the set B.

Range Subtraction

Given $R \in S \leftrightarrow T$ and $B \subseteq T$

The range subtraction of R by B is $\boxed{R \rhd\!\!\!- B}$

Remove those pairs from R whose second component is in B.

Remove all the entries associated with a person in the directory:

$$
\begin{array}{lll}
RemoveName(p) & \widehat{=} & \text{PRE} \\
 & & \quad p \in Person \\
 & & \text{THEN} \\
 & & \quad dir := \{p\} \lhd dir \\
 & & \text{END}
\end{array}
$$

Remove all the entries associated with a number in the directory:

$$
\begin{array}{lll}
RemoveNumber(n) & \widehat{=} & \text{PRE} \\
 & & \quad n \in PhoneNum \\
 & & \text{THEN} \\
 & & \quad dir := dir \rhd \{n\} \\
 & & \text{END}
\end{array}
$$

Relational Inverse

Given $R \in S \leftrightarrow T$

Relational inverse of R is $\boxed{R^{-1}}$

$$R^{-1} = \{ y \mapsto x \mid x \mapsto y \in R \}$$

Example:

$$\begin{aligned} directory^{-1} = \{ & 287573 \mapsto mary, \\ & 398620 \mapsto mary, \\ & 829483 \mapsto john, \\ & 398620 \mapsto jim \} \end{aligned}$$

$$directory^{-1}[\ \{398620\}\ ] = \{ mary,\ jim \}$$

Return all the people associated with a number in the directory:

$$peo \longleftarrow GetNames(n) \quad \widehat{=} \quad \begin{array}{l} \text{PRE} \\ \quad n \in PhoneNum \\ \text{THEN} \\ \quad peo \ := \ dir^{-1}[\ \{n\}\ ] \\ \text{END} \end{array}$$

Return all the people associated with a set of numbers:

$$peo \longleftarrow GetMultiNames(ns) \quad \widehat{=} \quad \begin{array}{l} \text{PRE} \\ \quad ns \subseteq PhoneNum \\ \text{THEN} \\ \quad peo \ := \ dir^{-1}[\ ns\ ] \\ \text{END} \end{array}$$

Recap

Cartesian product is the constructed type for pairs of elements.

A relation is a set of pairs.

Range of a relation, domain of a relation.

Relational image, relational inverse,

Domain restriction, subtraction

Range restriction, subtraction