A Homogeneous Universe of Constant Mass and Increasing Radius accounting for the Radial Velocity of Extra-galactic Nebulæ. By Abbé G. Lemaître.

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1. *Introduction.*

According to the theory of relativity, a homogeneous universe may exist such that all positions in space are completely equivalent; there is no centre of gravity. The radius of space R is constant; space is elliptic, *i.e.* of uniform positive curvature $1/R^2$; straight lines starting from a point come back to their origin after having travelled a path of length πR; the volume of space has a finite value $\pi^2 R^3$; straight lines are closed lines going through the whole space without encountering any boundary.

Two solutions have been proposed. That of de Sitter ignores the existence of matter and supposes its density equal to zero. It leads to special difficulties of interpretation which will be referred to later, but it is of extreme interest as explaining quite naturally the observed receding velocities of extra-galactic nebulæ, as a simple consequence of the properties of the gravitational field without having to suppose that we are at a point of the universe distinguished by special properties.

The other solution is that of Einstein. It pays attention to the evident fact that the density of matter is not zero, and it leads to a relation between this density and the radius of the universe. This relation forecasted the existence of masses enormously greater than any known at the time. These have since been discovered, the distances and dimensions of extra-galactic nebulæ having become known. From Einstein's formulæ and recent observational data, the radius of the universe is found to be some hundred times greater than the most distant objects which can be photographed by our telescopes.

Each theory has its own advantages. One is in agreement with the observed radial velocities of nebulæ, the other with the existence of matter, giving a satisfactory relation between the radius and the mass of the universe. It seems desirable to find an intermediate solution which could combine the advantages of both.

At first sight, such an intermediate solution does not appear to exist. A static gravitational field for a uniform distribution of matter without internal stress has only two solutions, that of Einstein and that of de Sitter. De Sitter's universe is empty, that of Einstein has been described as "containing as much matter as it can contain." It is remarkable that the theory can provide no mean between these two extremes.

The solution of the paradox is that de Sitter's solution does not really meet all the requirements of the problem. Space is homogeneous with constant positive curvature; space-time is also homogeneous, for all events are perfectly equivalent. But the partition of space-time into space and time disturbs the homogeneity. The co-ordinates used introduce a centre. A particle at rest at the centre of space describes a geodesic of the universe ; a particle at rest otherwhere than at the centre does not describe a geodesic. The co-ordinates chosen destroy the homogeneity and produce the paradoxical results which appear at the so-called " horizon " of the centre. When we use co-ordinates and a corresponding partition of space and time of such a kind as to preserve the homogeneity of the universe, the field is found to be no longer static ; the universe becomes of the same form as that of Einstein, with a radius no longer constant but varying with the time according to a particular law.

In order to find a solution combining the advantages of those of Einstein and de Sitter, we are led to consider an Einstein universe where the radius of space or of the universe is allowed to vary in an arbitrary way.

2. *Einstein Universe of Variable Radius. Field Equations. Conservation of Energy.*

As in Einstein's solution, we liken the universe to a rarefied gas whose molecules are the extra-galactic nebulæ. We suppose them so numerous that a volume small in comparison with the universe as a whole contains enough nebulæ to allow us to speak of the density of matter. We ignore the possible influence of local condensations. Furthermore, we suppose that the nebulæ are uniformly distributed so that the density does not depend on position. When the radius of the universe varies in an arbitrary way, the density, uniform in space, varies with time. Furthermore, there are generally interior stresses, which, in order to preserve the homogeneity, must reduce to a simple pressure, uniform in space and variable with time. The pressure, being two-thirds of the kinetic energy of the " molecules," is negligible with respect to the energy associated with matter ; the same can be said of interior stresses in nebulæ or in stars belonging to them. We are thus led to put $p = 0$.

Nevertheless it might be necessary to take into account the radiation-pressure of electromagnetic energy travelling through space ; this energy is weak but it is evenly distributed through the whole of space and might afford a notable contribution to the mean energy. We shall thus keep the pressure p in the general equations as the mean radiation-pressure of light, but we shall write $p = 0$ when we discuss the application to astronomy.

We denote the density of total energy by ρ, the density of radiation energy by $3p$, and the density of the energy condensed in matter by $\delta = \rho - 3p$. We identify ρ and $-p$ with the components T_4^4 and $T_1^1 = T_2^2 = T_3^3$ of the material energy tensor, and δ with T. Working out the contracted Riemann tensor for a universe with a line-element given by

$$ds^2 = -R^2 d\sigma^2 + dt^2, \qquad . \quad . \quad . \quad . \quad (1)$$

where $d\sigma$ is the elementary distance in a space of radius unity, and R is a function of the time t, we find that the field equations can be written

$$3\frac{R'^2}{R^2} + \frac{3}{R^2} = \lambda + \kappa\rho \quad . \quad . \quad . \quad (2)$$

and

$$2\frac{R''}{R} + \frac{R'^2}{R^2} + \frac{1}{R^2} = \lambda - \kappa p \quad . \quad . \quad . \quad (3)$$

Accents denote derivatives with respect to t. λ is the unknown cosmological constant, and κ is the Einstein constant whose value is $1{\cdot}87 \,.\, 10^{-27}$ in C.G.S. units (8π in natural units).

The four identities giving the expression of the conservation of momentum and of energy reduce to

$$\frac{d\rho}{dt} + \frac{3R'}{R}(\rho + p) = 0 \quad . \quad . \quad . \quad . \quad (4)$$

which is the energy equation. This equation can replace (3). As $V = \pi^2R^3$ it can be written

$$d(V\rho) + pdV = 0, \quad . \quad . \quad . \quad . \quad (5)$$

showing that *the variation of total energy plus the work done by radiation-pressure in the dilatation of the universe is equal to zero.*

3. *Universe of Constant Mass.*

If $M = V\delta$ remains constant, we write, a being a constant,

$$\kappa\delta = \frac{a}{R^3} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

As

$$\rho = \delta + 3p$$

we have

$$3d(pR^3) + 3pR^2dR = 0 \quad . \quad . \quad . \quad (7)$$

and, β being a constant of integration,

$$\kappa p = \frac{\beta}{R^4} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and therefore

$$\kappa\rho = \frac{a}{R^3} + \frac{3\beta}{R^4} \quad . \quad . \quad . \quad . \quad (9)$$

By substitution in (2) we have

$$\frac{R'^2}{R^2} = \frac{\lambda}{3} - \frac{1}{R^2} + \frac{\kappa\rho}{3} = \frac{\lambda}{3} - \frac{1}{R^2} + \frac{a}{3R^3} + \frac{\beta}{R^4} \quad . \quad . \quad (10)$$

and

$$t = \int \frac{dR}{\sqrt{\dfrac{\lambda R^2}{3} - 1 + \dfrac{a}{3R} + \dfrac{\beta}{R^2}}} \quad . \quad . \quad . \quad (11)$$

When α and β vanish, we obtain the de Sitter solution in Lanczos's form—

$$R = \sqrt{\frac{3}{\lambda}} \cosh \sqrt{\frac{\lambda}{3}}(t - t_0) \qquad (12)$$

The Einstein solution is found by making $\beta = 0$ and R constant. Writing $R' = R'' = 0$ in (2) and (3) we find

$$\frac{1}{R^2} = \lambda \qquad \frac{3}{R^2} = \lambda + \kappa\rho \qquad \rho = \delta$$

or

$$R = \frac{1}{\sqrt{\lambda}} \qquad \kappa\delta = \frac{2}{R^2} \qquad (13)$$

and from (6)

$$\alpha = \kappa\delta R^3 = \frac{2}{\sqrt{\lambda}} \qquad (14)$$

The Einstein solution does not result from (14) alone; it also supposes that the initial value of R' is zero. If we write

$$\lambda = \frac{1}{R_0^2} \qquad (15)$$

we have for $\beta = 0$ and $\alpha = 2R_0$

$$t = R_0\sqrt{3}\int\frac{dR}{R - R_0}\sqrt{\frac{R}{R + 2R_0}} \qquad (16)$$

For this solution the two equations (13) are of course no longer valid.

Writing

$$\kappa\delta = \frac{2}{R_E^2} \qquad (17)$$

we have from (14) and (15)

$$R^3 = R_E^2 R_0 \qquad (18)$$

The value of R_E, the radius of the universe computed from the mean density by Einstein's equation (17), has been found by Hubble to be

$$R_E = 8{\cdot}5 \times 10^{28} \text{ cm.} = 2{\cdot}7 \times 10^{10} \text{ parsec.} \qquad (19)$$

We shall see later that the value of R_0 can be computed from the radial velocities of the nebulæ; R can then be found from (18).

Finally, we shall show that a serious departure from (14) would lead to consequences not easily acceptable.

4. *Doppler Effect due to the Variation of the Radius of the Universe.*

From (1) we have for a ray of light

$$\sigma_2 - \sigma_1 = \int_{t_1}^{t_2}\frac{dt}{R} \qquad (20)$$

where σ_1 and σ_2 relate to spatial co-ordinates. We suppose that the light is emitted at the point σ_1 and observed at σ_2. A ray of light emitted slightly later starts from σ_1 at time $t_1 + \delta t_1$ and reaches σ_2 at time $t_2 + \delta t_2$. We have therefore

$$\frac{\delta t_2}{R_2} - \frac{\delta t_1}{R_1} = 0, \qquad \frac{\delta t_2}{\delta t_1} - 1 = \frac{R_2}{R_1} - 1 \quad . \quad . \quad (21)$$

where R_1 and R_2 are the values of the radius R at the time of emission t_1 and at the time of observation t_2. If δt_1 is the period of the emitted light, δt_2 is the period of the observed light. Now δt_1 is also the period of light emitted under the same conditions in the neighbourhood of the observer, because the period of light emitted under the same physical conditions has the same value everywhere when reckoned in proper time. Therefore

$$\frac{v}{c} = \frac{\delta t_2}{\delta t_1} - 1 = \frac{R_2}{R_1} - 1 \quad . \quad . \quad . \quad . \quad (22)$$

is the apparent Doppler effect due to the variation of the radius of the universe. *It equals the ratio of the radii of the universe at the instants of observation and emission, diminished by unity.*

v is that velocity of the observer which would produce the same effect. When the light source is near enough, we have the approximate formulæ

$$\frac{v}{c} = \frac{R_2 - R_1}{R_1} = \frac{dR}{R} = \frac{R'}{R}dt = \frac{R'}{R}r$$

where r is the distance of the source. We have therefore

$$\frac{R'}{R} = \frac{v}{cr} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

From a discussion of available data, we adopt

$$\frac{R'}{R} = 0{\cdot}68 \times 10^{-27}\ \text{cm.}^{-1} \quad . \quad . \quad . \quad (24)$$

and find from (16)

$$\frac{R'}{R} = \frac{1}{R_0\sqrt{3}}\sqrt{1 - 3y^2 + 2y^3} \quad . \quad . \quad . \quad (25)$$

where

$$y = \frac{R_0}{R} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Now from (18) and (26)

$$R_0^2 = R_E^2 y^3 \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and therefore

$$3\left(\frac{R'}{R}\right)^2 R_E^2 = \frac{1 - 3y^2 + 2y^3}{y^3} \quad . \quad . \quad . \quad (28)$$

With the adopted numerical data (24) and (19), we have

$$y = 0{\cdot}0465$$

giving

$$R = R_E\sqrt{y} = 0{\cdot}215 R_E = 1{\cdot}83 \times 10^{28}\ \text{cm.} = 6 \times 10^9\ \text{parsecs.}$$
$$R_0 = Ry = R_E y^{\frac{3}{2}} = 8{\cdot}5 \times 10^{26}\ \text{cm.} = 2{\cdot}7 \times 10^8\ \text{parsecs.}$$
$$= 9 \times 10^8\ \text{light-years.}$$

Integral (16) can easily be computed. Writing

$$x^2 = \frac{R}{R + 2R_0} \quad . \quad . \quad . \quad . \quad (29)$$

it can be written

$$t = R_0\sqrt{3}\int\frac{4x^2dx}{(1 - x^2)(3x^2 - 1)}$$
$$= R_0\sqrt{3}\log\frac{1 + x}{1 - x} + R_0\log\frac{\sqrt{3}x - 1}{\sqrt{3}x + 1} + C \quad . \quad (30)$$

If σ is the fraction of the radius of the universe travelled by light during time t, we have also

$$\sigma = \int\frac{dt}{R} = \sqrt{3}\int\frac{2dx}{3x^2 - 1} = \log\frac{\sqrt{3}x - 1}{\sqrt{3}x + 1} + C' \quad . \quad (31)$$

The following table gives values of σ and t for different values of R/R_0 :—

Table.—*Values of σ and t.*

$\frac{R}{R_0}$.	$\frac{t}{R_0}$.	σ Radians	σ Degrees.	$\frac{v}{c}$.
1	$-\infty$	$-\infty$	$-\infty$	19
2	−4·31	−0·889	−51°	9
3	−3·42	−0·521	−30	$5\frac{2}{3}$
4	−2·86	−0·359	−21	4
5	−2·45	−0·266	−15	3
10	−1·21	−0·087	− 5	1
15	−0·50	−0·029	− 1·7	$\frac{1}{3}$
20	0·00	0·000	0·0	0
25	0·39	0·017	1	..
∞	∞	0·087	5	..

The constants of integration are adjusted to make σ and t vanish for $R/R_0 = 20$ in place of 21·5. The last column gives the Doppler effect computed from (22). The approximate formula (23) would make v/c proportional to r and thus to σ. The error is only 0·005 for $v/c = 1$. The approximate formula may therefore be used within the limits of the visible spectrum.

5. *The Meaning of Equation* (14).

The relation (14) between the two constants λ and α has been adopted following Einstein's solution. It is the necessary condition that the quartic under the radical in (11) may have a double root R_0 giving on integration a logarithmic term. For simple roots, integration would give a square root, corresponding to a minimum of R as in de Sitter's solution (12). This minimum would generally occur at time of the order of R_0, say 10^9 years—*i.e.* quite recently for stellar evolution.

If the positive roots were to become imaginary, the radius would vary from zero upwards, the variation slowing down in the neighbourhood of the modulus of the imaginary roots. In both cases the time of variation of R in the same sense would be of the order of R_0 if the relation between λ and α were seriously different from (14).

6. *Conclusion.*

We have found a solution such that

(1°) The mass of the universe is a constant related to the cosmological constant by Einstein's relation

$$\sqrt{\lambda} = \frac{2\pi^2}{\kappa M} = \frac{1}{R_0}.$$

(2°) The radius of the universe increases without limit from an asymptotic value R_0 for $t = -\infty$.

(3°) The receding velocities of extragalactic nebulæ are a cosmical effect of the expansion of the universe. The initial radius R_0 can be computed by formulæ (24) and (25) or by the approximate formula

$$R_0 = \frac{rc}{v\sqrt{3}}.$$

This solution combines the advantages of the Einstein and de Sitter solutions.

Note that the largest part of the universe is for ever out of our reach. The range of the 100-inch Mount Wilson telescope is estimated by Hubble to be 5×10^7 parsecs, or about R/200. The corresponding Doppler effect is 3000 km./sec. For a distance of 0·087R it is equal to unity, and the whole visible spectrum is displaced into the infra-red. It is impossible to see ghost-images of nebulæ or suns, as even if there were no absorption these images would be displaced by several octaves into the infra-red and would not be observed.

It remains to find the cause of the expansion of the universe. We have seen that the pressure of radiation does work during the expansion. This seems to suggest that the expansion has been set up by the radiation itself. In a static universe light emitted by matter travels round space, comes back to its starting-point, and accumulates indefinitely. It seems that this may be the origin of the velocity of expansion R′/R which Einstein assumed to be zero and which in our interpretation is observed as the radial velocity of extra-galactic nebulæ.

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(*Communicated by Sir A. S. Eddington.*)

1. *Introduction.*

Eddington has suggested that the expansion of a universe in equilibrium may be started by the formation of condensations. A preliminary investigation by W. H. McCrea and G. C. McVittie seems to point out an effect of opposite sense according to the nature of the condensations.* I find that the formation of condensations and the degree of concentration of these condensations have no effect whatever on the equilibrium of the universe. Nevertheless, the expansion of the universe is due to an effect very closely related to the formation of condensations, which may be named the "stagnation" of the universe. When there is no condensation, the energy, or at least a notable part of it, may be able to wander freely through the universe. When condensations are formed this free kinetic energy has a chance to be captured by the condensations and then to remain bound to them. That is what I mean by a "stagnation" of the world—a diminution of the exchanges of energy between distant parts of it.

In order to investigate the effect of condensations in a universe homogeneous in the mean, I consider a definite condensation of supposed spherical symmetry, and I average the outside condensations so that they also may be thought of as having spherical symmetry. The condensation under investigation is limited by a spherical shell which is the neutral zone between it and neighbouring condensations; a point on this neutral zone is not more within the gravitational influence of the interior condensation than of the condensations outside. The expansion of the neutral zone gives a measure of the expansion of the

* Sir A. S. Eddington, *M.N.*, **90**, 668, 1930; W. H. McCrea and G. C. McVittie, *M.N.*, **91**, 128, 1930; G. C. McVittie, *M.N.*, **91**, 274, 1931.