A categorical approach to open and interconnected dynamical systems

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Abstract

In his 1986 Automatica paper Willems introduced the influential behavioural approach to control theory with an investigation of linear time-invariant (LTI) discrete dynamical systems. The behavioural approach places open systems at its centre, modelling by tearing, zooming, and linking. We show that these ideas are naturally expressed in the language of symmetric monoidal categories.

Our main result gives an intuitive sound and fully complete string diagram algebra for reasoning about LTI systems. These string diagrams are closely related to the classical notion of signal flow graphs, endowed with semantics as multi-input multi-output transducers that process discrete streams with an *infinite past* as well as an infinite future. At the categorical level, the algebraic characterisation is that of the prop of corelations.

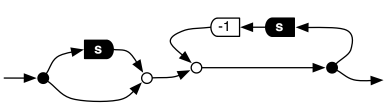
Using this framework, we derive a novel structural characterisation of controllability, and consequently provide a methodology for analysing controllability of networked and interconnected systems. We argue that this has the potential of providing elegant, simple, and efficient solutions to problems arising in the analysis of systems over networks, a vibrant research area at the crossing of control theory and computer science.

CCS Concepts: •**Theory of computation** → **Logic**; *Equational logic and rewriting*.

1. Introduction

The remit of this paper is the development of a sound and fully complete equational theory of linear time-invariant (LTI) dynamical systems. This theory is *graphical*, with its terms—modelling the LTI systems themselves—best represented as diagrams closely resembling the signal flow graphs of Shannon [18].

To acquaint ourselves with signal flow graphs, we begin with the example below, rendered in traditional, directed notation.

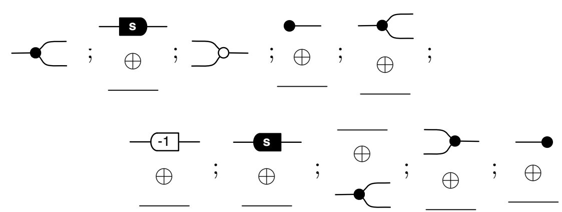


(1)

This system takes, as input on the left, a stream of values from a field k, e.g. the rational numbers, and on the right outputs a processed stream of values. The white circles are adders, the black circles are duplicators, the s gates are 1-step delays and the -1 gate is an instance of an amplifier that outputs -1 times its input. Processing is done synchronously according to a global clock.

For instance, assume that at time 0 the left s gate 'stores' the value 1 and the right s gate stores 2. Given an input of -1, the flow graph first adds the left stored value 1, and then adds -1×2, for an output of -2. Immediately after this time step the s gates, acting as delays, now store -1 and -2 respectively, and we repeat the process with the next input. Thus from this time 0 an input stream of $-1, 1, -1, 1\ldots$ results in an output stream of $-2, 2, -2, 2, \ldots$.

We can express (1) as a string diagram, a notation for the arrows of monoidal categories made popular by Joyal and Street [12], by forgetting the directionality of wires and composing the following basic building blocks using the operations of monoidal categories.



The building blocks come from the signature of an algebraic theory—a *symmetric monoidal theory* to be exact. The terms of this theory comprise the morphisms of a *prop*, a symmetric monoidal category in which the objects are the natural numbers. With an operational semantics suggested by the above example, the terms can also be considered as a process algebra for signal flow graphs. The idea of understanding complex systems by "tearing" them into more basic components, "zooming" to understand their individual behaviour and "linking" to obtain a composite system is at the core of the behavioural approach in control, originated by Willems [22]. The algebra of symmetric monoidal categories thus seems a good fit for a formal account of these compositional principles.

This paper is the first to make this link between monoidal categories and the behavioural approach to control explicit. Moreover, it is the first to endow signal flow graphs with their standard systems theoretic semantics in which the registers—the 's' gates—are permitted to hold *arbitrary* values at the beginning of a computation (in previous work [4, 6] they were initialised with 0). This extended notion of behaviour is not merely a theoretical curiosity: it gives the class of *complete LTI discrete dynamical systems* [22], which is practically the lingua franca of control theory. The inter-

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est of systems theorists is due to practical considerations: physical systems seldom evolve from zero initial conditions.

Technically, the semantics of diagrams are sets of *biinfinite* streams: those sequences of elements of k that are infinite in the past *as well* as in the future—that is, elements of $\mathsf{k}^{\mathbb{Z}}$. Starting with the operational description, one obtains a biinfinite trajectory by executing circuits forwards and backwards in time, for some initialisation of the registers. The dynamical system defined by a signal flow diagram is the set of trajectories obtained by considering all possible executions from all possible initialisations.

An equational theory also requires equations between the terms. We obtain the equations in two steps. First, we show there is a full, but not faithful, morphism from the prop $\mathsf{Cospan\,Mat\,k}[s, s^{-1}]$ of cospans of matrices over the ring $\mathsf{k}[s, s^{-1}]$ to the prop LTI of complete LTI discrete dynamical systems. Using the presentation of $\mathsf{Cospan\,Mat\,k}[s, s^{-1}]$ in [5, 23], the result is a sound, but not complete, proof system. The second ingredient is restricting our attention from cospans to jointly-epic cospans, or *corelations*. This gives a faithful morphism, allowing us to present the prop of corelations as a symmetric monoidal theory, and hence giving a sound and complete proof system for reasoning about LTIs (Theorem 4.9).

The advantages of the string diagram calculus over the traditional matrix calculus are manifold. The operational semantics make the notation intuitive, as does the compositional aspect: it is cumbersome to describe connection of systems using matrices, whereas with string diagrams you just connect the right terminals. Moreover, the calculus unifies the variety of distinct methods for representing LTI systems with matrix equations—built from kernel and image representations [20, 22]—into a single framework, heading off possibilities for ambiguity and confusion.

We hope, however, the greatest advantage will be the way these properties can be leveraged in analysis of controllability. In Theorem 6.4, we show that in our setting controllability has an elegant structural characterisation. Compositionality pays off here, with our proof system giving a new technique for reasoning about control of compound systems (Prop. 6.8). From the systems theoretic point of view, these results are promising since the compositional, diagrammatic techniques we bring to the subject seem well-suited to problems such as controllability of interconnections, of primary interest for multiagent and spatially interconnected systems [15].

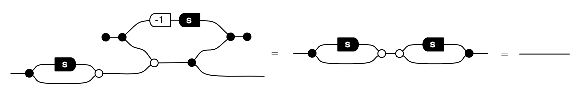
Summing up, our original technical contributions are:

- a characterisation of the class of LTI systems as a category of corelations of matrices
- a presentation of this category of corelations of matrices as a symmetric monoidal theory
- an operational semantics that agrees with the standard systems theoretic semantics of signal flow graphs
- a characterisation of controllability

Our work lies in the intersection of computer science, mathematics, and systems theory. From computer science, we use concepts of formal semantics of programming languages, with an emphasis on compositionality and a firm denotational foundation for operational definitions. From a mathematical perspective, we obtain presentations of several relevant domains, and identify the rich underlying algebraic structures. For systems theory, our insight is that mere matrices are not optimised for discussing behaviour; instead it is profitable to use signal flow graphs, which treat linear subspaces rather than linear transformations as the primitive concept and are thus closer to the idea of system as a set of trajectories. At the core is the maxim—perhaps best understood by computer scientists—that the right language allows deeper insights into the underlying structure.

Related work. Work on categorical approaches to control systems goes back at least to Goguen [10] and Arbib and Manes [1]. In recent years, there has been a resurgence of interest in the topic, including work by Baez and Erbele [2], Vagner, Spivak, and Lerman [19], as well as Bonchi, Zanasi, and the second author [4–6, 23].

Although previous work [5, 6, 23] made the connection between signal flow graphs and string diagrams, their operational semantics is more restrictive than that considered here, considering only trajectories with finite past and demanding that, initially, all the registers contain the value 0. Indeed, with this restriction, it is not difficult to see that the trajectories of (1) are those where the output is *the same* as the input. The input/output behaviour is thus that of a stateless wire. The equational presentation in this case is the theory $\mathbb{IH}_{k[s]}$ of interacting Hopf algebras [5], and indeed, in $\mathbb{IH}_{k[s]}$:



(2)

Note that (2) is *not sound* for circuits with our more liberal, operational semantics. Indeed, recall that when the registers of (1) initially hold values 1 and 2, the input $-1, 1, -1, 1, \ldots$ results in the output $-2, 2, -2, 2, \ldots$. This trajectory is not permitted by a stateless wire, so $\mathbb{IH}_{k[s]}$ is not sound for reasoning about LTI systems in general. The contribution of this paper is to provide sound and complete theory to do just that.

In terms of the algebraic semantics, the difference from previous work [4, 6] is that where there streams were handled with Laurent (formal power) series, here we use the aforementioned biinfinite streams. Indeed, this is the very extension that allows us to discuss non-controllable behaviours; in [2, 4, 6, 23] all definable behaviours were controllable.

Structure of the paper. In §3 we develop a categorical account of complete LTI discrete dynamical systems. This serves as a denotational semantics for the graphical language, introduced in §4, where we also derive the equational characterisation. In §5 we relate this to the operational semantics. We conclude in §6 with a structural account of controllability.

2. Preliminaries

We assume familiarity with basic concepts of linear algebra and category theory.

A **prop** is a strict symmetric monoidal category where the set of objects is the natural numbers $\mathbb{N}$, and monoidal product ($\oplus$) on objects is addition. Homomorphism of props are identity-on-objects strict symmetric monoidal functors.

A **symmetric monoidal theory** (SMT) is a presentation of a prop: a pair (Σ, E) where Σ is a set of **generators** $\sigma\colon m \to n$, where m is the **arity** and n the **coarity**. A Σ-term is a obtained from Σ, identity $\mathrm{id}\colon 1 \to 1$ and symmetry $\mathrm{tw}\colon 2 \to 2$ by composition and monoidal product, according to the grammar

$$\tau \ ::= \ \sigma \mid \mathrm{id} \mid \mathrm{tw} \mid \tau \,;\, \tau \mid \tau \oplus \tau$$

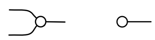
where ; and $\oplus$ satisfy the standard typing discipline that keeps track of the domains (arities) and codomains (coarities)

$$\frac{\tau : m \to d \quad \tau' : d \to n}{\tau \,;\, \tau' : m \to n} \qquad \frac{\tau : m \to n \quad \tau' : m' \to n'}{\tau \oplus \tau' : m + m' \to n + n'}$$

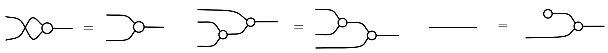
The second component E of an SMT is a set of **equations**, where an equation is a pair (τ, μ) of Σ-terms with compatible types, i.e. $\tau, \mu\colon m \to n$ for some $m, n \in \mathbb{N}$.

Given an SMT (Σ, E), the prop $\mathbf{S}_{(\Sigma,E)}$ has as arrows the Σ-terms quotiented by the smallest congruence that includes the laws of symmetric monoidal categories and equations E. We sometimes abuse notation by referring to $\mathbf{S}_{(\Sigma,E)}$ as an SMT. Given an arbitrary prop $\mathbb{X}$, a **presentation** of $\mathbb{X}$ is an SMT (Σ, E) s.t. $\mathbb{X} \cong \mathbf{S}_{(\Sigma,E)}$.

String diagrams play an important role in our work. Given generators Σ, we consider string diagrams to be arrows of $\mathbf{S}_{(\Sigma,\varnothing)}$, that is, syntactic objects constructed by composition from the generators, quotiented by the laws of symmetric monoidal categories. For example, consider generators $2 \to 1$ and $0 \to 1$, which we draw below, with 'dangling wires' accounting for arities and coarities:



Armed with the graphical notation, we can present sets of equations as equations between diagrams. For example, the SMT of commutative monoids consists of these generators together with equations



that respectively are commutativity, associativity, and the unit law.

A **split mono** is a morphism $m\colon X \to Y$ such that there exists $m'\colon Y \to X$ with $m'm = \mathrm{id}_X$. For the remainder of the paper k is a field: for concreteness one may take this to be the rationals $\mathbb{Q}$, the reals $\mathbb{R}$ or the booleans $\mathbb{Z}_2$.

3. Linear time-invariant dynamical systems

This section focusses on the mathematical domains of interest for the remainder of the paper. We rely on definitions of the *behavioural approach* in control, which is informed by compositional considerations [20]. The concepts are standard in systems theory. Our categorical insights are, to the best of our knowledge, original.

Following Willems [22], a **dynamical system** $(T, W, \mathscr{B})$ is: a **time axis** T, a **signal space** W, and a **behaviour** $\mathscr{B} \subseteq W^T$. We refer to $w \in \mathscr{B}$ as **trajectories**. In this paper we are interested in discrete trajectories that are **biinfinite**: infinite in past and future. Our time axis is thus the integers $\mathbb{Z}$. The signal space is k^d, where d is the number of **terminals** of a system. These, in engineering terms, are the interconnection variables that enable interaction with an environment.

The dynamical systems of concern to us are thus specified by some natural number d and a subset $\mathscr{B}$ of $(\mathsf{k}^d)^{\mathbb{Z}}$. The sets $(\mathsf{k}^d)^{\mathbb{Z}}$ are k-vector spaces, with pointwise addition and scalar multiplication. We restrict attention to *linear* systems, meaning that $\mathscr{B}$ is required to be a k-*linear subspace*—i.e. closed under addition and multiplication by k-scalars—of $(\mathsf{k}^d)^{\mathbb{Z}}$.

We partition terminals into a *domain* and *codomain* of m and n terminals respectively, writing $\mathscr{B} \subseteq (\mathsf{k}^m)^{\mathbb{Z}} \oplus (\mathsf{k}^n)^{\mathbb{Z}} \cong (\mathsf{k}^d)^{\mathbb{Z}}$. This may seem artificial, in the sense that the assignment is arbitrary. In particular, it is crucial not to confuse the domains (codomains) with inputs (outputs). In spite of the apparent contrivedness of choosing such a partition, Willems and others have argued that it is vital for a sound theory of system *decomposition*; indeed, it enables the "tearing" of Willems' tearing, zooming and linking [20].

Once the domains and codomains have been chosen, systems are linked by connecting terminals. In models of physical systems this means variable coupling or sharing [20]; in our discrete setting where behaviours are subsets of a cartesian product—i.e. relations—it amounts to relational composition. Since behaviours are both relations and linear subspaces, a central underlying mathematical notion—as in previous work [2, 4]—is a linear relation.

Definition 3.1. *The monoidal category* $\mathsf{LinRel}_{\mathsf{k}}$ *of* k*-linear relations has* k*-vector spaces as objects, and as arrows from* V *to* W*, linear subspaces of* $V \oplus W$*, considered as* k*-vector spaces. Composition is relational: given* $A\colon U \to V$*,* $B\colon V \to W$*,* $A \,;\, B\colon U \to W$ *is the relation*

$$\{\, (u, w) \mid \exists v \in V \text{ s.t. } (u, v) \in A \text{ and } (v, w) \in B \,\}$$

that is easily checked to be a linear subspace. Finally, the monoidal product on both objects and morphisms is direct sum.

A behaviour is **time-invariant** when for every trajectory $w \in \mathscr{B}$ and any fixed $i \in \mathbb{Z}$, the trajectory whose value at every time $t \in \mathbb{Z}$ is $w(t+i)$ is also in $\mathscr{B}$. Time-invariance brings with it a connection with the algebra of polynomials. Following the standard approach in control theory, going back to Rosenbrock [16], we work with polynomials over an indeterminate s as well as its formal inverse s^{-1}—i.e. the elements of the ring $\mathsf{k}[s, s^{-1}]$.[1]

The indeterminate s acts on a given biinfinite stream $w \in \mathsf{k}^{\mathbb{Z}}$ as a one-step delay, and s^{-1} as its inverse, a one step acceleration:

$$(s \cdot w)(t) \overset{\text{def}}{=} w(t-1), \quad (s^{-1} \cdot w)(t) \overset{\text{def}}{=} w(t+1).$$

We can extend this, in the obvious linear, pointwise manner, to an action of any polynomial $p \in \mathsf{k}[s, s^{-1}]$ on w. Since $\mathsf{k}^{\mathbb{Z}}$ is a k-vector space, any such p defines a k-linear map $\mathsf{k}^{\mathbb{Z}} \to \mathsf{k}^{\mathbb{Z}}$ $(w \mapsto p \cdot w)$.

Given this, we can view $n \times m$ matrices over $\mathsf{k}[s, s^{-1}]$ as k-linear maps from $(\mathsf{k}^m)^{\mathbb{Z}}$ to $(\mathsf{k}^n)^{\mathbb{Z}}$. This viewpoint can be explained succinctly as a functor from the prop $\mathsf{Mat}\,\mathsf{k}[s, s^{-1}]$, defined below, to the category of k-vector spaces and linear transformations $\mathsf{Vect}_{\mathsf{k}}$.

Definition 3.2. *The prop* $\mathsf{Mat}\,\mathsf{k}[s, s^{-1}]$ *has as arrows* $m \to n$ *the* $n \times m$*-matrices over* $\mathsf{k}[s, s^{-1}]$*. Composition is matrix multiplication, and the monoidal product of* A *and* B *is* $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$*. The symmetries are permutation matrices.*

The functor of interest

$$\theta\colon\ \mathsf{Mat}\,\mathsf{k}[s, s^{-1}] \longrightarrow \mathsf{Vect}_{\mathsf{k}}$$

takes a natural number n to $(\mathsf{k}^n)^{\mathbb{Z}}$, and an $n \times m$ matrix to the induced linear transformation $(\mathsf{k}^m)^{\mathbb{Z}} \to (\mathsf{k}^n)^{\mathbb{Z}}$. Note that θ is faithful.

The final restriction on the set of behaviours is called *completeness*, and is a touch more involved. For $t_0, t_1 \in \mathbb{Z}$, $t_0 \le t_1$, write $w|_{[t_0,t_1]}$ for the restriction of $w : \mathbb{Z} \to \mathsf{k}^n$ to the set $[t_0, t_1] = \{t_0, t_0+1, \ldots, t_1\}$. Write $\mathscr{B}|_{[t_0,t_1]}$ for the set of the restrictions of all trajectories $w \in \mathscr{B}$ to $[t_0, t_1]$. Then $\mathscr{B}$ is **complete** when $w|_{[t_0,t_1]} \in \mathscr{B}|_{[t_0,t_1]}$ for all $t_0, t_1 \in \mathbb{Z}$ implies $w \in \mathscr{B}$. This topological condition is important as it characterises the linear time-invariant behaviours that are kernels of the action of $\mathsf{Mat}\,\mathsf{k}[s, s^{-1}]$; see Theorem 3.5.

Definition 3.3. *A* **linear time-invariant (LTI) behaviour** *comprises a domain* $(\mathsf{k}^m)^{\mathbb{Z}}$*, a codomain* $(\mathsf{k}^n)^{\mathbb{Z}}$*, and a subset* $\mathscr{B} \subseteq (\mathsf{k}^m)^{\mathbb{Z}} \oplus (\mathsf{k}^n)^{\mathbb{Z}}$ *such that* $(\mathbb{Z}, \mathsf{k}^m \oplus \mathsf{k}^n, \mathscr{B})$ *is a complete, linear, time-invariant dynamical system.*

The algebra of LTI behaviours is captured concisely as a prop.

Proposition 3.4. *There exists a prop* LTI *with morphisms* $m \to n$ *the LTI behaviours with domain* $(\mathsf{k}^m)^{\mathbb{Z}}$ *and codomain* $(\mathsf{k}^n)^{\mathbb{Z}}$*. Composition is relational. The monoidal product is direct sum.*

The proof of Proposition 3.4 relies on *kernel representations* of LTI systems. The following result lets us pass between behaviours and polynomial matrix algebra.

[1] The introduction of the formal inverse s^{-1} is a departure from previous work [4, 6] that dealt with Laurent streams (finite in the past, infinite in the future), and algebraically with the field of polynomial fractions. As we will see below, there is a natural action of $\mathsf{k}[s, s^{-1}]$ on biinfinite streams, but it does not make sense, in general, to define the action of a polynomial fraction on a biinfinite stream.

Theorem 3.5 (Willems [22, Th. 5]). *Let $\mathscr{B}$ be a subset of $(\mathsf{k}^n)^{\mathbb{Z}}$ for some $n \in \mathbb{N}$. Then $\mathscr{B}$ is an LTI behaviour iff there exists $M \in \mathsf{Mat}\,\mathsf{k}[s, s^{-1}]$ such that $\mathscr{B} = \ker(\theta M)$.*

The prop $\mathsf{Mat}\,\mathsf{k}[s, s^{-1}]$ is equivalent to the category $\mathsf{FMod}\,\mathsf{k}[s, s^{-1}]$ of finite-dimensional free $\mathsf{k}[s, s^{-1}]$-modules. Since $\mathsf{FMod}\,R$ over a principal ideal domain (PID) R has finite colimits [5], and $\mathsf{k}[s, s^{-1}]$ is a PID, $\mathsf{Mat}\,\mathsf{k}[s, s^{-1}]$ has finite colimits, and thus it has pushouts.

We can therefore define the prop $\mathsf{Cospan}\,\mathsf{Mat}\,\mathsf{k}[s, s^{-1}]$ where arrows are (isomorphism classes of) cospans of matrices: arrows $m \to n$ comprise a natural number d together with a $d \times m$ matrix A and a $d \times n$ matrix B; we write this $m \xrightarrow{A} d \xleftarrow{B} n$. Composition is given by pushout, and isomorphic cospans are identified; for details see Bénabou [3].

We can then extend θ to the functor

$$\Theta\colon\ \mathsf{Cospan}\,\mathsf{Mat}\,\mathsf{k}[s, s^{-1}] \longrightarrow \mathsf{LinRel}_{\mathsf{k}}$$

where on objects $\Theta(n) = \theta(n) = (\mathsf{k}^n)^{\mathbb{Z}}$, and on arrows

$$m \xrightarrow{A} d \xleftarrow{B} n$$

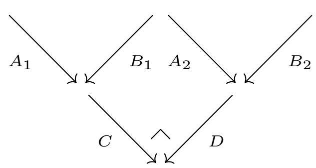
maps to

$$\{(\mathbf{x}, \mathbf{y}) \,|\, (\theta A)\mathbf{x} = (\theta B)\mathbf{y}\} \subseteq (\mathsf{k}^m)^{\mathbb{Z}} \oplus (\mathsf{k}^n)^{\mathbb{Z}}. \qquad (3)$$

It is straightforward to prove that this is well-defined.

Proposition 3.6. Θ *is a functor.*

Proof. Identities are clearly preserved; it suffices to show that composition is too. Consider the diagram below, where the pushout is calculated in $\mathsf{Mat}\,\mathsf{k}[s, s^{-1}]$.



To show that Θ preserves composition we must verify that

$$\{(\mathbf{x}, \mathbf{y}) \,|\, \theta C A_1 \mathbf{x} = \theta D B_2 \mathbf{y}\} = \{(\mathbf{x}, \mathbf{z}) \,|\, \theta A_1 \mathbf{x} = \theta B_1 \mathbf{z}\} ; \{(\mathbf{z}, \mathbf{y}) \,|\, \theta A_2 \mathbf{z} = \theta B_2 \mathbf{y}\}.$$

The inclusion $\subseteq$ follows from properties of pushouts in $\mathsf{Mat}\,\mathsf{k}[s, s^{-1}]$ (see [5, Prop. 5.7]). To see $\supseteq$, we need to show that if there exists $\mathbf{z}$ such that $\theta A_1 \mathbf{x} = \theta B_1 \mathbf{z}$ and $\theta A_2 \mathbf{z} = \theta B_2 \mathbf{y}$, then $\theta C A_1 \mathbf{x} = \theta D B_2 \mathbf{y}$. But $\theta C A_1 \mathbf{x} = \theta C B_1 \mathbf{z} = \theta D A_2 \mathbf{z} = \theta D B_2 \mathbf{y}$. ☐

Rephrasing the definition of Θ on morphisms (3), the behaviour consists of those $(\mathbf{x}, \mathbf{y})$ that satisfy

$$\theta \begin{bmatrix} A & -B \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \mathbf{0},$$

so one may say—ignoring for a moment the terminal domain/codomain assignment—that

$$\Theta(\xrightarrow{A}\xleftarrow{B}) = \ker \theta \begin{bmatrix} A & -B \end{bmatrix}.$$

With this observation, as a consequence of Theorem 3.5, Θ has as its image (essentially) the prop LTI. This proves Proposition 3.4. We may thus consider Θ a functor onto the codomain LTI; denote this corestriction $\overline{\Theta}$. We thus have a full functor:

$$\overline{\Theta}\colon\ \mathsf{Cospan}\,\mathsf{Mat}\,\mathsf{k}[s, s^{-1}] \longrightarrow \mathsf{LTI}\,.$$

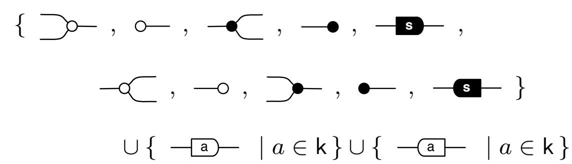
Remark 3.7. *It is important for the sequel to note that* $\overline{\Theta}$ *is* not *faithful. For instance,* $\overline{\Theta}(1 \xrightarrow{[1]} 1 \xleftarrow{[1]} 1) = \overline{\Theta}(1 \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} 2 \xleftarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} 1)$, *yet the cospans are not isomorphic. The task of the next section is to develop a setting where these cospans are nonetheless* equivalent.

4. Presentation of LTI

In this section we give a presentation of LTI as an SMT. This means that (*i*) we obtain a syntax—conveniently expressed using string diagrams—for specifying every LTI behaviour, and (*ii*) a sound and fully complete equational theory for reasoning about them.

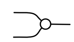
4.1 Syntax

We start by describing the graphical syntax of dynamical systems, the arrows of the category $\mathbb{S} = \mathbf{S}_{(\Sigma, \varnothing)}$, where Σ is the set of generators:

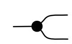


(4)

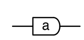
For each generator, we give its **denotational semantics**, an LTI behaviour, thereby defining a prop morphism $[\![-]\!] : \mathbb{S} \to \mathsf{LTI}$.

 $\mapsto \{\, (\left(\begin{smallmatrix}\tau\\ \upsilon\end{smallmatrix}\right), \tau + \upsilon) \mid \tau, \upsilon \in \mathsf{k}^{\mathbb{Z}} \,\} : 2 \to 1$

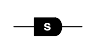
 $\mapsto \{\, ((), 0) \,\} \subseteq \mathsf{k}^{\mathbb{Z}} : 0 \to 1$

 $\mapsto \{\, (\tau, \left(\begin{smallmatrix}\tau\\ \tau\end{smallmatrix}\right)) \mid \tau \in \mathsf{k}^{\mathbb{Z}} \,\} : 1 \to 2$

 $\mapsto \{\, (\tau, ()) \mid \tau \in \mathsf{k}^{\mathbb{Z}} \,\} : 1 \to 0$



$\mapsto \{\, (\tau, a \cdot \tau) \mid \tau \in \mathsf{k}^{\mathbb{Z}} \,\} : 1 \to 1 \quad (a \in \mathsf{k})$



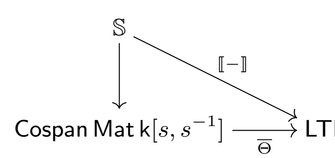
$\mapsto \{\, (\tau, s \cdot \tau) \mid \tau \in \mathsf{k}^{\mathbb{Z}} \,\} : 1 \to 1$

The denotations of the mirror image generators are the opposite relations. Parenthetically, we note that a finite set of generators is possible over a finite field, or the field $\mathbb{Q}$ of rationals, *cf.* §4.2.

The following result guarantees that the syntax is fit for purpose: every behaviour in LTI has a syntactic representation in $\mathbb{S}$.

Proposition 4.1. $[\![-]\!] : \mathbb{S} \to \mathsf{LTI}$ *is full.*

Proof. The fact that $[\![-]\!]$ is a prop morphism is immediate since $\mathbb{S}$ is free on the SMT $(\Sigma, \varnothing)$ with no equations. Fullness follows from the fact that $[\![-]\!]$ factors as the composite of two full functors:

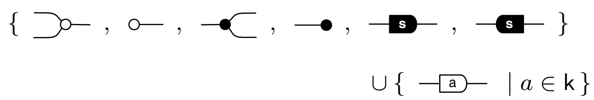


The functor $\overline{\Theta}$ is full by definition. The existence and fullness of the functor $\mathbb{S} \to \mathsf{Cospan}\,\mathsf{k}[s, s^{-1}]$ follows from [23, Th. 3.41]. We give details in the next two subsections. ☐

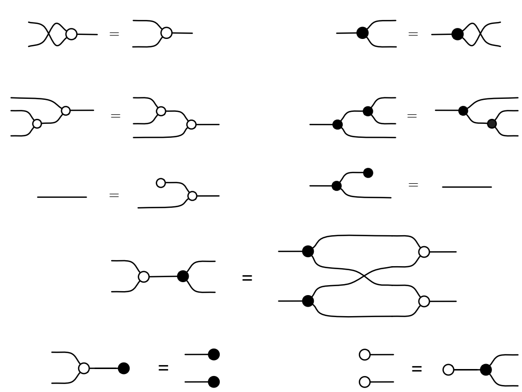
Having defined the syntactic prop $\mathbb{S}$ to represent arrows in LTI, our task for this section is to identify an equational theory that *characterises* LTI: i.e. one that is sound and fully complete. The first step is to use the existence of Θ: with the results of [5, 23] we can obtain a presentation for $\mathsf{Cospan}\,\mathsf{Mat}\,\mathsf{k}[s, s^{-1}]$. This is explained in the next two subsections: in §4.2 we present $\mathsf{Mat}\,\mathsf{k}[s, s^{-1}]$ and in §4.3, the equations for cospans of matrices.

4.2 Presentation of Mat $\mathsf{k}[s, s^{-1}]$

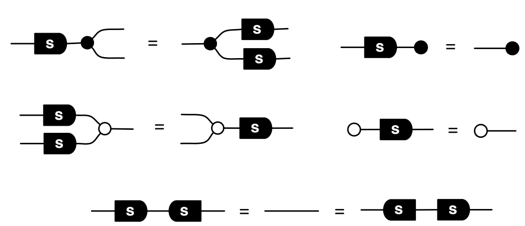
To obtain a presentation of Mat $\mathsf{k}[s, s^{-1}]$ as an SMT we only require some of the generators:



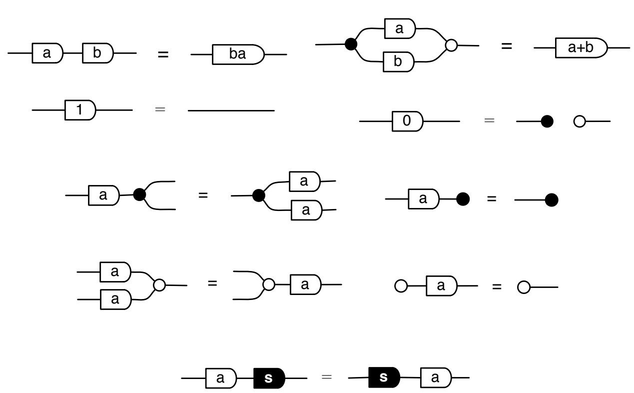
and the following equations. First, the white and the black structure forms a (bicommutative) bimonoid:



Next, the formal indeterminate s is compatible with the bimonoid structure and has its mirror image as a formal inverse.



Finally, we insist that the algebra of k be compatible with the bimonoid structure and commute with s.

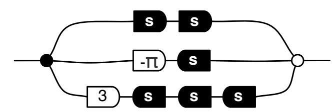


The three sets of equations above form the theory of Hopf algebras. Write $\mathbb{HA}_{\mathsf{k}[s,s^{-1}]}$ for the prop induced by the SMT consisting of the equations above. The following follows from [23, Prop. 3.9].

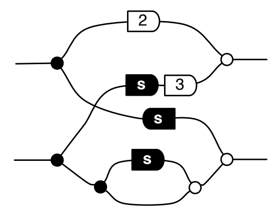
Proposition 4.2. Mat $\mathsf{k}[s, s^{-1}] \cong \mathbb{HA}_{\mathsf{k}[s,s^{-1}]}$.

Arrows of Mat $\mathsf{k}[s, s^{-1}]$ are matrices with polynomial entries, but it may not be obvious to the reader how polynomials arise with the string diagrammatic syntax. We illustrate this below.

Example 4.3. *Any polynomial $p = \sum_{i=u}^{v} a_i s^i$, where $u \le v \in \mathbb{Z}$ and with coefficients $a_i \in \mathsf{k}$, can be written graphically using the building blocks of $\mathbb{HA}_{\mathsf{k}[s,s^{-1}]}$. Rather than giving a tedious formal construction, we illustrate this with an example for $\mathsf{k} = \mathbb{R}$. A term for $3s^{-3} - \pi s^{-1} + s^2$ is:*



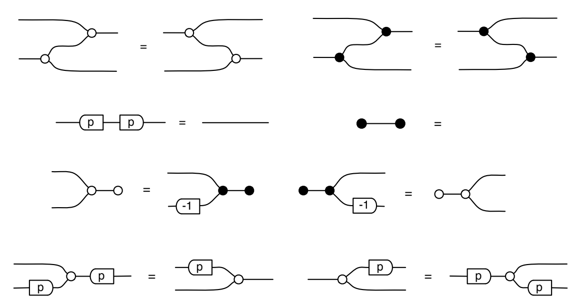
As an arrow $1 \to 1$ in Mat $\mathbb{R}[s, s^{-1}]$, the above term represents a 1×1-matrix over $\mathbb{R}[s, s^{-1}]$. To demonstrate how higher-dimensional matrices can be written, we also give a term for the 2×2-matrix $\begin{bmatrix} 2 & 3s \\ s^{-1} & s+1 \end{bmatrix}$:



The above examples are intended to be suggestive of a normal form for terms in $\mathbb{HA}_{\mathsf{k}[s,s^{-1}]}$; for further details see [23].

4.3 Presentation of Cospan Mat $\mathsf{k}[s, s^{-1}]$

To obtain the equational theory of Cospan Mat $\mathsf{k}[s, s^{-1}]$ we need the full set of generators (4), along with the equations of $\mathbb{HA}_{\mathsf{k}[s,s^{-1}]}$, their mirror images, and the following



where p ranges over the nonzero elements of $\mathsf{k}[s, s^{-1}]$ (see Example 4.3). Note that in the second equation on the right-hand side, we use the so-called 'empty diagram', or blank space, to represent the identity map on the monoidal unit, 0.

The equations of $\mathbb{HA}_{\mathsf{k}[s,s^{-1}]}$ ensure the generators of $\mathbb{HA}_{\mathsf{k}[s,s^{-1}]}$ behave as morphisms in Mat $\mathsf{k}[s, s^{-1}]$, while their mirror images ensure the remaining generators behave as morphisms in the opposite category Mat $\mathsf{k}[s, s^{-1}]^{\mathrm{op}}$. The additional equations above govern the interaction between these two sets of generators, axiomatising pushouts in Mat $\mathsf{k}[s, s^{-1}]$. Let $\mathbb{IH}^{\mathsf{Csp}}$ denote the resulting SMT. The procedure for obtaining the equations from a distributive law of props is explained in [23, §3.3].

Proposition 4.4 (Zanasi [23, Th. 3.41])**.**

$$\text{Cospan Mat } \mathsf{k}[s, s^{-1}] \cong \mathbb{IH}^{\mathsf{Csp}}.$$

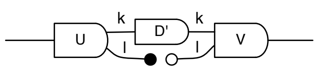
Using Prop. 4.4 and the existence of $\overline{\Theta}$, the equational theory of $\mathbb{IH}^{\mathsf{Csp}}$ is a sound proof system for reasoning about LTI. Due to the fact that $\overline{\Theta}$ is not faithful (see Remark 3.7), however, the system is not complete. Achieving completeness is our task for the remainder of this section.

4.4 Corelations

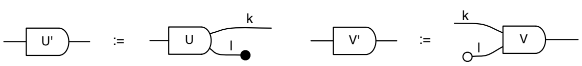
In the category of sets, relations can be identified with jointly-monic spans of functions; that is, those spans $X \xleftarrow{p} R \xrightarrow{q} Y$ where the induced map $R \xrightarrow{\langle p,q \rangle} X \times Y$ is injective. Corelations are a dual concept: we consider cospans $X \xrightarrow{i} S \xleftarrow{j} Y$ where the induced map $X + Y \xrightarrow{[i,j]} S$ is surjective. To make sense of this more generally, one needs a category with a factorisation system. In this subsection we identify a factorisation system in $\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$, and show that the induced prop $\mathsf{Corel}\,\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$ of corelations is isomorphic to LTI. We then give a presentation of $\mathsf{Corel}\,\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$ and arrive at a sound and fully complete equational theory for LTI.

Proposition 4.5. *Every morphism $R \in \mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$ can be factored as $R = BA$, where A is an epi and B is a split mono.*

Proof. Given any matrix R, the Smith normal form [13, Section 6.3] gives us $R = VDU$, where U and V are invertible, and D is diagonal. In graphical notation we can write it thus:



This implies we may write it as $R = U'; D'; V'$, where U' is a split epimorphism, D' diagonal of full rank, and V' a split monomorphism. Explicitly, the construction is given by



Recall that $\mathsf{k}[s,s^{-1}]$ is a PID, so the full rank diagonal matrix D' is epi. It can be checked that $R = V'(D'U')$ is an epi-split mono factorisation. □

A more careful examination of Proposition 4.5 yields:

Corollary 4.6. *Let $\mathcal{E}$ be the subcategory of epis and $\mathcal{M}$ the subcategory of split monos. The pair $(\mathcal{E}, \mathcal{M})$ is a factorisation system on $\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$, with $\mathcal{M}$ stable under pushout.*

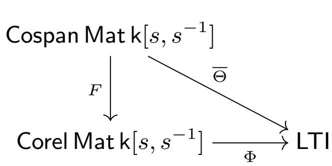
Given finite colimits and an $(\mathcal{E}, \mathcal{M})$-factorisation system with $\mathcal{M}$ stable under pushouts, we may define a category of corelations. The morphisms are isomorphism classes of cospans $X \xrightarrow{i} S \xleftarrow{j} Y$ where the copairing $[i,j]\colon X + Y \to S$ is in $\mathcal{E}$. Composition is given by pushout, as in the category of cospans, followed by factorising the copairing of the resulting cospan. This is a dualisation of the well-known construction of relations from spans [11]; further details can be found in [9].

Definition 4.7. *The prop $\mathsf{Corel}\,\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$ has as morphisms equivalence classes of jointly-epic cospans in $\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$.*

We have a full morphism

$$F\colon \mathsf{Cospan}\,\mathsf{Mat}\,\mathsf{k}[s,s^{-1}] \longrightarrow \mathsf{Corel}\,\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$$

mapping a cospan to its jointly-epic counterpart given by the factorisation system. Then $\overline{\Theta}$ factors through F as follows:



The morphism Φ along the base of this triangle is an isomorphism of props, and this is our main technical result, Theorem 4.9. The proof relies on the following beautiful result of systems theory.

Proposition 4.8 (Willems [22, p.565]). *Let M, N be matrices over $\mathsf{k}[s,s^{-1}]$. Then $\ker \theta M \subseteq \ker \theta N$ iff $\exists$ a matrix X s.t. $XM = N$.*

Further details and a brief history of the above proposition can be found in Schumacher [17, pp.7–9].

Theorem 4.9. *There is an isomorphism of props*

$$\Phi\colon \mathsf{Corel}\,\mathsf{Mat}\,\mathsf{k}[s,s^{-1}] \longrightarrow \mathsf{LTI}$$

taking a corelation $\xrightarrow{A}\xleftarrow{B}$ to $\overline{\Theta}(\xrightarrow{A}\xleftarrow{B}) = \ker \theta[A \;\, -B]$.

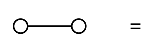
Proof. For functoriality, start from $\theta\colon \mathsf{Mat}\,\mathsf{k}[s,s^{-1}] \to \mathsf{Vect}_{\mathsf{k}}$. Now (i) Vect_k has an epi-mono factorisation system, (ii) θ maps epis to epis and (iii) split monos to monos, so θ preserves factorisations. Since it is a corollary of Prop. 3.6 that θ preserves colimits, it follows that θ extends to $\Psi\colon \mathsf{Corel}\,\mathsf{Mat}\,\mathsf{k}[s,s^{-1}] \to \mathsf{Corel}\,\mathsf{Vect}_{\mathsf{k}}$. But $\mathsf{Corel}\,\mathsf{Vect}_{\mathsf{k}}$ is isomorphic to $\mathsf{LinRel}_{\mathsf{k}}$ (see [9]). By Theorem 3.5, the image of Ψ is LTI, and taking the corestriction to gives us precisely Φ, which is therefore a full morphism of props.

As corelations $n \to m$ are in one-to-one correspondence with epis out of $n + m$, to prove faithfulness it suffices to prove that if two epis R and S with the same domain have the same kernel, then there exists an invertible matrix U such that $UR = S$. This is immediate from Proposition 4.8: if $\ker R = \ker S$, then we can find U, V such that $UR = S$ and $VS = R$. Since R is an epimorphism, and since $VUR = VS = R$, we have that $VU = 1$ and similarly $UV = 1$. This proves that any two corelations with the same image are isomorphic, and so Φ is full and faithful. □

4.5 Presentation of $\mathsf{Corel}\,\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$

Thanks to Theorem 4.9, the task of obtaining a presentation of LTI is that of obtaining one for $\mathsf{Corel}\,\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$. To do this, we start with the presentation $\mathbb{IH}^{\mathsf{Csp}}$ for $\mathsf{Cospan}\,\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$ of §4.3; the task of this section is to identify the additional equations that equate exactly those cospans that map via F to the same corelation.

In fact, only one new equation is required, the "white bone law":



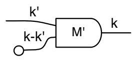
(5)

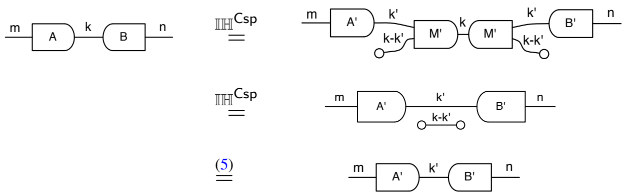
where we have carefully drawn the empty diagram to the right of the equality symbol. Expressed in terms of cospans, (5) asserts that $0 \to 1 \leftarrow 0$ and $0 \to 0 \leftarrow 0$ are identified: indeed, the two clearly yield the same corelation. The intuition here is that cospans $X \xrightarrow{i} S \xleftarrow{j} Y$ map to the same corelation if their respective copairings $[i,j]\colon X + Y \to S$ have the same jointly-epic parts. More colloquially, this allows us to 'discard' any part of the cospan that is not connected to the terminals. This is precisely what (5) represents. Further details on this viewpoint can be found in [7].

Let $\mathbb{IH}^{\mathsf{Cor}}$ be the SMT obtained from the equations of $\mathbb{IH}^{\mathsf{Csp}}$ together with equation (5).

Theorem 4.10. $\mathsf{Corel}\,\mathsf{Mat}\,\mathsf{k}[s,s^{-1}] \cong \mathbb{IH}^{\mathsf{Cor}}$.

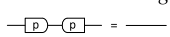
Proof. Since equation (5) holds in $\mathsf{Corel}\,\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$, we have a full morphism $\mathbb{IH}^{\mathsf{Cor}} \to \mathsf{Corel}\,\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$; it remains to show that it is faithful. It clearly suffices to show that in the equational theory $\mathbb{IH}^{\mathsf{Cor}}$ one can prove that every cospan is equal to its corelation. Suppose then that $m \xrightarrow{A} k \xleftarrow{B} n$ is a cospan and $m \xrightarrow{A'} k' \xleftarrow{B'} n$ its corelation. Then, by definition, there exists a split mono $M\colon k' \to k$ such that $MA' = A$ and $MB' = B$. Moreover, by the construction of the epi-split mono factorisation in

Mat $\mathsf{k}[s, s^{-1}]$, M is of the form  where $M' \colon k \to k$ is invertible. We can now give the derivation in $\mathbb{IH}^{\mathsf{Cor}}$:

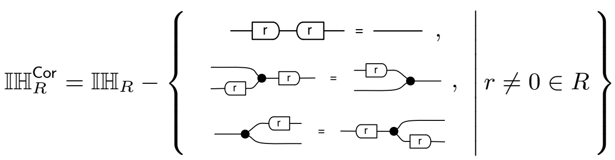


□

We therefore have a sound and fully complete equational theory for LTI systems, and also a normal form for each LTI system: every such system can be written, in an essentially unique way, as a jointly-epic cospan of terms in $\mathbb{HA}_{\mathsf{k}[s,s^{-1}]}$ in normal form.

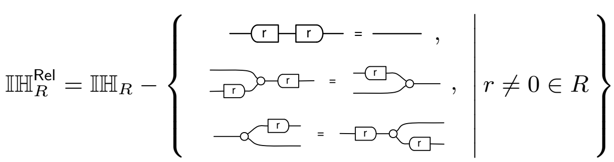
Remark 4.11. *$\mathbb{IH}^{\mathsf{Cor}}$ can also be described as having the equations of $\mathbb{IH}_{\mathsf{k}[s,s^{-1}]}$ [5, 23], but* without  *, and related*

Our results generalise; given any PID R we have (informally speaking):



$\cong$ Corel Mat R

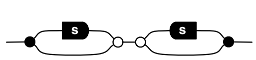
and, because of the transpose duality of matrices:



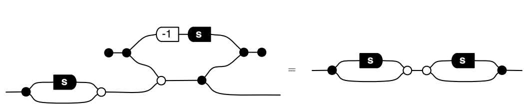
$\cong$ Rel Mat R.

Remark 4.12. *The omitted equations each associate a cospan with a span that, in terms of behaviour, has the effect of passing to a sub-behaviour*[2]*. Often this is a strict sub-behaviour, hence the failure of soundness of $\mathbb{IH}$ identified in the introduction.*

For example, consider the system $\mathcal{B}$ represented by the cospan



We met this system in the introduction; indeed the following derivation can be performed in $\mathbb{IH}^{\mathsf{Cor}}$:



The trajectories are $w = (w_1, w_2) \in \mathsf{k}^{\mathbb{Z}} \oplus \mathsf{k}^{\mathbb{Z}}$ where $(s+1) \cdot w_1 = (s+1) \cdot w_2$; that is, they satisfy the difference equation

$$w_1(t-1) + w_1(t) - w_2(t-1) - w_2(t) = 0.$$

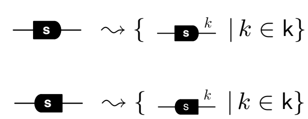
As we saw in the introduction, however, an equation of $\mathbb{IH}_{\mathsf{k}[s]}$ omitted from $\mathbb{IH}^{\mathsf{Cor}}_{\mathsf{k}[s,s^{-1}]}$ equates this to the identity system —— *, with the identity behaviour those sequences of the form $w = (w_1, w_1) \in \mathsf{k}^{\mathbb{Z}} \oplus \mathsf{k}^{\mathbb{Z}}$. The identity behaviour is strictly smaller than $\mathcal{B}$; e.g., $\mathcal{B}$ contains (w_1, w_2) with $w_1(t) = (-1)^t$ and $w_2(t) = 0$.*

The similarity between our equational presentations of $\mathbb{IH}^{\mathsf{Cor}}_{\mathsf{k}[s,s^{-1}]}$ and that of $\mathbb{IH}_{\mathsf{k}[s]}$ given in [4, 6] is remarkable, considering the differences between the intended semantics of signal flow graphs that string diagrams in those theories represent, as well as the underlying mathematics of streams, which for us are elements of the k vector space $\mathsf{k}^{\mathbb{Z}}$ and in [4, 6] are Laurent series. We contend that this is evidence of the *robustness* of the algebraic approach: the equational account of how various components of signal flow graphs interact is, in a sense, a higher-level specification than the technical details of the underlying mathematical formalisations.

5. Operational semantics

In this section we relate the denotational account given in previous sections with an operational view.

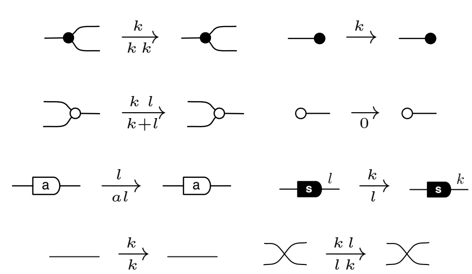
Operational semantics is given to Σ^*-terms—that is, to arrows of the prop $\mathbb{S}^* = \mathbf{S}_{(\Sigma^*,\varnothing)}$—where Σ^* is obtained from set of generators in (4) by replacing the formal variables s and s^{-1} with a family of registers, indexed by the scalars of k:



The idea is that at any time during a computation the register holds the signal it has received on the previous 'clock-tick'. There are no equations, apart from the laws of symmetric monoidal categories.

Next we introduce the structural rules: the transition relations that occur at each clock-tick, turning one Σ^*-term into another. Each transition is given two labels, written above and below the arrow. The upper refers to the signals observed at the 'dangling wires' on the left-hand side of the term, and the lower to those observed on the right hand side. Indeed, transitions out of a term of type $m \to n$ have upper labels in k^m and lower ones in k^n.

Because—for the purposes of the operational account—we consider these terms to be syntactic, we must also account for the twist  and identity —— . A summary of the structural rules is given below; the rules for the mirror image generators are symmetric, in the sense that upper and lower labels are swapped.



$$\frac{s \xrightarrow[\mathbf{v}]{\mathbf{u}} s' \quad t \xrightarrow[\mathbf{w}]{\mathbf{v}} t'}{s \,;\, t \xrightarrow[\mathbf{w}]{\mathbf{u}} s' \,;\, t'} \qquad \frac{s \xrightarrow[\mathbf{v_1}]{\mathbf{u_1}} s' \quad t \xrightarrow[\mathbf{v_2}]{\mathbf{u_2}} t'}{s \oplus t \xrightarrow[\mathbf{v_1}\,\mathbf{v_2}]{\mathbf{u_1}\,\mathbf{u_2}} s' \oplus t'}$$

Here $k, l, a \in \mathsf{k}$, s, t are Σ^*-terms, and $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{u_1}, \mathbf{u_2}, \mathbf{v_1}, \mathbf{v_2}$ are vectors in k of the appropriate length. Note that the only generators that change as a result of computation are the registers; it follows this is the only source of state in any computation.

[2] In fact the 'largest controllable sub-behaviour' of the system. We explore controllability in Section 6.

The structural rules are identical to those given in [6, §2]. Differently from [6], however, we can be more liberal with our assumptions about the *initial states* of computations. In [6] each computation starts off with all registers initialised with the $0 (\in \mathsf{k})$ value. For us, systems can be initialised with arbitrary elements of k.

Let $\tau \colon m \to n \in \mathbb{S}$ be a Σ-term. Fixing an ordering of the delays  in τ allows us to identify the set of delays with a finite ordinal $[d]$. A **register assignment** is then simply a function $\sigma : [d] \to \mathsf{k}$. We may instantiate the Σ-term τ with the register assignment σ to obtain the Σ^*-term $\tau_\sigma \in \mathbb{S}^*$ of the same type: for all $i \in [d]$ simply replace the ith delay with a register in state $\sigma(i)$.

A computation on τ initialised at σ is an infinite sequence of register assignments and transitions:

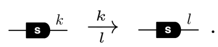
$$\tau_\sigma \xrightarrow[\mathbf{v_1}]{\mathbf{u_1}} \tau_{\sigma_1} \xrightarrow[\mathbf{v_2}]{\mathbf{u_2}} \tau_{\sigma_2} \xrightarrow[\mathbf{v_3}]{\mathbf{u_3}} \ldots$$

The **trace** induced by this computation is the sequence

$$(\mathbf{u_1}, \mathbf{v_1}), (\mathbf{u_2}, \mathbf{v_2}), \ldots$$

of elements of $\mathsf{k}^m \oplus \mathsf{k}^n$.

To relate the operational and denotational semantics, we introduce the notion of biinfinite trace: a trace with an infinite past as well as future. To define these, we use the notion of a *reverse* computation: a computation using the operational rules above, but with the rules for delay having their left and right hand sides swapped:



Definition 5.1. *Given $\tau \colon m \to n \in \mathbb{S}$, a* **biinfinite trace on** *τ is a sequence $w \in (\mathsf{k}^m)^{\mathbb{Z}} \oplus (\mathsf{k}^n)^{\mathsf{k}}$ such that there exists*

(i) a register assignment σ;
(ii) an infinite forward trace *ϕ_σ of a computation on τ initialised at σ; and,*
(iii) an infinite backward trace *ψ_σ of a reverse computation on τ initialised at σ,*

obeying

$$w(t) = \begin{cases} \phi(t) & t \geq 0; \\ \psi(-(t+1)) & t < 0. \end{cases}$$

We write $\mathrm{bit}(\tau)$ *for the set of all biinfinite traces on τ.*

The following result gives a tight correspondence between the operational and denotational semantics, and follows via a straightforward structural induction on τ.

Lemma 5.2. *For any $\tau \colon m \to n \in \mathbb{S}$, we have*

$$[\![\tau]\!] = \mathrm{bit}(\tau)$$

as subsets of $(\mathsf{k}^m)^{\mathbb{Z}} \times (\mathsf{k}^n)^{\mathbb{Z}}$.

6. Controllability

Suppose we are given a current and a target trajectory for a system. Is it always possible, in finite time, to steer the system onto the target trajectory? If so, the system is deemed controllable, and the problem of controllability of systems is at the core of control theory. The following definition is due to Willems [21].

Definition 6.1. *A system $(T, W, \mathscr{B})$ (or simply the behaviour $\mathscr{B}$) is* **controllable** *if for all $w, w' \in \mathscr{B}$ and all times $t_0 \in \mathbb{Z}$, there exists $w'' \in \mathscr{B}$ and $t_0' \in \mathbb{Z}$ such that $t_0' > t_0$ and w'' obeys*

$$w''(t) = \begin{cases} w(t) & t \leq t_0 \\ w'(t - t_0') & t \geq t_0'. \end{cases}$$

As mentioned previously, a novel feature of our graphical calculus is that it allows us to consider non-controllable behaviours.

Example 6.2. *Consider the system in the introduction, further elaborated in Remark 4.12. As noted previously, the trajectories of this system are precisely those sequences $w = (w_1, w_2) \in \mathsf{k}^{\mathbb{Z}} \oplus \mathsf{k}^{\mathbb{Z}}$ that satisfy the difference equation*

$$w_1(t-1) + w_1(t) - w_2(t-1) - w_2(t) = 0.$$

To see that the system is non-controllable, note that

$$w_1(t-1) - w_2(t-1) = -(w_1(t) - w_2(t)),$$

so $(w_1 - w_2)(t) = (-1)^t c_w$ for some $c_w \in \mathsf{k}$. This c_w is a time-invariant property of any trajectory. Thus if w and w' are trajectories such that $c_w \neq c_{w'}$, then it is not possible to transition from the past of w to the future of w' along some trajectory in $\mathcal{B}$.

Explicitly, taking $w(t) = ((-1)^t, 0)$ and $w'(t) = ((-1)^t 2, 0)$ suffices to show $\mathcal{B}$ is not controllable.

6.1 A categorical characterisation

We now show that controllable systems are precisely those representable as *spans* of matrices. This novel characterisation leads to new ways of reasoning about controllability of composite systems.

Among the various equivalent conditions for controllability, the existence of *image representations* is most useful for our purposes.

Proposition 6.3 (Willems [20, p.86])**.** *An LTI behaviour $\mathscr{B}$ is controllable iff $\exists\, M \in \mathsf{Mat}\,\mathsf{k}[s, s^{-1}]$ such that $\mathscr{B} = \mathrm{im}\,\theta M$.*

Restated in our language, Prop. 6.3 states that controllable systems are precisely those representable as *spans* of matrices.

Theorem 6.4. *Let $m \xrightarrow{A} d \xleftarrow{B} n$ be a corelation in $\mathsf{Corel\,Mat}\,\mathsf{k}[s, s^{-1}]$. Then $\Phi(\xrightarrow{A}\xleftarrow{B})$ is controllable iff $\exists\, R : e \to m$, $S : e \to n$ s.t.*

$$m \xleftarrow{R} e \xrightarrow{S} n = m \xrightarrow{A} d \xleftarrow{B} n$$

as morphisms in $\mathsf{Corel\,Mat}\,\mathsf{k}[s, s^{-1}]$.

Proof. To begin, note that the behaviour of a span is its joint image. That is, $\Phi(\xleftarrow{R}\xrightarrow{S})$ is the composite of linear relations $\ker\theta[\mathrm{id}_m \;\; -R]$ and $\ker\theta[S \;\; -\mathrm{id}_n]$, which comprises all $(\mathbf{x}, \mathbf{y}) \in (\mathsf{k}^m)^{\mathbb{Z}} \oplus (\mathsf{k}^n)^{\mathbb{Z}}$ s.t. $\exists\, \mathbf{z} \in (\mathsf{k}^e)^{\mathbb{Z}}$ with $\mathbf{x} = \theta R\mathbf{z}$ and $\mathbf{y} = \theta S\mathbf{z}$. Thus

$$\Phi(\xleftarrow{R}\xrightarrow{S}) = \mathrm{im}\,\theta \begin{bmatrix} R \\ S \end{bmatrix}.$$

The result then follows immediately from Prop. 6.3. □

In terms of the graphical theory, this means that a term in the form $\mathbb{HA}_{\mathsf{k}[s,s^{-1}]} ; \mathbb{HA}^{op}_{\mathsf{k}[s,s^{-1}]}$ ('cospan form') is controllable iff we can find a derivation, using the rules of $\mathbb{IH}^{\mathsf{Cor}}$, that puts it in the form $\mathbb{HA}^{op}_{\mathsf{k}[s,s^{-1}]} ; \mathbb{HA}_{\mathsf{k}[s,s^{-1}]}$ ('span form'). This provides a general, easily recognisable representation for controllable systems.

Span representations also lead to a test for controllability: take the pullback of the cospan and check whether the system described by it coincides with the original one. Indeed, note that as $\mathsf{k}[s, s^{-1}]$ is a PID, the category $\mathsf{Mat}\,\mathsf{k}[s, s^{-1}]$ has pullbacks. A further consequence of Th. 6.4, together with Prop. 4.8, is the following.

Proposition 6.5. *Let $m \xrightarrow{A} d \xleftarrow{B} n$ be a cospan in $\mathsf{Mat}\,\mathsf{k}[s, s^{-1}]$, and write the pullback of this cospan $m \xleftarrow{R} e \xrightarrow{S} n$. Then the behaviour of the pullback span $\Phi(\xleftarrow{R}\xrightarrow{S})$ is the maximal controllable sub-behaviour of $\Phi(\xrightarrow{A}\xleftarrow{B})$.*

Proof. Suppose we have another controllable behaviour $\mathscr{C}$ contained in $\ker\theta[A \;\; -B]$. Then this behaviour is the Φ-image of some span $m \xleftarrow{R'} e' \xrightarrow{S'} n$. As $\mathrm{im}\,\theta \begin{bmatrix} R' \\ S' \end{bmatrix}$ lies in $\ker\theta[A \;\; -B]$, the universal property of the pullback gives a map $e' \to e$ such that the relevant diagram commutes. This implies that the controllable behaviour $\mathscr{C} = \operatorname{im}\theta \begin{bmatrix} R' \\ S' \end{bmatrix}$ is contained in $\operatorname{im}\theta \begin{bmatrix} R \\ S \end{bmatrix}$, as required. □

Corollary 6.6. *Suppose that an LTI behaviour $\mathscr{B}$ has cospan representation*

$$m \xrightarrow{A} d \xleftarrow{B} n.$$

Then $\mathscr{B}$ is controllable iff the Φ-image of the pullback of this cospan in $\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$ is equal to $\mathscr{B}$.

Moreover, taking the pushout of this pullback span gives another cospan. The morphism from the pushout to the original cospan, given by the universal property of the pushout, describes the way in which the system fails to be controllable.

Graphically, the pullback may be computed by using the axioms of the theory of interacting Hopf algebras $\mathbb{IH}_{\mathsf{k}[s,s^{-1}]}$ [5, 23]. For example, the pullback span of the system of Ex. 6.2 is simply the identity span, as derived in equation (2) of the introduction. In the traditional matrix calculus for control theory, one derives this by noting the system has kernel representation $\ker\theta \begin{bmatrix} s+1 & -(s+1) \end{bmatrix}$, and eliminating the common factor $s+1$ between the entries. Either way, we conclude that the maximally controllable subsystem of $1 \xrightarrow{[s+1]} 1 \xleftarrow{[s+1]} 1$ is simply the identity system $1 \xrightarrow{[1]} 1 \xleftarrow{[1]} 1$.

6.2 Control and interconnection

From this vantage point we can make useful observations about controllable systems and their composites: we simply need to ask whether we can rewrite them as spans.

Example 6.7. *Suppose that $\mathscr{B}$ has cospan representation $m \xrightarrow{A} d \xleftarrow{B} n$. Then $\mathscr{B}$ is easily seen to be controllable when A or B is invertible. Indeed, if A is invertible, then $m \xleftarrow{A^{-1}B} n \xrightarrow{\mathrm{id}_n} n$ is an equivalent span; if B is invertible, then $m \xleftarrow{\mathrm{id}_m} m \xrightarrow{B^{-1}A} n$.*

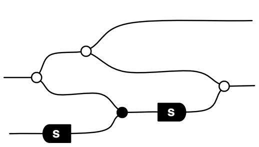
More significantly, the compositionality of our framework aids understanding of how controllability behaves under the interconnection of systems—an active field of investigation in current control theory. We give an example application of our result.

First, consider the following proposition.

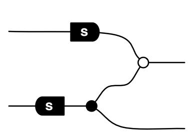
Proposition 6.8. *Let $\mathscr{B}, \mathscr{C}$ be controllable systems, given by the respective Φ-images of the spans $m \xleftarrow{B_1} d \xrightarrow{B_2} n$ and $n \xleftarrow{C_1} e \xrightarrow{C_2} l$. Then the composite $\mathscr{C} \circ \mathscr{B} : m \to l$ is controllable if $\Phi(\xrightarrow{B_2}\xleftarrow{C_1})$ is controllable.*

Proof. Replacing $\xrightarrow{B_2}\xleftarrow{C_1}$ with an equivalent span gives a span representation for $\mathscr{C} \circ \mathscr{B}$. □

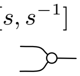
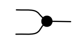
Example 6.9. *Consider LTI systems*



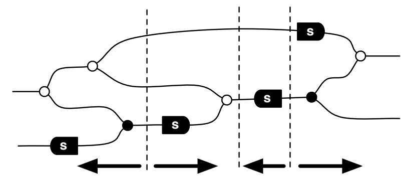
and



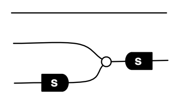
.

These systems are controllable because each is represented by a span in $\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$. Indeed, recall that each generator of $\mathsf{LTI} = \mathsf{Corel}\,\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$ arises as the image of a generator in $\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$ or $\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]^{\mathrm{op}}$; for example, the white monoid map  *represents a morphism in $\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$, while the black monoid map*  *represents a morphism in $\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]^{\mathrm{op}}$. The above diagrams are spans as we may partition the diagrams above so that each generator in $\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]^{\mathrm{op}}$ lies to the left of each generator in $\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$.*

To determine controllability of the interconnected system

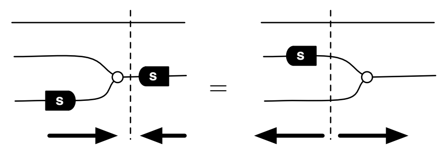


Prop. 6.8 states that it is enough to consider the controllability of the subsystem



.

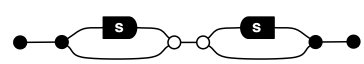
The above diagram gives a representation of the subsystem as a cospan in $\mathsf{Mat}\,\mathsf{k}[s,s^{-1}]$. We can prove it is controllable by rewriting it as a span using an equation of LTI*:*



.

Thus the composite system is controllable.

Remark 6.10. *Note the converse of Proposition 6.8 fails. For a simple example, consider the system*



This is equivalent to the empty system, and so trivially controllable. The central span, however, is not controllable (Example 6.2).

6.3 Comparison to matrix methods

The facility with which the graphical calculus formalises and solves such controllability issues is especially appealing in view of potential applications in the analysis of controllability of *systems over networks* (see [15]). To make the reader fully appreciate such potential, we sketch how complicated such analysis is using standard algebraic methods and dynamical system theory even for the highly restrictive case of two systems that compose to make a single-input, single-output system. See also pp. 513–516 of [8], where a generalization of the result of Prop. 6.8 is given in a polynomial- and operator-theoretic setting.

In the following we abuse notation by writing a matrix for its image under the functor θ. The following is a useful result for analysing the controllability of kernel representations applying only to the single-input single-output case.

Proposition 6.11 (Willems [20, p.75])**.** *Let $\mathscr{B} \subseteq (\mathsf{k}^2)^{\mathbb{Z}}$ be a behaviour given by the kernel of the matrix $\theta \begin{bmatrix} A & B \end{bmatrix}$, where A and B are column vectors with entries in $\mathsf{k}[s,s^{-1}]$. Then $\mathscr{B}$ is controllable if and only if the greatest common divisor $\gcd(A,B)$ of A and B is 1.*

Using the notation of Prop. 6.8, the trajectories of $\mathscr{B}$ and $\mathscr{C}$ respectively are those $(w_1, w_2) \in (\mathsf{k}^n)^{\mathbb{Z}} \oplus (\mathsf{k}^m)^{\mathbb{Z}}$ and $(w_2', w_3) \in$

$(\mathsf{k}^m)^{\mathbb{Z}} \oplus (\mathsf{k}^p)^{\mathbb{Z}}$ such that

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \ell_1 \quad \text{and} \quad \begin{bmatrix} w_2' \\ w_3 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \ell_2 \tag{6}$$

for some $\ell_1 \in (\mathsf{k}^b)^{\mathbb{Z}}$, $\ell_2 \in (\mathsf{k}^c)^{\mathbb{Z}}$. These are the explicit image representations of the two systems. We assume without loss of generality that the representations (6) are *observable* (see [20]); this is equivalent to $\gcd(B_1, B_2) = \gcd(C_1, C_2) = 1$. Augmenting (6) with the interconnection constraint $w_2 = B_2\ell_1 = C_1\ell_2 = w_2'$ we obtain the representation of the interconnection:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_2' \\ w_3 \\ 0 \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \\ 0 & C_1 \\ 0 & C_2 \\ B_2 & -C_1 \end{bmatrix} \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} . \tag{7}$$

Prop. 6.8 concerns the controllability of the set $\mathscr{C} \circ \mathscr{B}$ of trajectories (w_1, w_3) for which there exist trajectories w_2, w_2', ℓ_1, ℓ_2 such that (7) holds.

To obtain a representation of such behavior the variables ℓ_1, ℓ_2, w_2 and w_2' must be eliminated from (7) via algebraic manipulations (see the discussion on p. 237 of [21]). Denote $G = \gcd(B_1, C_2)$, and write $C_2 = GC_2'$ and $B_1 = GB_1'$, where $\gcd(B_1', C_2') = 1$. Without entering in the algebraic details, it can be shown that a kernel representation of the projection of the behavior of (7) on the variables w_1 and w_3 is

$$\begin{bmatrix} C_2'B_2 & -B_1'C_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_3 \end{bmatrix} = 0 . \tag{8}$$

We now restrict to the single-input single-output case. Recalling Prop. 6.11, the behavior represented by (8) is controllable if and only if $\gcd(C_2'B_2, B_1'C_1) = 1$.

Finally then, to complete our alternate proof of the single-input single-output case of Prop. 6.8, note that $\overline{\Theta}(\xrightarrow{B_2}\xleftarrow{C_1})$ is controllable if $\gcd(B_2, C_1) = 1$. Given the observability assumption, this implies $\gcd(C_2'B_2, B_1'C_1) = 1$, and so the interconnected behaviour $\mathscr{C} \circ \mathscr{B}$ represented by (8) is controllable.

In the multi-input, multi-output case stating explicit conditions on the controllability of the interconnection given properties of the representations of the individual systems and their interconnection is rather complicated. This makes the simplicity of Prop. 6.8 and the straightforward nature of its proof all the more appealing.

7. Conclusion

Willems concludes [20] with

> Thinking of a dynamical system as a behavior, and of interconnection as variable sharing, gets the physics right.

In this paper we have shown that the algebra of symmetric monoidal categories gets the mathematics right.

We characterised the prop LTI of linear time invariant dynamical systems as the prop of corelations of matrices over $\mathsf{k}[s, s^{-1}]$ and used this fact to present it as a symmetric monoidal theory. As a result, we obtained the language of string diagrams as a syntax for LTI systems. From the point of view of formal semantics, the syntax was endowed with denotational and operational interpretations that are closely related, as well as a sound and complete system of equations for diagrammatic reasoning.

We harnessed the compositional nature of the language to provide a new characterisation of the fundamental notion of controllability, and argued that this approach is well-suited to some of the problems that are currently of interest in systems theory, for example in systems over networks, where compositionality seems to be a vital missing ingredient.

The power of compositionality will be of no surprise to researchers in concurrency theory or formal semantics of programming languages. Our theory–which, as we show in the final section, departs radically from traditional techniques—brings this insights to control theory. By establishing links between our communities, our ambition is to open up control theory to formal specification and verification.

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