Coinductive Reasoning for Contextual Graph-Rewriting

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Plan of the talk

1. Deriving bisimulation congruences
   - motivation
   - groupoidal relative pushouts (GRPOs)
   - labelled transition systems (LTSs) and congruence theorems

2. Cospans as generalised contexts

3. Bisimulation for graph rewriting
Deriving Congruences

Many syntactic formalisms for concurrency and mobility

Unification efforts:

1. Milner et al ‘90s-now: action calculi, bigraphs

2. Sewell, Leifer, Milner, Sassone and Sobocinski: meta theory of process calculi

where do the labels of a process calculus’ operational semantics come from?
Labels in a LTS

*Slogan:* Labels should be smallest contexts which allow reaction/interaction

- eg. simple CCS-style calculus $a \xrightarrow{-|a|} 0$

- *Sewell (1998):* Detailed syntactic analysis of simplified process calculi

- *Leifer and Milner (2000):* General notion of smallest context - the relative pushout

- *Sassone and Sobocinski (2002):* 2-categorical generalisation of theory to allow handling of structural congruence
Reactive Systems

Idea: category in which arrows are “contexts” and objects are “holes”

Definition: G-category is a 2-category in which all 2-cells are iso

Idea: the 2-cells in a reactive system are “proofs of structural congruence”
Reactive Systems

A reactive system consists of

1. a G-category $\mathcal{C}$
2. a collection $\mathcal{D}$ of reactive contexts
3. a collection of reaction rules

$$\mathcal{R} \subseteq \bigcup_{\mathcal{C} \in \mathcal{C}} \mathcal{C}(0,C) \times \mathcal{C}(0,C)$$

if there exists $\langle l, r \rangle, d \in \mathcal{D}$ and

$$\rho: dl \Rightarrow a, \rho': a' \Rightarrow dr$$
Given $\alpha : ca \Rightarrow db$

$\langle I_5, e, f, g, \beta, \gamma, \delta \rangle$

$\delta b \cdot g \beta \cdot \gamma a = \alpha$
Minimality

$\varphi : e' \Rightarrow he$

$\psi : hf \Rightarrow f'$

$\tau : g'h \Rightarrow g$

$\tau e \cdot g' \varphi \cdot \gamma' = \gamma$

$\delta' \cdot g' \psi \cdot \tau^{-1}f = \delta$

$\psi b \cdot h\beta \cdot \varphi a = \beta'$
Essential Uniqueness

$\exists! \, \xi : h \rightarrow h'$

$\xi e \bullet \varphi = \varphi'$

$\psi \bullet \xi^{-1} f = \psi'$

$\tau' \bullet g' \xi = \tau$
LTS

- **Nodes:** \([a] : 0 \rightarrow I_1\)
- **Labels:** \([a] \xrightarrow{[f]} [a']\)

\[\exists \langle l, r \rangle \in \mathcal{R} \quad \exists f \in \mathcal{C} \quad \exists d \in \mathcal{D} \quad \exists \alpha : f a \Rightarrow dl \quad \exists \alpha' : dr \Rightarrow a'\]

\[\begin{array}{c}
\text{and} \\
I_2 \\
\alpha \\
I_3 \\
\text{is a GIPO}
\end{array}\]

\[\begin{array}{c}
I_4 \\
f \\
d
\end{array}\]
Properties of LTS

- Bisimulation is a congruence
- Trace equivalence is a congruence
- Failures equivalence is a congruence
Plan of the talk

1. Deriving bisimulation congruences
2. Cospans as generalised contexts
   - cospan bicategories
   - adhesive categories
   - existence of GRPOs
3. Bisimulation for graph rewriting
Cospan Bicategories

Given $\mathbf{C}$, $\text{Cospan}(\mathbf{C})$ has

- Objects: that of $\mathbf{C}$
- Arrows: cospans
- 2-cells:

\[
\begin{align*}
I_1 & \xrightarrow{f} C & \xleftarrow{g} I_2 \\
I_1 & \xrightarrow{f} C \\
& \xleftarrow{h} C' & \xrightarrow{g'} I_2
\end{align*}
\]
Composition & Assoc.

- Identities: \[ I_1 \xrightarrow{id} I_1 \xleftarrow{id} I_1 \]

- Composition by pushout

\[ C + I_2 D \]

\[ I_1 \xrightarrow{f} C \xleftarrow{g} I_2 \xrightarrow{f'} D \xleftarrow{g'} I_3 \]

- \[ a : (C + I_2 D) + I_3 E \rightarrow C + I_2 (D + I_3 E) \]
- \[ e_l : (I_1 + I_1 C) \rightarrow C \]
- \[ e_r : (C + I_2 I_2) \rightarrow C \]

satisfying coherence
Van Kampen Square

pushout stable under pullback
Van Kampen Square

- If back faces are pullbacks
- top face pushout implies front faces pullbacks
Van Kampen Square

- Given a cube with back faces pullbacks
- top face pushout iff front faces pullbacks
A category $\mathcal{C}$ is adhesive when

1. It has pushouts along monos
2. It has pullbacks
3. pushouts along monos are VK squares
Examples

- Set, Graph
- C adhesive, so is C/C and C/C
- C adhesive, so is [D,C]
- C and D adhesive, so is CxD
Examples

- Any topos
- $\text{Par} = */\text{Set}$ is adhesive, not extensive, not topos
Left-Linear Cospans

When $\mathbf{C}$ is adhesive $\text{LLC}(\mathbf{C})$ is the bicategory

- objects as in $\mathbf{C}$
- arrows cospans $\begin{array}{c} I_1 \xrightarrow{m} \text{C} \xleftarrow{g} I_2 \end{array}$
GRPOs for cospans

**Theorem:** Suppose that $\mathcal{C}$ is an adhesive category, $\text{LLC}(\mathcal{C})$ has redex-GRPOs.
GRPOs in LLC(C)

Given redex square...
GRPOs of Cospans

... find minimal factorisation
Construction

\[ Y = A \cup_X B \]
Example 1

All morphisms mono
Example 2

\[ o_A \text{ not mono} \]
Example 3

\( o_A \) and \( o_L \) not mono
Plan of the talk

1. Deriving bisimulation congruences
2. Cospans as generalised contexts
3. Bisimulation for graph rewriting
   - double pushout graph rewriting
   - graph rewriting as reactive system
   - using GRPOs to construct LTSs
Dp-Rewriting

Ehrig (1973)

A framework for graph-rewriting

Rewrite rules $p$ are spans

$L \leftarrow^l K \rightarrow^r R$
Rewriting

1. Find $f$
2. Find $E$, $v$ and $g$ making left pushout
3. Construct right pushout
Graph Rewriting as Reactive System

For every span $L \xleftarrow{l} K \xrightarrow{r} R$
let $\langle 0 \to L \xleftarrow{l} K, 0 \to R \xrightarrow{r} K \rangle \in \mathcal{R}$

Lemma:
- $\rightarrow$ double-pushout rewrite
- $\rightarrow\Rightarrow$ reaction relation in reactive system

$C \rightarrow D$ iff $C_0^0 \rightarrow D_0^0$
LTS for graph rewriting

The resulting LTS has:

- **Nodes**: graphs (up-to-iso) with output interface (possibly non-mono)
- **Labels**: smallest graph contexts (up-to-iso) which allow reaction

*Theorem*: Bisimulation, trace equivalence, failures equivalence are congruences
Related Work

*Ehrig and Konig (2004)*

- LTS for graph rewriting
- up-to-iso not taken into account
- all interfaces mono
- when restricting our GRPOs to linear cospans we derive *the same* LTS, provided one quotients their labels by iso
- their congruence theorem follows from congruence theorem for GRPOs
Advantages of LTS

- Transfer of concepts from process algebra to graph rewriting
- Process calculus on graphs?
  - the class of adhesive categories seems to cover many “graph-like” categories
Bigraphs


Structures with two types of connectivity

- Physical: one node is physically within the domain of another
- Logical: one node has a connection to another node

GRPOs exist
One may view bigraphs as (almost) cospans over the category of place-link graphs

Construction of GRPOs follows
Conclusion

- Construction of labels for an interesting class of reactive systems
- Two applications so far, more in the future?