Congruences for contextual graph rewriting

Pawel Sobocinski
IT University, Copenhagen
FGUC 4/9/04

joint work with Vladimiro Sassone
1. Deriving bisimulation congruences
   - groupoidal relative pushouts (GRPOs)
   - labelled transition systems (LTSs) and congruence theorems
2. Cospans as generalised contexts
3. Bisimulation for graph rewriting
Labels in a lts

*Slogan*: Labels should be smallest contexts which allow reaction/interaction

eg. simple CCS-style calculus \( a \xrightarrow{\overline{a}} 0 \)

*Sewell (1998)*: Detailed syntactic analysis of simplified process calculi

*Leifer and Milner (2000)*: General notion of smallest context - the relative pushout

*Sassone and Sobocinski (2002)*: 2-categorical generalisation of theory to allow handling of structural congruence
Reactive systems

Idea: category in which arrows are contexts and objects are types

Definition: G-category is a 2-category in which all 2-cells are iso

Idea: the 2-cells in a reactive system are “proofs of structural congruence”
A reactive system consists of

1. a G-category $\mathcal{C}$
2. a collection $\mathbf{D}$ of reactive contexts
3. a collection of reaction rules

$$\mathcal{R} \subseteq \bigcup_{C \in \mathcal{C}} \mathcal{C}(0,C) \times \mathcal{C}(0,C)$$
A reactive system consists of

1. a G-category $\mathbf{C}$
2. a collection $\mathbf{D}$ of reactive contexts
3. a collection of reaction rules

$$\mathcal{R} \subseteq \bigcup_{C \in \mathbf{C}} \mathbf{C}(0,C) \times \mathbf{C}(0,C)$$

$$\rho : dl \Rightarrow a, \rho' : a' \Rightarrow dr$$

if there exists

$$\langle l, r \rangle, d \in \mathbb{D}$$

and

$$a \rightarrow a'$$
Given $\alpha: ca \Rightarrow db$
Given $\alpha : ca \Rightarrow db$

$\langle I_5, e, f, g, \beta, \gamma, \delta \rangle$

$\delta b \cdot g \beta \cdot \gamma a = \alpha$
Nodes: \([a]: 0 \rightarrow I_1\)

Labels: \([a] \xrightarrow{[f]} [a']\)
Nodes: \([a]: 0 \rightarrow I_1\)

Labels: \([a] \xrightarrow{[f]} [a']\)

\[\exists \langle l, r \rangle \in R \quad \exists f \in \mathbb{C} \quad \exists d \in \mathbb{D} \quad \exists \alpha : fa \Rightarrow dl \quad \exists \alpha' : dr \Rightarrow a'\]
Nodes: \([a] : 0 \rightarrow I_1\)

Labels: \([a] \xrightarrow{[f]} [a']\)

\[\exists \langle l, r \rangle \in \mathcal{R} \quad \exists f \in \mathcal{C} \quad \exists d \in \mathcal{D} \quad \exists \alpha : fa \Rightarrow dl \quad \exists \alpha' : dr \Rightarrow a'\]

and \(I_2\)

\(I_4\)

\(I_3\) is a GIPO
Properties of Lts

- Bisimilarity is a congruence
- Trace equivalence is a congruence
- Failures equivalence is a congruence
Plan of the talk

1. Deriving bisimulation congruences
2. Cospans as generalised contexts
   - cospan bicategories
   - adhesive categories
   - existence of GRPOs
3. Bisimulation for graph rewriting
Cospan bicategories

Given \( \mathbf{C} \), \( \text{Cospan}(\mathbf{C}) \) has
- Objects: that of \( \mathbf{C} \)
- Arrows: cospans
- 2-cells:

\[
\begin{array}{ccc}
I_1 & \xrightarrow{f} & C & \xleftarrow{g} & I_2 \\
\downarrow & & \downarrow & & \downarrow \\
C' & \xleftarrow{g} & I_2
\end{array}
\]

\[
\begin{array}{ccc}
I_1 & \xrightarrow{f} & C & \xrightarrow{g} & I_2 \\
\downarrow & & \downarrow & & \downarrow \\
I_1 & \xrightarrow{f'} & C' & \xleftarrow{g'} & I_2
\end{array}
\]
Composition & assoc.

Identities:  \( I_1 \xrightarrow{id} I_1 \xleftarrow{id} I_1 \)

Composition by pushout

\[
C +_{I_2} D
\]

\[
\begin{align*}
I_1 & \xrightarrow{f} C \xleftarrow{g} I_2 \\
C & \xrightarrow{i_1} I_2 \\
I_2 & \xrightarrow{f'} D \xleftarrow{g'} I_3
\end{align*}
\]
Composition & assoc.

Identities: \( I_1 \xrightarrow{id} I_1 \xleftarrow{id} I_1 \)

Composition by pushout

\[
\begin{array}{ccc}
I_1 & \xrightarrow{f} & C & \xleftarrow{g} & I_2 & \xrightarrow{f'} & D & \xleftarrow{g'} & I_3 \\
& ^{i_1} & _{i_2} & & & & & & \\
C + I_2 D & & & & & & & & \\
\end{array}
\]

\( a : (C + I_2 D) + I_3 E \to C + I_2 (D + I_3 E) \)

\( e_l : (I_1 + I_1 C) \to C \) satisfying coherence

\( e_r : (C + I_2 I_2) \to C \)
A suitable underlying cat

Motivation: find a natural class of categories which allows construction of GRPOs in cospans.

- adhesive categories (Lack Sobocinski ‘04)
- “well-behaved” pushouts along monomorphisms
Examples

- Set, Graph
- C adhesive, so is $C/C$ and $C/C$
- C adhesive, so is $[D,C]$
- C and D adhesive, so is $CxD$
Left-linear cospans

When $\mathbf{C}$ is adhesive LLC($\mathbf{C}$) is the bicategory

objects as in $\mathbf{C}$

arrows cospans $\xymatrix{I_1 \ar@<2ex>[r]^m & \mathbf{C} & I_2 \ar@<2ex>[l]^g}$
Theorem: Suppose that $\mathbf{C}$ is an adhesive category, LLC($\mathbf{C}$) has GRPOs.
GRPOs in LLC(C)

Given redex square...
Given redex square...
GRPOs of cospans

... find minimal factorisation

\[
\begin{array}{c}
I_4 \\
\downarrow G \\
I_2 \\
\downarrow \gamma \\
E \rightarrow I_5 \\
\downarrow \delta \\
I_3 \\
\downarrow F \\
I_1 \\
\downarrow \beta \\
A \leftarrow B \\
\end{array}
\]
GRPOs of cospans

... find minimal factorisation

![Diagram of GRPOs of cospans]
Characterising GIPOs

Borrowed contexts
(Ehrig and Konig ‘04)
Characterising GIPOs

Borrowed contexts
(Ehrig and Konig ‘04)
Plan of the talk

1. Deriving bisimulation congruences
2. Cospans as generalised contexts
3. Bisimulation for graph rewriting
   - double pushout graph rewriting
   - graph rewriting as reactive system
   - using GRPOs to construct LTSs
Dpo-Rewriting

Ehrig (1973)

A framework for graph-rewriting

Rewrite rules $\rho$ are spans

\[ L \leftarrow l K \rightarrow r R \]
Lemma: a reactive system on Cospan(Graph) is the same thing as a dpo rewriting system.
LTS for graph rewriting

The resulting LTS has:

- Nodes: graphs (up-to-iso) with output interface (possibly non-mono)
- Labels: smallest graph contexts (up-to-iso) which allow reaction

**Theorem:** Bisimulation, trace equivalence, failures equivalence are congruences
Ehrig and Konig (2004)

- LTS for graph rewriting
- all interfaces mono
- when restricting our GRPOs to linear cospans we derive the same (concrete) LTS
- their congruence theorem follows from congruence theorem for GRPOs
Advantages of LTS

- Transfer of concepts from process algebra to graph rewriting
- Process calculus on graphs?
  - the class of adhesive categories seems to cover many “graph-like” categories
Conclusion

Construction of labels for an interesting class of reactive systems

Other applications:
- bigraphs
- right-linear cospans?
- Petri nets
