MA676: Bayesian Methods – Additional Exercises 2

1. Suppose that, $X_1, \ldots, X_n$, a random sample of size n is taken from a Poisson distribution for which the value of the mean θ is unknown, and that the prior distribution of θ is a gamma distribution, with density given by

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}, \quad \theta > 0.$$

Show that the Bayes estimator of θ under squared error loss will be a weighted average having the form $a_n\bar{X}_n + (1 - a_n)\mu$ where μ is the prior mean of θ, and show that $a_n \to 1$ as $n \to \infty$.

2. A Bayes estimator is required for θ under the loss function

$$L(a, \theta) = e^{c(a-\theta)} - c(a - \theta) - 1,$$

where c is a positive constant. As the constant c varies, the loss function varies from very asymmetric to almost symmetric. This is called the LINEX (LINear-EXponential) loss.

By minimising the expected loss, show that the Bayes estimator is

$$\hat{\theta} = \frac{-1}{c}\log E\left(e^{-c\theta}|\mathbf{x}\right),$$

where the expectation is under the posterior distribution $\pi(\theta|\mathbf{x})$.

Suppose that $X_1, \ldots, X_n$ is a random sample from the normal distribution $N(\theta,\ \sigma^2)$ where σ^2 is known. A priori θ follows the normal distribution with mean μ and variance τ^2 where both μ and τ^2 are known. Find the Bayes estimators of θ under the above loss function and under the squared error loss. Compare the two estimators. **Hint:** Recall the definition and the expression for the moment generating function of a normal distribution.

3. Suppose that $X_1, \ldots, X_n$ is a sample from the negative binomial distribution which has the probability mass function,

$$f(x|r, \theta) = \binom{r + x - 1}{x}\theta^r(1 - \theta)^x, \quad x = 0, 1, \ldots;\ 0 < \theta < 1,$$

where $r > 0$ is a known integer. Suppose also that a-priori θ has a beta(α, β) distribution with the pdf,

$$\pi(\theta) = \frac{1}{B(\alpha, \beta)}\theta^{\alpha-1}(1 - \theta)^{\beta-1},\ 0 < \theta < 1.$$

Find the posterior distribution of θ and the Bayes estimator under squared and absolute error loss, assume that $r = 2$ and $x = 5$.

4. Prove that the expected square and absolute error loss is minimised by the posterior mean and median, respectively **without differentiation**.